LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT FOR THE SESSION 1886-87 (NOV. 12TH, 1886, TO NOV. 9TH, 1887).

CASH ACCOUNT.

Cr.

		£.	s.	d.
Balance from 1885-86:—				
General Fund	£158 14 7			
Life Compositions Fund...	10 10 0			
De Morgan Medal Fund...	7 18 1			
		177	2	8
Interest on Capital—				
Lord Rayleigh's Fund ...	53 18 5			
Life Compositions Fund...	23 16 2			
Invested Surplus Fund ...	10 3 10			
De Morgan Medal Fund ...	3 2 10			
		91	1	3
10 Entrance Fees		10	10	6
143 Subscriptions—				
1 for 1881–82	£1 1 0			
1 for 1882–83	1 1 0			
3 for 1883–84	3 3 0			
11 for 1884–85	11 11 0			
58 for 1885–86	60 18 0			
65 for 1886–87	68 5 0			
4 for 1887–88	4 4 0			
		150	3	0
Sales of *Proceedings*		54	6	11
5 Life Compositions		52	10	0

Dr.

		£.	s.	d.
Printing *Proceedings*, &c.		186	12	0
Purchase of Journals		8	1	0
Binding		11	6	9
Postage, Stationery, and Sundries		16	19	4
Rent		10	0	0
Coals and Gas		1	10	0
Attendance—				
British Association ...	£1 5 0			
Mr. Stewardson	10 10 0			
		11	15	0
Life Compositions Fund—				
Purchase of £61. 18*s*. Three per Cent. Consols		63	0	0
De Morgan Medal Fund—				
2nd Medal	£9 7 6			
Postage and carriage of dies...	0 2 0			
		9	9	6
Balance at Bank—				
General Fund	£215 9 4			
De Morgan Medal Fund	1 11 5			
		217	0	9

	£.	s.	d.		£.	s.	d.
Cash at Bank	215	9	4	4 Subscriptions for 1887-88 paid in advance...	4	4	0
88 Subscriptions owing—							
4 for 1883-4 ... 4 4 0							
13 for 1884-5 ... 13 13 0							
22 for 1885-6 ... 23 2 0							
*49 for 1886-7 ... 51 9 0	92	8	0	Balance	303	13	4
	£307	17	4		£307	17	4

* Viz., 116, the number of Subscribers for 1886-87, less 2 Subscriptions paid in advance in 1885-86, and 65 paid in 1886-87.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	1	11	5	Balance	1	11	5

CAPITAL ACCOUNT.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
General Fund—							
1. Life Compositions Fund	847	0	0	865	12	3	Three per Cent. Consols.
2. Lord Rayleigh's Fund	1000	0	0	870	0	0	Guaranteed Five per Cent. Great Indian Peninsula Railway Stock.
3. Invested Surplus Fund	350	0	0	350	16	3	New Three per Cents.
De Morgan Medal Fund	103	5	3	104	19	8	Reduced Three per Cents.

Audited and found correct,

17th November, 1887. (Signed) **A. B. BASSET.**

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XIX.

FROM NOVEMBER, 1887, TO NOVEMBER, 1888.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1889.

LONDON:
PRINTED BY C. F. HODGSON AND SON,
GOUGH SQUARE, FLEET STREET.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XIX.

TWENTY-FOURTH SESSION, 1887—1888.

November 10*th*, 1887.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

PRESENTATION OF THE DE MORGAN MEDAL.

Prof. Sylvester, being incapacitated by an accident from attending in person, deputed Mr. J. Hammond to act as his representative.

The President prefaced the presentation with the following remarks, to which Mr. Hammond made a suitable reply.

"It is my duty, acting for the Society, to present to you, on behalf of Professor Sylvester, our De Morgan Medal. This award is but a just recognition of those vast mathematical attainments and discoveries which will make his name immortal. To detail on this occasion, or even to enumerate, achievements so varied is impracticable. The mere phrase 'Modern Algebra' is suggestive enough. But in every department of pure mathematics the marks of his strong hand are seen. Take, for instance, the Theory of Equations. Here, among other things, we owe to him the clearing up of the doctrine of Newton's Rules, and to him we owe those researches on Linear Solution which have recently culminated in his papers on the Hamiltonian Numbers. Were it desirable to dwell at length on his labours, Elimi-

nation, Partition, Operation, n-ads, and other topics would crowd in on us; but one cannot forbear from remarking that he is as happy in the invention of new weapons as he is dexterous in the wielding of the older ones, and from hoping that his new and splendidly successful Theory of Reciprocants may not be his last boon to science. This presentation on the part of the Society is, after all, only the expression of an admiration which is felt in all civilised lands, and which will be felt in after times, so long at least as men continue to admire grand powers nobly employed."

The following gentlemen were elected members:—E. G. Gallop, M.A., Fellow of Trinity College, Cambridge; E. W. Hobson, M.A., Fellow of Christ's College, Cambridge; R. W. Hogg, M.A., Fellow of St. John's College, Cambridge, Mathematical Master at Christ's Hospital; and A. M. Nash, M.A., formerly Scholar of Queen's College, Oxford, Professor in the Calcutta University.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Mr. A. G. Greenhill, seconded by Mr. W. J. C. Sharp, and carried.

The Treasurer subsequently announced that the application of the Society to the Privy Council for a Charter of Incorporation had failed.

At the request of the Chairman, Mr. A. B. Basset consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of members since the General Meeting, held November 11th, 1886, had increased from 184 to 192.

The Society had lost one member by death, viz., Mr. Frank Scott Haydon, B.A., of the Record Office.

The following communications had been made:—

Certain Operators in connection with Symmetric Functions: Mr. Lachlan.

On the Transformations of the General Elliptic Element $\frac{\delta x}{\sqrt{U_x}}$, where $U_x = x-\alpha . x-\beta . x-\gamma . x-\delta \; (=ax^4+4bx^3+6cx^2+4dx+e)$: Mr. Robert Russell.

Discussion of a Multilinear Operator, with applications to the Theories of Invariants and Reciprocants: Captain MacMahon, R.A.

The Theory of Screws in Elliptic Space (Fourth Note): Mr. A. Buchheim.

The Rectification of certain Curves: Mr. R. A. Roberts.

The Rectification of a Sphero-Conic: Mr. H. F. Burstall.

Third Paper on Reciprocants: Mr. L. J. Rogers.

The "Sine-Triple-Angle" Circle: Mr. Tucker.

The Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants: Mr. E. B. Elliott.

Circular Notes: Mr. Tucker.

The Problem of the Duration of Play: Captain P. A. MacMahon, R.A.

Notes on Two Annihilators in the Theory of Elliptic Functions: Mr. J. Griffiths.

Conjugate Tucker Circles: Mr. Tucker.

On the Incorrectness of the Rules for contracting the processes of finding the Square and Cube Roots of a Number: Professor M. J. M. Hill.

On the Complex Angle: Mr. J. J. Walker.

On the Equation of Riccati: the President.

The "Orthocentroidal" Triangle: Mr. Tucker.

On Polygons inscribed in a Quadric and circumscribed about two Confocal Quadrics: Mr. R. A. Roberts.

On the Binomial Equation $x^p - 1 = 0$, Quinquisection: Prof. Lloyd Tanner.

Symmetrical Determinant Formulæ in Elliptic Functions: Mr. L. J. Rogers.

A Note on Curves: Mr. H. M. Taylor.

Some Generalisations of Formulæ connected with the change of the Independent Variable in a Differential Expression, with application to a new class of Reciprocants: Mr. C. Leudesdorf.

A Metrical Property of Plane Curves: Mr. R. Lachlan.

Note on the Weierstrass Functions: Mr. A. G. Greenhill.

Second Paper on change of the Independent Variable, with applications to some Functions of the Reciprocant kind: Mr. C. Leudesdorf.

Note on Knots: the Treasurer.

On the Intersections of a Circle and a Plane Curve: Prof. Genese.

A new Theory of Harmonic Polygons: Rev. T. C. Simmons.

On some Properties of Simplicissima, with especial regard to the related Spherical Loci: Mr. W. J. C. Sharp.

On Briot and Bouquet's Theory of the Differential Equation $F\left(u, \frac{du}{dz}\right) = 0$: Prof. Cayley.

The Cosine Orthocentres of a Plane Triangle and a Cubic through them: Mr. Tucker.

Note on a Tetrahedron: Dr. Wolstenholme.

General Theory of Dupin's Space-extension of the Focal Properties of Conic Sections: Dr. J. Larmor.

Sur une propriété de la Sphère et son extension aux Surfaces quelconques: M. Maurice d'Ocagne.

On the Motion of Two Spheres in a Liquid, and allied Problems: Mr. A. B. Basset.

Second Note on Elliptic Transformation Annihilators: Mr. J. Griffiths.

Note on the Linear Covariants of the Binary Quintic: Mr. A. Buchheim.

The Motion of a Sphere in a Viscous Liquid: Mr. A. B. Basset.

On the Reversion of Series in connection with Reciprocants: Captain P. A. MacMahon, R.A.

Explanation of Illustrations of a Preliminary Note on Diameters of Cubics: Mr. J. J. Walker.

The same Journals had been subscribed for as in the preceding Session.

The following additions had been made to the list of exchanges:— "Annals of Mathematics," edited by Ormond Stone, of Leander McCormick Observatory, University of Virginia; and "Circolo Matematico," of Palermo.

The meeting next proceeded to the election of the new Council.

The Scrutators (Messrs. G. Heppel and H. Perigal) having examined the Balloting Lists, declared the following gentlemen duly elected:—President, Sir J. Cockle, Knt., F.R.S.; Vice-Presidents, J. W. L. Glaisher, F.R.S., Prof. Hart, M.A., Lord Rayleigh, F. and Sec. R.S.; Treasurer, A. B. Kempe, F.R.S.; Hon. Secs., M. Jenkins, M.A., R. Tucker, M.A.; other Members, A. Buchheim, M.A., E. B. Elliott, M.A., A. G. Greenhill, M.A., J. Hammond, M.A., J. Larmor, D.Sc., C. Leudesdorf, M.A., Captain P. A. MacMahon, R.A., S. Roberts, F.R.S., and J. J. Walker, F.R.S.

The following communications were made:—

On Pure Ternary Reciprocants and Functions allied to them: Mr. E. B. Elliott.

On the General Linear Differential Equation of the Second Order: the President.

On the Stability of a Liquid Ellipsoid which is rotating about a principal axis under the influence of its own Attraction: Mr. A. B. Basset.

(1) On Modular Equations, and (2) Geometry of the Quartic: Mr. R. Russell.

The Differential Equations satisfied by Concomitants of Quantics: Mr. A. R. Forsyth.

On the Stability or Instability of certain Fluid Motions—II.: Lord Rayleigh.

Notes on a system of Three Conics touching at one point: Dr. Wolstenholme.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLIII., No. 258.

"Educational Times," for November.

"Bulletin des Sciences Mathématiques," T. XI., Nov., 1887.

"Bulletin de la Société Mathématique de France," T. XV., No. 6, et avant-dernier.

"L'Académie Royale de Belgique—Bulletins," 3e serie, Tomes 9—13; Annuaires, 1886 and 1887.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VIII., No. 1.

"Bollettino delle Pubblicazioni Italiane, ricevute per diritto di Stampa," Nos. 43 and 44.

"Memorias de la Sociedad Cientifica 'Antonio Alzate,'" Tomo I., No. 3.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Anno XXV., Fasc. 4—12.

"Memorie delle Regia Accademia di Scienze, Lettere, ed Arti in Modena," Serie I., Tomo XX., Parte III.; Serie II., Vol. IV.

"Königlich-Sächsischen Gesellschaft der Wissenschaften—Abhandlungen der Mathematisch-physischen Classe," Band XIV., Nos. 1—4; Leipzig.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Vol. XXXII., No. 1.

"Annales de l'École Polytechnique de Delft," T. III., 2me livr.; Leide, 1887.

On Pure Ternary Reciprocants, and Functions allied to them.

By Mr. E. B. ELLIOTT.

[*Read Nov.* 10*th*, 1887.]

1. In the present paper reference will from time to time be made to the two here mentioned. They will be quoted for shortness as Paper I. and Paper II., respectively,—

I. *On Ternary and n-ary Reciprocants* (*Proceedings*, Vol. XVII., pp. 172—196).

II. *On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants* (*Proceedings*, Vol. XVIII., pp. 142—164).

The notation used will be that of Paper II. Thus, for instance, z_{rs} will throughout denote $\frac{1}{r!\,s!}\,\frac{d^{r+s}z}{dx^r dy^s}$. The main conclusion of Paper II. was, it will be remembered, that pure ternary reciprocants are those homogeneous and doubly isobaric functions of derivatives, such as z_{rs}, which have four annihilators called Ω_1, Ω_2, V_1, V_2. Of these the first two are

$$\Omega_1 = \Sigma\left\{(m+1)\,z_{m+1,\,n-1}\frac{d}{dz_{mn}}\right\},\ n \nless 1,\ m+n \nless 2 \dots\dots\dots (1),$$

$$\Omega_2 = \Sigma\left\{(n+1)\,z_{m-1,\,n+1}\frac{d}{dz_{mn}}\right\},\ m \nless 1,\ m+n \nless 2 \dots\dots\dots (2),$$

while the two others [*cf.* Paper II., § 9 (v.)] may be most compactly written

$$V_1 = \Sigma\left\{\Sigma\,(rz_{rs}z_{m+1-r,\,n-s})\,\frac{d}{dz_{mn}}\right\} \dots\dots\dots\dots\dots\dots\dots (3),$$

the inner summation in which is limited by $r+s \nless 2$, $r \ngtr m+1$, $s \ngtr n$, $r+s \ngtr m+n-1$, and the outer by $m+n \nless 3$, and

$$V_2 = \Sigma\left\{\Sigma\,(sz_{rs}z_{m-r,\,n+1-s})\,\frac{d}{dz_{mn}}\right\} \dots\dots\dots\dots\dots\dots\dots (4),$$

limited by $r+s \nless 2$, $r \ngtr m$, $s \ngtr n+1$, $r+s \ngtr m+n-1$, $m+n \nless 3$. Within limits as stated in each case, the summations are all supposed to be taken over all the range of positive integral (including zero) values of m, n, r, s.

It s proposed to base most of what follows on the consideration of

functions, homogeneous and isobaric both in first and second suffixes, which, though subject to annihilation by both V_1 and V_2, are not annihilated also both by Ω_1 and Ω_2, and consequently, not being full invariants of the emanants $(z_{20}, z_{11}, z_{02} \between (u, v)^2$, &c., are not reciprocants.

2. It will be well to have before us the matrix

$$
\left|\begin{array}{lllllllllllllll}
z_{20} \\
z_{11} \\
z_{02} \\
z_{30}, & z_{20} \\
z_{21}, & z_{11}, & z_{20} \\
z_{12}, & z_{02}, & z_{11} \\
z_{03}, & & z_{02} \\
z_{40}, & z_{30}, & & z_{20} \\
z_{31}, & z_{21}, & z_{30}, & z_{11}, & z_{20} \\
z_{22}, & z_{12}, & z_{21}, & z_{02}, & z_{11}, & z_{20} \\
z_{13}, & z_{03}, & z_{12}, & & z_{02}, & z_{11} \\
z_{04}, & & z_{03}, & & & z_{02} \\
z_{50}, & z_{40}, & & z_{30}, & & & z_{20} \\
z_{41}, & z_{31}, & z_{40}, & z_{21}, & z_{30}, & & z_{11}, & z_{20} \\
z_{32}, & z_{22}, & z_{31}, & z_{12}, & z_{21}, & z_{30}, & z_{02}, & z_{11}, & z_{20} \\
z_{23}, & z_{13}, & z_{22}, & z_{03}, & z_{12}, & z_{21}, & & z_{02}, & z_{11}, & z_{20} \\
z_{14}, & z_{04}, & z_{13}, & & z_{03}, & z_{12}, & & & z_{02}, & z_{11} \\
z_{05}, & & z_{04}, & & & z_{03}, & & & & z_{02} \\
z_{60}, & z_{50}, & & z_{40}, & & & z_{30}, & & & & z_{20} \\
z_{51}, & z_{41}, & z_{50}, & z_{31}, & z_{40}, & & z_{21}, & z_{30}, & & & z_{11}, & z_{20} \\
z_{42}, & z_{32}, & z_{41}, & z_{22}, & z_{31}, & z_{40}, & z_{12}, & z_{21}, & z_{30}, & & z_{02}, & z_{11}, & z_{20} \\
z_{33}, & z_{23}, & z_{32}, & z_{13}, & z_{22}, & z_{31}, & z_{03}, & z_{12}, & z_{21}, & z_{30}, & & z_{02}, & z_{11}, & z_{20} \\
z_{24}, & z_{14}, & z_{23}, & z_{04}, & z_{13}, & z_{22}, & & z_{03}, & z_{12}, & z_{21}, & & & z_{02}, & z_{11}, & z_{20} \\
z_{15}, & z_{05}, & z_{14}, & & z_{04}, & z_{13}, & & & z_{03}, & z_{12}, & & & & z_{02}, & z_{11} \\
z_{06}, & & z_{05}, & & & z_{04}, & & & & z_{03}, & & & & & z_{02} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right.
$$

From this we can immediately extract any number of determinants which are functions obeying the laws stated at the end of the last article. In fact, we have the

Theorem.—*A determinant of any order n, obtained by selecting from the matrix a row which extends to the n^{th} column and no further, and any $n-1$ preceding rows, is a homogeneous and doubly isobaric function of the derivatives, and is annihilated by both V_1 and V_2.* Call this Prop. I.

For instance, any one of the first three rows is such a determinant of one term, any three of the rows 4 to 7 give such a determinant of the third order, the five rows 8 to 12 with any previous row give one of the fifth order, &c., &c.

That such determinants are all homogeneous needs no proof; that they are separately isobaric in first and second suffixes follows from the fact that the differences of the two partial weights of the constituents in two chosen columns and any the same row are both independent of the particular row; and that they are annihilated by V_1 and V_2 is made clear as follows.

Adopt for the moment the notation c_{rs} to denote the constituent in the r^{th} row and s^{th} column of the matrix. It is easy to see that, the summations extending to all values of the number r,

$$V_1 = \Sigma\left[\{2z_{20}c_{r2}+z_{11}c_{r3}+3z_{30}c_{r4}+2z_{21}c_{r5}+z_{12}c_{r6} \\ +4z_{40}c_{r7}+3z_{31}c_{r8}+2z_{22}c_{r9}+z_{13}c_{r10}+\dots\}\frac{d}{dc_{r1}}\right],$$

and $$V_2 = \Sigma\left[\{z_{11}c_{r2}+2z_{02}c_{r3}+z_{21}c_{r4}+2z_{12}c_{r5}+3z_{03}c_{r6} \\ +z_{31}c_{r7}+2z_{22}c_{r8}+3z_{13}c_{r9}+4z_{04}c_{r10}+\dots\}\frac{d}{dc_{r1}}\right];$$

whence it follows that V_1, operating on the first column, produces from it a sum of multiples of succeeding columns; and similarly for V_2. Moreover, if any other column than the first be chosen, subsequent columns can be selected in which its constituents are followed by other constituents exactly in the same arrangement as are the same constituents where they appear in the first column. Thus, the operation of V_1 on any column produces a column which is a sum of multiples of following columns; and similarly for V_2. V_1 and V_2 then both annihilate all determinants which can be obtained by associating complete rows of the matrix.

3. It is of great importance to remark that, whatever be the function operated on, the following four surprisingly simple equiva-

lences of operators hold :—

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \quad\text{.............................}(5),$$

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \quad\text{.............................}(6),$$

$$\Omega_2 V_1 - V_1 \Omega_2 = V_2 \quad\text{..........................}(7),$$

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\text{..........................}(8).$$

To prove the first of these, use the expressions of § 1 for Ω_1 and V_1 Selecting the terms which give $\frac{d}{dz_{mn}}$ in $\Omega_1 V_1 - V_1 \Omega_1$, we find that if $n > 0$ the coefficient of $\frac{d}{dz_{mn}}$ is

$$\Sigma\{r(r+1)\, z_{r+1,\, s-1},\ z_{m+1-r,\, n-s}\} + \Sigma\,\{r\,(m+2-r)\, z_{rs} z_{m+2-r,\, n-1-s}\}$$
$$-(m+1)\,\Sigma\,\{r z_{rs} z_{m+2-r,\, n-1-s}\},$$

the first range of summation being limited by

$$r \not> m+1, \quad s \not< 1 \not> n,$$

the second by $\quad r \not> m+1, \quad s \not> n-1,$

the third by $\quad r \not> m+2, \quad s \not> n-1,$

and all three by $\quad r+s \not< 2 \not> m+n-1.$

Now the ranges of the second and third summations, though apparently different, are really the same, since the value $m+2$ of r, which belongs to the third though not to the second, adds to the second only a zero term in virtue of the coefficient $m+2-r$. Thus the coefficient is equal to

$$\Sigma\,\{r\,(r+1)\, z_{r+1,\, s-1} z_{m+1-r,\, n-s}\} \text{ over the range } r \not> m+1,\ s \not< 1 \not> n$$

$$-\Sigma\,\{r\,(r-1)\, z_{rs}\, z_{m+2-r,\, n-1-s}\} \quad \text{over the range } r \not> m+2,\ s \not> n-1,$$

the ranges being further limited by $r+s \not< 2 \not> m+n-1$. But, if in the latter summation, and the conditions determining its limits, we put $r+1$ for r and $s-1$ for s, we produce exactly the former summation and the conditions by which it is limited. Thus the difference of the summations, *i.e.*, the coefficient of $\frac{d}{dz_{mn}}$ in $\Omega_1 V_1 - V_1 \Omega_1$, vanishes.

The case of $n = 0$ has here been omitted. No such symbol as $\frac{d}{dz_{m0}}$, however, occurs in Ω_1, while in V_1 the coefficients of such symbols

contain only such derivatives as $z_{m'0}$, and are consequently annihilated by Ω_1. It follows that the coefficient of $\frac{d}{dz_{m0}}$ in $\Omega_1 V_1 - V_1 \Omega_1$ vanishes.

The proof is therefore complete, that

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \qquad \text{(5)}.$$

In precisely the same way, or merely by interchange of first and second suffixes throughout,

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \qquad \text{(6)}.$$

In reducing $\Omega_2 V_1 - V_1 \Omega_2$ we must consider separately the coefficients of symbols like $\frac{d}{dz_{0n}}$ and those of the more general symbols $\frac{d}{dz_{mn}}$, where m is not zero. We have, firstly,

$$\begin{aligned}
\text{Co. } \frac{d}{dz_{0n}} &\text{ in } \Omega_2 V_1 - V_1 \Omega_2 \\
&= \Omega_2 \text{ Co. } \frac{d}{dz_{0n}} \text{ in } V_1 \\
&= \Omega_2 \Sigma (z_{1s} z_{0,n-s}) \\
&= \Sigma \left\{ (s+1)\, z_{0,s+1} z_{0,n-s} \right\},\ s \nless 1 \ngtr n-2;
\end{aligned}$$

and, secondly, for values of m exceeding zero,

$$\text{Co. } \frac{d}{dz_{mn}} \text{ in } \Omega_2 V_1 - V_1 \Omega_2$$

$$= \Sigma \left\{ r\,(s+1)\, z_{r-1,s+1} z_{m+1-r,n-s} \right\} + \Sigma \left\{ r\,(n-s+1)\, z_{rs} z_{m-r,n-s+1} \right\}$$
$$- (n+1)\, \Sigma \left\{ r z_{rs} z_{m-r,n-s+1} \right\},$$

all three summations being limited by

$$r+s \nless 2 \ngtr m+n-1,$$

the first also by $\quad r \nless 1 \ngtr m+1, \quad s \ngtr n,$

the second by $\quad r \ngtr m, \quad s \ngtr n,$

and the third by $\quad r \ngtr m, \quad s \ngtr n+1.$

The value $n+1$ of s, which belongs to the range of the third but not to that of the second summation, gives rise only to a zero term if added to that range, in virtue of the coefficient $n-s+1$. Thus the

difference of the second and third parts of Co. $\dfrac{d}{dz_{mn}}$ is

$$-\Sigma\left\{rs\,z_{rs}\,z_{m-r,\,n-s+1}\right\},$$

over the range limited by

$$r+s \nless 2 \ngtr m+n-1, \quad r \ngtr m, \quad s \ngtr n+1.$$

Now in the first summation put $r+1$ for r, and $s-1$ for s, thus making the limits of that summation identical with these limits. We obtain the result

$$\begin{aligned}\text{Co. } \frac{d}{dz_{mn}} \text{ in } \Omega_2 V_1 - V_1 \Omega_2 &= \Sigma\left[\left\{(r+1)\,s - rs\right\} z_{rs}\,z_{m-r,\,n-s+1}\right]\\ &= \Sigma\left\{s z_{rs}\,z_{m-r,\,n-s+1}\right\},\end{aligned}$$

limited by $r+s \nless 2 \ngtr m+n-1$, $r \ngtr m$, $s \ngtr n+1$, a form with which the previously found coefficient of $\dfrac{d}{dz_{0n}}$ is strictly in accord.

Thus $$\Omega_2 V_1 - V_1 \Omega_2 = \Sigma\left\{\Sigma\left(s z_{rs}\,z_{m-r,\,n-s+1}\right)\right\},$$

over the inner range limited as above, and the outer limited by $m+n \nless 3$,

$$= V_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence also, lastly, by interchange of first and second suffixes throughout,

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

4. The conclusions which can be drawn from the four symbolical identities (5) to (8) are numerous and important. Attention is in the first place called to one which affects primarily the theory of invariants and seminvariants of a system of quantics, but which will be seen later (§ 12) to have also an important bearing on the theory of reciprocants.

PROP. II.—*From any seminvariant I of the system of quantics*

$$(z_{20}, z_{11}, z_{02} \between u, v)^2, \quad (z_{30}, z_{21}, z_{12}, z_{03} \between u, v)^3, \text{ \&c.,}$$

another seminvariant of the system may be generated by operating with V_1 upon it.

For, if $\Omega_1 I = 0$, $V_1 \Omega_1 I = 0$, and therefore, by (5), $\Omega_1 V_1 I = 0$, *i.e.*, $V_1 I$ is annihilated by Ω_1, and is a seminvariant.

The proposition is of course a purely algebraical one with regard to the quantics, whatever be their coefficients, being quite independent of any notion as to those coefficients being derivatives of a function z with regard to x and y.

If the seminvariant I be of degree i, the seminvariant $V_1 I$ thus generated is of degree $i+1$. The first partial weight of $V_1 I$, *i.e.*, the sum of first suffixes in each of its terms, exceeds that of I by unity, and the second partial weight, sum of second suffixes, is the same in I and $V_1 I$. This second partial weight is the weight in the ordinary language of binary quantics. Thus, adopting that ordinary language, the weight of $V_1 I$ is the same as that of I, while its degree exceeds the degree of I by unity.

One example of this use of the operator V_1 will suffice for the present. Choose for I the seminvariant $ac-b^2$ of the n-ic,

$$(z_{n0},\ z_{n-1,1}\ \ldots\ z_{0n}\between u,\ v)^n,$$

i.e., take $$I = 2nz_{n0}z_{n-2,2}-(n-1)\,z^2_{n-1,1}.$$

We deduce from this the seminvariant

$$\begin{aligned}
\tfrac{1}{2}V_1 I &= nz_{n-2,2}\left(\text{Co.}\frac{d}{dz_{n0}}\text{ in }V_1\right)+nz_{n0}\left(\text{Co.}\frac{d}{dz_{n-2,2}}\text{ in }V_1\right)\\
&\qquad -(n-1)\,z_{n-1,1}\left(\text{Co.}\frac{d}{dz_{n-1,1}}\text{ in }V_1\right)\\
&= nz_{n-2,2}\sum_{r\not<2}^{r\not>n-1}(rz_{r0}z_{n+1-r,0})\\
&\quad +nz_{n0}\left\{\sum_{r\not<2}^{r\not>n-1}(rz_{r0}z_{n-1-r,2})+\sum_{r\not<1}^{r\not>n-2}(rz_{r1}z_{n-1-r,1})\right.\\
&\qquad\qquad \left.+\sum_{r\not<0}^{r\not>n-3}(rz_{r2}z_{n-1-r,0})\right\}\\
&\quad -(n-1)\,z_{n-1,1}\left\{\sum_{r\not<2}^{r\not>n-}(rz_{r0}z_{n-r,1})+\sum_{r\not<1}^{r\not>n-2}(rz_{r1}z_{n-r,0})\right\}\quad\ldots(9).
\end{aligned}$$

In particular, taking $n=3$, from the seminvariant

$$2\,(3z_{30}z_{12}-z^2_{21}),$$

we obtain in this manner

$$6z_{12}z^2_{20}+3z_{30}\,(2z_{30}z_{02}+z^2_{11})-6z_{21}z_{30}z_{11},$$

a seminvariant from which, upon subtraction of

$$3z_{30}\,(z^2_{11}-4z_{30}z_{02}),$$

which is another of the same degree and weights, and division by 6, we obtain another of three terms only, viz.,

$$z_{12}z_{20}^2 - z_{21}z_{20}z_{11} + 3z_{30}z_{20}z_{02},$$

or, again, by adding $\quad z_{30}(z_{11}^2 - 4z_{20}z_{02}),$

the seminvariant $\quad z_{12}z_{20}^2 - z_{21}z_{20}z_{11} + z_{30}(z_{11}^2 - z_{20}z_{02}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$

i.e.

$$\begin{vmatrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \end{vmatrix}$$

of which V_1 and V_2 are annihilators. This we shall meet with again presently.

From any seminvariant formed by the method of this article, repeated operation with Ω_2 will, of course, enable us to write down all the coefficients of a corresponding covariant.

It is almost unnecessary to add that, in virtue of $\Omega_2V_2 - V_2\Omega_2 = 0$, *$V_2$ is in like manner a generator of seminvariants of the same quantics read from right to left, from other such seminvariants.* This may be quoted as Prop. III.

5. Of other results of the equivalences (5) to (8), the following will be useful for present purposes.

Prop. IV.—*If a function R of the derivatives is annihilated by V_1, so also is Ω_1R.*

For, by (5), $V_1\Omega_1R = \Omega_1V_1R = 0$. In like manner, by (6).

Prop. V.—*If a function R is annihilated by V_2, so also is Ω_2R.*

Prop. VI.—*If a function R is annihilated by both V_1 and V_2, so also are both Ω_1R and Ω_2R.*

That Ω_1R is annihilated by V_1, and Ω_2R by V_2, is told us by the two last propositions. That Ω_1R is annihilated by V_2 is true since, by (8),

$$V_2\Omega_1R = \Omega_1V_2R - V_1R = 0;$$

and, similarly, that Ω_2R is annihilated by V_1 is seen to be necessary.

Prop. VII.—*If V_2 and Ω_1 both annihilate a function, so too does V_1.* This is an immediate consequence of (8). Similarly by (7).

Prop. VIII.—*If V_1 and Ω_2 both annihilate a function, so too does V_2.*

6. We are now in a position to construct an important class of covariants of the emanants $(z_{20}, z_{11}, z_{02})(u, v)^2$, &c.

Take P, a pure function of the derivatives, which is annihilated both by V_1 and V_2, but not by both Ω_1 and Ω_2. If homogeneous and doubly isobaric—and to such functions it will be well to confine attention—it is exceedingly likely to be a coefficient of a covariant of the emanants; but, even if it be not such a coefficient, a covariant, all whose coefficients have the same property as itself, may be obtained from it as follows.

By repeated operation with Ω_1 form the series of pure functions $\Omega_1 P$, $\Omega_1^2 P$, $\Omega_1^3 P$, ... These, like P, will by Prop. VI. be annihilated by V_1 and V_2. Now, each of these functions is of second partial weight one lower than the immediately preceding. One of them, $\Omega_1^n P$, must therefore be presently arrived at, which and all its successors vanish. The last non-vanishing one, $\Omega_1^{n-1} P$, is then annihilated by Ω_1, that is to say, is a seminvariant of the emanants. Call it P_0.

Operate on P_0 repeatedly by Ω_2 till a vanishing result $\Omega_2^{m+1} P_0$ is obtained. Then, writing, in accordance with this fact,

$$\Omega_2 P_0 = mP_1,\ \Omega_2 P_1 = (m-1)\, P_2,\ \dots\ \Omega_2 P_{m-1} = P_m,\ \Omega_2 P_m = 0,$$

we obtain a covariant of the emanants,

$$(P_0,\ P_1,\ P_2,\ \dots\ P_m)(u,\ v)^m \ \dots\dots\dots\dots\dots\dots\dots (11),$$

all whose coefficients are annihilated by V_1 and by V_2.

If the degree and partial weights of P_0 are i, w_1, w_2,

those of P_1 are i, $w_1 - 1$, $w_2 + 1$,

those of P_2 are i, $w_1 - 2$, $w_2 + 2$,

&c.,

and finally, those of P_m are i, $w_1 - m$, $w_2 + m$.

Thus, we have $w_1 - m = w_2$ and $w_2 + m = w_1$,

each of which is identical with

$$m = w_1 - w_2.$$

(In particular, if $w_1 = w_2$, $m = 0$; and the covariant reduces to a single term—an invariant of the emanants, and consequently a reciprocant.)

An instance of such covariants is the cubic

$$A_0 u^3 + 3A_1 u^2 v + 3A_2 uv^2 + A_3 v^3 \ \dots\dots\dots\dots\dots\dots (12),$$

where A_0 is the seminvariant (10)—where A_0, A_1, A_2, A_3 are, in fact, the determinants obtained by omitting the fourth, third, second, first

rows, respectively, from the matrix

$$\left\|\begin{matrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \\ z_{03}, & & z_{02} \end{matrix}\right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

It may be here remarked that the conditions $A_0 = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 0$, two only of which can be independent, are the differential equations of the third order obtained by eliminating the constants from the general equation of a quadric surface.*

7. It is convenient to have a name for covariants of the class introduced in the last article. Let us speak of them as *Reciprocantive Covariants*, and of their leading coefficients, such as P_0, as *Reciprocantive Seminvariants*, of the emanants. The names are justified by the immediately following proposition, as well as by other facts to be adduced later.

PROP. IX.—*Any invariant of a Reciprocantive Covariant of the emanants is a Pure Ternary Reciprocant.*

For, being a function only of $P_0, P_1, \ldots P_m$, all of which are annihilated by V_1 and by V_2, it is itself annihilated by each of those operators; and, being a covariant of a covariant of the emanants, it is a covariant of the emanants themselves.

An example immediately to be considered leads us to supplement this theorem by another which might at first sight appear unnecessary to state, though clearly true; viz.,

PROP. X.—*If any function of seminvariants of the same or different Reciprocantive Covariants be annihilated by Ω_2, it is a Pure Ternary Reciprocant.*

8. In exemplification of this method of constructing pure ternary reciprocants, let us consider two simple cases.

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* In fact the four results of differentiating three times partially the equation

$$a + 2bx + 2cy + dx^2 + 2exy + fy^2 + 2gz + 2hxz + 2kyz + lz^2 = 0$$

may be written

$$\begin{aligned} (g + hx + ky + lz)\, z_{30} + (h + lz_{10})\, z_{20} & & &= 0, \\ (g + hx + ky + lz)\, z_{21} + (h + lz_{10})\, z_{11} &+ (k + lz_{01})\, z_{20} &&= 0, \\ (g + hx + ky + lz)\, z_{12} + (h + lz_{10})\, z_{02} &+ (k + lz_{01})\, z_{11} &&= 0, \\ (g + hx + ky + lz)\, z_{03} \qquad\qquad\quad &+ (k + lz_{01})\, z_{02} &&= 0. \end{aligned}$$

(α) The quadratic emanant

$$z_{20}u^2+z_{11}uv+z_{02}v^2$$

is itself a reciprocantive covariant. Its one invariant,

$$z_{20}z_{02}-\tfrac{1}{4}z^2_{11} \equiv H, \text{ say} \qquad (14),$$

is the one reciprocant involving second derivatives only (*cf.* Paper I., § 12, or Paper II., § 11).

(β) Take the cubic reciprocantive covariant

$$A_0u^3+3A_1u^2v+3A_2uv^2+A_3v^3 \qquad (12),$$

where A_0, A_1, A_2, A_3 have the values given in (13) above.

Its coefficients are connected by the linear relations

$$z_{30}A_3-z_{21}A_2+z_{12}A_1-z_{03}A_0 = 0,$$
$$z_{20}A_3-z_{11}A_2+z_{02}A_1 \qquad = 0,$$
$$z_{20}A_2-z_{11}A_1+z_{02}A_0 = 0,$$

of which the second and third tell us that

$$\frac{A_0A_2-A_1^2}{z_{20}} = \frac{A_0A_3-A_1A_2}{z_{11}} = \frac{A_1A_3-A_2^2}{z_{02}} = R, \text{ say} \qquad (15).$$

The first of these identical forms of R shows that it is annihilated by Ω_1, and the third that it is annihilated by Ω_2. Thus, by Prop. X., R is a reciprocant. It is of order 5 and of partial weights 6, 6, and is, in fact, the resultant of the quadratic and cubic emanants (*cf.* Paper II., § 11).

The one invariant of (12), its discriminant

$$\begin{aligned}\Delta &= (A_0A_2-A_1^2)(A_1A_3-A_2^2)-\tfrac{1}{4}(A_0A_3-A_1A_2)^2 \\ &= R^2(z_{20}z_{02}-\tfrac{1}{4}z_{11}), \text{ by (15)}, \\ &= R^2H \qquad (16),\end{aligned}$$

gives no new reciprocant.

9. Facts with regard to the transformation of functions such as we are considering by cyclical changes of dependent and independent variables will now be investigated. In the first place, it is easy to see that—

Prop. XI.—*If Q be any homogeneous isobaric pure function of the derivatives of z, whose degree is i and first partial weight w_1, and which is annihilated by* V_1 (not necessarily also by V_2), *the transformed ex-*

pression for $Qz_{10}^{-i-w_1}$ *in terms of the derivatives of x is homogeneous and of no dimensions in the first derivatives* x_{10}, x_{01}.

For, by Paper II., (11) and (13), we have, under the conditions stated,

$$\frac{d}{dx_{01}}\left(\frac{Q}{z_{10}^{i+w_1}}\right) = -\frac{z_{01}}{z_{10}^{i+w_1}}\Omega_1 Q \text{......................} (17),$$

and

$$\frac{d}{dx_{10}}\left(\frac{Q}{z_{10}^{i+w_1}}\right) = -\frac{1}{z_{10}^{i+w_1}}\Omega_1 Q \text{......................} (18),$$

whence

$$\left(z_{01}\frac{d}{dx_{10}} - \frac{d}{dx_{01}}\right)\left(\frac{Q}{z_{10}^{i+w_1}}\right) = 0,$$

i.e., since (Paper I., § 5),

$$\frac{z_{10}}{-1} = \frac{z_{01}}{x_{10}} = \frac{-1}{x_{01}} \text{..........................} (19),$$

$$\left(x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}}\right)\left(\frac{Q}{z_{10}^{i+w_1}}\right) = 0.$$

In like manner, by Paper II., (18) and (21), it is proved that—

PROP. XII.—*If* Q' *be a homogeneous isobaric pure function, of degree* i *and second partial weight* w_2, *of the derivatives of* z, *which is annihilated by* V_2 (not necessarily by V_1), *the y-transform of* $Q'z_{01}^{-i-w_2}$ *is homogeneous and of no dimensions in the first derivatives* y_{10}, y_{01}.

Now, take for Q a function annihilated by Ω_1 and having other properties as in Prop. XI. Equations (17) and (18) have in this case vanishing right-hand members, and tell us that the x-transform of $Qz_{10}^{-i-w_1}$ is pure.

Again, take for Q' a function having properties as in Prop. XII., and besides annihilated by Ω_2. We see, in like manner, that the y-transform of $Q'z_{10}^{-i-w_2}$ is pure.

These conclusions are, in consequence of the absence of requirement that Q be annihilated by V_2 or Q' by V_1, somewhat more general than their important cases:—

PROP. XIII.—*The x-transform of* $P_0 z_{10}^{-i-w_1}$, *where* P_0 *is a reciprocantive seminvariant of degree* i *and first partial weight* w_1, *is a pure function.*

PROP. XIV.—*The y-transform of* $P_m z_{01}^{-i-w_1}$, *where* P_m *is the result of interchanging first and second suffixes in a reciprocantive seminvariant* P_0, *and* i, w_1 *are the degree and second partial weight of* P_m, *is a pure function.* (N.B.—The second partial weight w_1 of P_m is, of course, the first partial weight of P_0.)

These two propositions are really identical, as will become clear later when we determine the actual expressions of the pure transforms.

10. Let us next employ (17) and (18) to aid in discussing the transformation of coefficients other than the first and last in a reciprocantive covariant

$$(P_0, P_1, P_2, \dots P_m)(u, v)^m.$$

If w_1 be the first partial weight of P_0, w_1-r is that of P_r. Hence, by (17),

$$\frac{d}{dx_{01}}\left(\frac{P_r}{z_{10}^{i+w_1-r}}\right) = -\frac{z_{01}}{z_{10}^{i+w_1-r}}\Omega_1 P_r$$

$$= x_{10}x_{01}^{i+w_1-r-1}\,rP_{r-1} \quad \dots\dots\dots\dots\dots\dots\dots (20),$$

by (19) and the law of eduction of one coefficient of a covariant from the preceding. Again, by (18),

$$\frac{d}{dx_{10}}\left(\frac{P_r}{z_{10}^{i+w_1-r}}\right) = -\frac{1}{z_{10}^{i+w_1-r}}\Omega_1 P_r$$

$$= -x_{01}^{i+w_1-r}\,rP_r \quad \dots\dots\dots\dots\dots\dots\dots\dots (21).$$

Now
$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, P_2, \dots P_m)(u, v)^m,$$

may be written

$$\left(\frac{P_0}{z_{10}^{i+w_1}}, \frac{P_1}{z_{10}^{i+w_1-1}}, \frac{P_2}{z_{10}^{i+w_1-2}}, \dots \frac{P_m}{z_{10}^{i+w_2}}\right)\left(u, \frac{v}{z_{10}}\right)^m,$$

w_1 and w_2 meaning the first and second partial weights of P_0, and having their difference equal to m; and, by (19), this may be also written

$$(P_0x_{01}^{i+w_1}, P_1x_{01}^{i+w_1-1}, P_2x_{01}^{i+w_1-2}, \dots P_mx_{01}^{i+w_2})(u, vx_{01})^m \quad \dots\dots (22).$$

It suggests itself, in connection with Props. XIII. and XIV., and results (20) and (21) above, to seek values of u and v that this may be annihilated by $\dfrac{d}{dx_{01}}$ and $\dfrac{d}{dx_{10}}$, *i.e.*, that its x-transform may be a pure function.

Now, by (20), which is best made use of in the form

$$\frac{d}{dx_{01}}(P_rx_{01}^{i+w_1-r}) = r\frac{x_{10}}{x_{01}^2}(P_{r-1}x_{01}^{i+w_1-r+1}),$$

the result of differentiating (22) partially with regard to x_{01} is

$$\left\{\frac{d}{dx_{01}}(u^m)+m\frac{x_{10}}{x_{01}^2}u^{m-1}(vx_{01})\right\}P_0x_{01}^{i+w_1}$$

$$+\left\{m\frac{d}{dx_{01}}(u^{m-1}vx_{01})+m(m-1)\frac{x_{10}}{x_{01}^2}u^{m-2}(vx_{01})^2\right\}P_1x_{01}^{i+w_1-}$$

$$+\left\{\frac{m(m-1)}{1.2}\frac{d}{dx_{01}}\left[u^{m-2}(vx_{01})^2\right]\right.$$

$$+\ldots\ldots; \qquad \left.+\frac{m(m-1)(m-2)}{1.2}\frac{x_{10}}{x_{01}^2}u^{m-3}(vx_{01})^3\right\}P_2x_{01}^{i+w_1-2}$$

and, using (21) in the form

$$\frac{d}{dx_{10}}(P_rx_{01}^{i+w_1-r})=-\frac{r}{x_{01}}(P_{r-1}x_{01}^{i+w_1-r+1}),$$

the result of partially differentiating (22) with regard to x_{10} is

$$\left\{\frac{d}{dx_{10}}(u^m)-m\frac{1}{x_{01}}u^{m-1}(vx_{01})\right\}P_0x_{01}^{i+w_1}$$

$$+\left\{m\frac{d}{dx_{10}}(u^{m-1}vx_{01})-m(m-1)\frac{1}{x_{01}}u^{m-2}(vx_{01})^2\right\}P_1x_{01}^{i+w_1-}$$

$$+\left\{\frac{m(m-1)}{1.2}\frac{d}{dx_{10}}\left[u^{m-2}(vx_{01})^2\right]\right.$$

$$+\ldots\ldots; \qquad \left.-\frac{m(m-1)(m-2)}{1.2}\frac{1}{x_{01}}u^{m-3}(vx_{01})^3\right\}P_2x_{01}^{i+w_1-2}$$

and in both these results of differentiation, it is readily seen that the coefficients of $P_0x_{01}^{i+w_1}$, $P_1x_{01}^{i+w_1-1}$, $P_2x_{01}^{i+w_1-2}$, ... are all made to vanish upon putting

$$u=\frac{x_{10}}{x_{01}}, \quad v=\frac{1}{x_{01}},$$

i.e.,
$$u=-z_{01}, \quad v=z_{10}.$$

Hence the conclusion—

PROP. XV.—*If* $(P_0, P_1, \ldots P_m)(u, v)^m$ *be a reciprocantive covariant, the x-transform of*

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \ldots P_m)(-z_{01}, z_{10})^m$$

is a pure function, i being the degree and w_1 the first partial weight of P_0.

Also we have, in like manner,—

PROP. XVI.—*Under exactly the same circumstances the y-transform of*

$$\frac{1}{z_{01}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

is a pure function.

11. Required now the pure function of the derivatives of x to which, by Prop. XV.,

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

is equal. This may, as above, be written

$$\left(\frac{P_0}{z_{10}^{i+w_1}}, \frac{P_1}{z_{10}^{i+w_1-1}}, \dots \frac{P_m}{z_{10}^{i+w_1}}\right)\left(\frac{x_{10}}{x_{01}}, 1\right)^m \dots\dots\dots\dots\dots\dots (23).$$

Now, in Paper II., § 10, it was seen that

$$\frac{z_{rs}}{z_{10}^{1+r}} = -x_{sr} + \text{terms with } \frac{1}{x_{01}} \text{ as a factor.}$$

Consequently, if P be a homogeneous isobaric function of the suffixed z's, whose degree and first partial weight are i and w_1, and if $P'(x)$ denote the same function of the suffixed x's with suffixes reversed in order (x_{sr} for z_{rs}, &c.),

$$\frac{P}{z_{10}^{i+w_1}} = (-1)^i P'(x) + \text{terms with } \frac{1}{x_{01}} \text{ as a factor.}$$

Now, P'_r is P_{m-r}. Thus (23) becomes

$$(-1)^i\left[P_m(x)+\dots,\ P_{m-1}(x)+\dots,\ \dots\ P_0(x)+\dots\right]\left(\frac{x_{10}}{x_{01}}, 1\right)^m,$$

each $+\dots$ in which indicates that terms with $\frac{1}{x_{01}}$ as factor are omitted where it occurs.

But it has been proved (Prop. XV.) that this form is independent of x_{10} and x_{01}. Thus we may give these first derivatives any values we please. Make x_{01} then infinitely great compared with other magnitudes occurring in the expression. The form taken is

$$(-1)^i P_0(x).$$

This, consequently, is the transform of

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

required.

In exactly the same way, the y-transform of

$$\frac{1}{z_{01}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m,$$

already proved to be pure, is

$$(-1)^{i+m}P_m(y).$$

The two results together are most compactly stated

$$(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m = (-1)^i\frac{P_0(x)}{x_{01}^{i+w_1}} = (-1)^{i+m}\frac{P_m(y)}{y_{10}^{i+w_1}} \dots (24),$$

by use of (19), and the analogous qualities

$$\frac{y_{10}}{-1} = \frac{y_{01}}{z_{10}} = \frac{-1}{z_{01}}.$$

In (24) is also contained information as to the pure functions to which Props. XIII. and XIV. have told us that $P_0 z_{10}^{-i-w_1}$ and $P_m z_{01}^{-i-w_1}$ are equal. The first, by putting z, x, y for x, y, z, is seen to be $(-1)^{i+w_1}P_m(x)$; and the second, by putting y, z, x for x, y, z, to be $(-1)^{i+w_1}P_0(y)$.

In words, the results (24) may be stated as follows :—

PROP. XVII.—*A first cyclical transformation of dependent and independent variables in a reciprocantive covariant with* $-z_{10}$, z_{01} *inserted for its variables, produces from it, but for a sign factor and a power of a first derivative, the reciprocantive seminvariant which is its leading coefficient; and a second cyclical transformation produces, but for factors as before, the same reciprocantive seminvariant with first and second suffixes interchanged throughout.* Or, of course, the facts may be stated beginning from the seminvariant P_0, or from the reversed seminvariant P_m.

One of many conclusions from (24) with regard to mixed ternary reciprocants seems worth mentioning. Taking the product of the three members of (24), all written with z as the dependent variable, we see that

$$\frac{P_0P_m(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m}{(z_{10}, z_{01})^{i+w_1}}$$

is an absolute mixed ternary reciprocant, or its numerator a mixed

ternary reciprocant of index $i+w_1$. For instance,

$$z_{20}z_{02}\left(z_{20}z_{01}^2-z_{11}z_{01}z_{10}+z_{02}z_{10}^2\right)$$

is a ternary reciprocant of index 3; and, referring to (12) and (13),

$$A_0A_3\left(A_0z_{01}^3-3A_1z_{01}^2z_{10}+3A_2z_{01}z_{10}^2-A_3z_{10}^3\right),$$

i.e.
$$\begin{vmatrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \end{vmatrix} \times \begin{vmatrix} z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \\ z_{03}, & & z_{02} \end{vmatrix} \times \begin{vmatrix} z_{10}^3, & z_{30}, & z_{20} & \\ 3z_{10}^2z_{01}, & z_{21}, & z_{11}, & z_{20} \\ 3z_{10}z_{01}^2, & z_{12}, & z_{02}, & z_{11} \\ z_{01}^3, & z_{03}, & & z_{02} \end{vmatrix},$$

is one of index 8.

12. Let us return from this discussion of the extent to which what the present paper has defined as reciprocantive seminvariants and covariants possess the fundamental property of ternary reciprocants, to the consideration of methods afforded by the propositions of § 5 for the determination of pure ternary reciprocants.

The method of §§ 6—8 is a powerful one; but it is based upon the knowledge of homogeneous isobaric functions annihilated by V_1 and V_2, and though in § 2 we have before us an infinite number of such functions, we have no indication that the system is a complete one. The method in question is not then yet rendered thoroughly systematic for the determination of *all* pure ternary reciprocants.

Another process to be now briefly explained follows closely one of known power for the systematic calculation of invariants of ternary quantics, and has the advantage of theoretical completeness. I exemplified it, without full confidence in or any statement of its generality, in Paper II., § 12, by obtaining the pure ternary reciprocant of type 3, 4, 4,

$$6\left(z_{02}^2z_{40}+z_{20}^2z_{04}\right)-3z_{11}\left(z_{02}z_{31}+z_{20}z_{13}\right)+z_{22}\left(z_{11}^2+2z_{02}z_{20}\right)$$
$$-3\left\{2z_{02}\left(3z_{30}z_{12}-z_{21}^2\right)-z_{11}\left(9z_{30}z_{03}-z_{12}z_{21}\right)+2z_{20}\left(3z_{03}z_{21}-z_{12}^2\right)\right\}.$$

(I would remark that V in the article referred to is mis-written for V_1.)

Required the pure ternary reciprocants of degree i and (equal) partial weights w_1, w_1.

Let R be such a reciprocant. It is (Paper I.) an invariant of the emanants

$$\left(z_{20}, z_{11}, z_{02}\between u, v\right)^2, \quad \left(z_{30}, z_{21}, z_{12}, z_{03}\between u, v\right)^3, \text{ \&c.}$$

If, then, $I_1, I_2, I_3, \dots$ be a complete system of linearly independent invariants of the given type of the system, it is necessary that

$$R \equiv a_1 I_1 + a_2 I_2 + a_3 I_3 + \dots \qquad \dots\dots\dots\dots\dots\dots\dots (25)$$

for some values or other of the numerical multipliers $a_1, a_2, a_3, \dots$

Now (Prop. II.), V_1 operating on any seminvariant I produces another seminvariant, of degree and first partial weight exceeding those of I by unity, and of second partial weight equal to that of I. If, then, $J_1, J_2, J_3, \dots$ be a complete system of linearly independent seminvariants of the emanants whose degree and partial weights are $i+1$, w_1+1, w_1, we must have

$$V_1 R \equiv (\lambda_1 a_1 + \mu_1 a_2 + \nu_1 a_3 + \dots)\, J_1 + (\lambda_2 a_1 + \mu_2 a_2 + \nu_2 a_3 + \dots)\, J_2$$
$$+ (\lambda_3 a_1 + \mu_3 a_2 + \nu_3 a_3 + \dots)\, J_3 + \dots$$

for some known, vanishing or non-vanishing, values of the multipliers λ, μ, ν. For R to be annihilated by V_1, we have then the conditions

$$\left.\begin{array}{l} \lambda_1 a_1 + \mu_1 a_2 + \nu_1 a_3 + \dots = 0 \\ \lambda_2 a_1 + \mu_2 a_2 + \nu_2 a_3 + \dots = 0 \\ \lambda_3 a_1 + \mu_3 a_2 + \nu_3 a_3 + \dots = 0 \\ \dots \quad \dots \quad \dots \quad \dots \end{array}\right\} \qquad \dots\dots\dots\dots\dots\dots (26).$$

If these be satisfied, not only will V_1 but also V_2 be an annihilator of R. For (Prop. VIII.), since $\Omega_2 R = 0$ and $V_1 R = 0$, $V_2 R = 0$. The linear conditions (26) are consequently all those which $a_1, a_2, a_3, \dots$ have to satisfy in order that R may be a reciprocant. If the number of these conditions be less by one or any greater deficiency than the number of a's, *i.e.*, than the number of invariants I in (25), a pure ternary reciprocant R, or a number having that deficiency for a superior limit of linearly independent pure ternary reciprocants, is determined. Hence

Prop. XVIII.—*If the number of linearly independent invariants of degree i and partial weights w_1, w_1 of the quadratic cubic, &c. emanants exceed the number of seminvariants of degree $i+1$ and partial weights w_1+1, w_1, the excess is equal to, or at any rate a superior limit to, the number of linearly independent pure ternary reciprocants of type i, w_1, w_1.*

The Differential Equations satisfied by Concomitants of Quantics.

By A. R. Forsyth, M.A., F.R.S.

[*Read Nov.* 10*th*, 1887.]

In a memoir published in 1872,* Clebsch established the important result, that in the case of a form in n variables, the concomitants of the form or of a system of such forms involve in the aggregate $n-1$ classes of variables. Thus, for instance, those of a ternary form involve two classes which may be geometrically interpreted as point and line coordinates in a plane; those of a quaternary form involve three classes which may be geometrically interpreted as point, line, and plane coordinates in space.

1. Now these concomitants satisfy characteristic differential equations, the most important of which belong to one or other of three classes. The first of these is constituted by the equations with bilinear operators in conjugate (contragredient) variables, and they arise from the assumption of definite special forms as normal forms of concomitants. They were first given by Gordan, in his memoir "Ueber Combinanten,"† for the simplest association of contragredient variables; and Clebsch, in his Göttingen memoir,‡ pointed out the generalisation obvious after his earlier results there obtained. The general result is, that any mixed concomitant Θ involving two conjugate sets of contragredient variables p and q, which are by their nature such that $\Sigma p_i q_i$ does not change when the original variables are subjected to linear transformations, can be replaced by a set of mixed concomitants $[\Theta]$ each of which satisfies the partial differential equation

$$\Sigma \frac{\partial^2}{\partial p_i \partial q_i} = 0.$$

The second of the classes of differential equations is constituted by those involving differential operators in variables which belong to one and the same class; and they arise from the fact that the variables of one class are, with the exception of two sets, not inde-

* "Ueber eine Fundamentalaufgabe der Invarianten-theorie," *Abh. d. K. Akad. d. W. zu Göttingen*, t. xvii.; *Math. Ann.*, t. v., pp. 427—434.

† *Math. Ann.*, t. v. (1872); see specially pp. 102—106. ‡ l.c., p. 39.

pendent of one another. The differential equations are affected in form in accordance with the possible changing but equivalent forms of the function concerned, which can be exhibited by the use of the permanent relations among the non-independent variables. These equations are indicated in Clebsch's memoir, the general result being that, when there is a permanent relation of the form

$$\Sigma Cpp' \ldots = 0,$$

then the function taken in a definite normal form satisfies an associated differential equation

$$\Sigma C \frac{\partial^r}{\partial p \, \partial p' \ldots} = 0.$$

It thus appears that the investigation of the differential equations is effectively the investigation of the identical relations among the variables of all the classes which enter into the expression of the function.

The third of the classes of differential equations, perhaps the only class to which the term "characteristic" should be applied, arises from the invariantive nature of the functions; they are, of course, affected by the normal forms adopted for the functions, but these normal forms are not the governing element.

It is unnecessary to regard further here the first class of equations mentioned; we proceed to investigate in turn the remaining two classes.

2. Consider a manifoldness of order $n-1$, which will therefore involve n variables. These may be denoted by $x_{11}, x_{12}, x_{13}, \ldots x_{1n}$; and other sets of variables cogredient with these may be denoted by $x_{21}, x_{22}, x_{23}, \ldots, x_{2n}$; $x_{31}, x_{32}, x_{33}, \ldots, x_{3n}$; and so on. Then the different classes of variables are constituted by the aggregates of minors of different orders of the determinant

$$\begin{vmatrix} x_{11}, & x_{12}, & \ldots\ldots, & x_{1n} \\ x_{21}, & x_{22}, & \ldots\ldots, & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1}, & x_{n2}, & \ldots\ldots, & x_{nn} \end{vmatrix}.$$

Those of a linear manifoldness of order $n-2$, say a hyper-plane (*i.e.*, a plane in the case of quaternary quantics), are obtained by suppressing the last row, and taking the ensuing n determinants each of $(n-1)^2$ constituents; those of a linear manifoldness of order $n-3$

(*i.e.*, a line in the case of quaternary quantics), by suppressing the last two rows and taking the ensuing $\frac{1}{2}n(n-1)$ determinants each of $(n-2)^2$ constituents; and so on, until we reach the linear manifoldness of order zero and have the n hyper-point coordinates. It is, as usual, assumed that the n hyper-point coordinates are independent of one another.

3. In the first place, it is evident that the number of coordinates, different from and linearly (though not functionally) independent of one another, of a linear manifoldness of order $r-1$ is the same as that of the coordinates of a linear manifoldness of order $n-r-1$; in fact, by a well-known theorem in determinants,* any determinant made up of r^2 hyper-point coordinates as constituents can be replaced by one made up of $(n-r)^2$ corresponding hyper-plane coordinates. Owing to this reciprocal duality we are enabled to deduce, from the identical relations between the variables of one linear manifoldness, the relations between the variables of the linear manifoldness of complementary order. For we can replace each of the former variables in terms of the hyper-point coordinates by its expression in terms of the hyper-plane coordinates, and the relation is changed into one which involves variables in hyper-plane coordinates alone of complementary order. We now, in each of this last set of identities, change each hyper-plane coordinate into a hyper-point coordinate, and so obtain identical relations between variables in terms of hyper-point coordinates of complementary order, that is, identical relations between the variables of the linear manifoldness of complementary order. Hence we have the theorem:

Any identical relation between the variables of a linear manifoldness of order $r-1$ gives an identical relation between the variables of a linear manifoldness of complementary order $n-r-1$, when each of the variables of the first manifoldness is replaced by the corresponding conjugate variable of the second manifoldness.

Examples will be given in the next section.

4. There are apparently two cases to be considered; but the difference between them is slight and applies rather to the results than to the methods of investigation. The first case is that in which n is odd, the second that in which n is even. There exists for the latter a manifoldness of order $\frac{1}{2}n-1$, the variables of which are self-conjugate; there is no such manifoldness for the former. The method of

* It is proved by Clebsch, l. c., § 2.

investigation is the same for all cases, and there results in general a number of relations, depending on the order of the manifoldness with which the variables are associated. But in that particular case when the variables are self-conjugate the fundamental relations are all of such a kind that the consequent differential equations are assigned by Clebsch to that class, here (§ 1) called the first.

It should further be remarked that the relations deduced are not functionally independent of one another, though they may be so linearly. For instance, in the case of quinary forms, we have (hyperline) variables given by

$$\left\| \begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1, & y_2, & y_3, & y_4, & y_5 \end{matrix} \right\| ;$$

they may be denoted by (12), (13), ..., (45), respectively. Then we have, as relations among them,

$$0 = (12)(34) + (23)(14) + (31)(24),$$
$$0 = (12)(35) + (23)(15) + (31)(25),$$
$$0 = (12)(45) + (24)(15) + (41)(25),$$
$$0 = (13)(45) + (34)(15) + (41)(35),$$
$$0 = (23)(45) + (34)(25) + (42)(35),$$

five linearly independent relations. But the elimination, from the last four of them, of the quantities (15), (25), (35), (45) gives us

$$\begin{vmatrix} (23), & (31), & (12), & 0 \\ (24), & (41), & 0, & (12) \\ (34), & 0, & (41), & (13) \\ 0, & (34), & (42), & (23) \end{vmatrix} = 0,$$

or, what is the same thing,

$$\{(12)(34) + (31)(24) + (23)(14)\}^2 = 0,$$

and so leads to the first relation.

In connection with these identities among the variables related to quinary forms, we have a simple illustration of the theorem of the last section; for, passing from these variables of a linear manifoldness of order 1 to the variables of a linear manifoldness of order 2, accord-

ing to the theorem, the relations among the latter variables are

$$0 = (345)(125)+(145)(235)+(245)(315),$$
$$0 = (354)(124)+(154)(234)+(254)(134),$$
$$0 = (453)(123)+(153)(243)+(253)(413),$$
$$0 = (452)(132)+(152)(342)+(352)(412),$$
$$0 = (451)(231)+(251)(341)+(351)(421),$$

five relations linearly independent. Eliminating from the last four of them the four quantities (234), (341), (412), (123), which are the conjugates of (15), (25), (35), (45), respectively, we are, as before, led to

$$\{(345)(125)+(145)(235)+(245)(315)\}^2 = 0,$$

that is, to the first relation.

5. In order to obtain all the relations among the variables of a linear manifoldness of any order $n-1$, derived from a manifoldness of order $n-1$, the method that appears most effective is founded on a theorem due to Sylvester.* We form two determinants each of r^2 constituents given by

$$\Theta = \begin{vmatrix} x_{1\alpha}, & x_{1\beta}, & \dots\dots, & x_{1\theta} \\ x_{2\alpha}, & x_{2\beta}, & \dots\dots, & x_{2\theta} \\ \dots & \dots & \dots & \dots \\ x_{r\alpha}, & x_{r\beta}, & \dots\dots, & x_{r\theta} \end{vmatrix},$$

$$\Phi = \begin{vmatrix} x_{1\psi}, & x_{1\chi}, & \dots\dots, & x_{1\lambda} \\ x_{2\psi}, & x_{2\chi}, & \dots\dots, & x_{2\lambda} \\ \dots & \dots & \dots & \dots \\ x_{r\psi}, & x_{r\chi}, & \dots\dots, & x_{r\lambda} \end{vmatrix},$$

where $\alpha, \beta, \dots, \theta$ are any r integers out of the series $1, 2, \dots, n$ and $\psi, \chi, \dots, \lambda$ are any other r integers from the same series. (We may assume $2r$ to be not greater than n, on account of the theorem of § 3, which makes the relations for the linear manifoldness of order $n-r-1$ a mere transcription of those for the linear manifoldness of

* *Phil. Mag.*, Ser. 4, Vol. II. (1851), p. 143; see also Brioschi, *La teorica dei determinanti* (Pavia, 1854), pp. 46, 47, where the general proof given differs from Sylvester's; and Baltzer, *Theorie und Anwendung der Determinanten*, § 3, 9 and § 4, 4.

order $r-1$.) In the first instance, we take no one of the integers $\alpha, \beta, \ldots, \theta$ to be the same as any one of the set $\psi, \chi, \ldots, \lambda$; it will appear immediately that there is associated with the aggregate of $2r$ integers, one group of relations, and that therefore the number of such groups is

$$\frac{n!}{\overline{n-2r}!\,2r!}.$$

When we select s out of r columns of either Θ or Φ, it can be done in

$$\frac{r!}{s!\,\overline{r-s}!} = r_s$$

ways, with which may be associated the integers $1, 2, \ldots, p, \ldots, r_s$. Let $\Theta_{m,p}$ denote the result of substituting in Θ for the s columns of Θ, corresponding to the associated combination integer m, the s columns of Φ, corresponding to the associated combination integer p; and let $\Phi_{p,m}$ denote the result of these interchanges in Φ. Then Sylvester's theorem gives the result

$$\Theta\Phi = \sum_{m=1}^{m=r_s} \Theta_{m,p}\,\Phi_{p,m},$$

and also, though not explicitly mentioned, the evidently similar result

$$\Phi\Theta = \sum_{p=1}^{p=r_s} \Phi_{p,m}\,\Theta_{m,p}.$$

The former of these is true for each of the r_s values of p, the latter for each of the r_s values of m; and there are thus apparently $2r_s$ equations. But these equations are not linearly independent; in fact, we have from the first set the result

$$r_s\Theta\Phi = \sum_{p=1}^{p=r_s} \left\{ \sum_{m=1}^{m=r_s} \Theta_{m,p}\,\Phi_{p,m} \right\},$$

and from the second set the same result, viz.,

$$r_s\Phi\Theta = \sum_{m=1}^{m=r_s} \left\{ \sum_{p=1}^{p=r_s} \Phi_{p,m}\,\Theta_{m,p} \right\}.$$

The equations must therefore be reduced in number by unity, and thus the number of independent equations is $2r_s-1$.

This is true for all the possible values of s; but, as the equations have been taken in the duplicate form, the only values of s which need be retained are $1, 2, \ldots$ up to the integral part of $\frac{1}{2}r$. For the effect of replacing s columns of Θ by s columns of Φ is the same as that of

replacing the remaining $n-s$ columns of Φ by the remaining $n-s$ columns of Θ. Thus the total number of equations is

$$2r_1-1 + 2r_2-1 + 2r_3-1+\dots;$$

in the case when r is even the number is easily found to be $2^r-\frac{1}{2}r-1$, and in that when r is odd it is $2^r-\frac{1}{2}(r+3)$. This number will be increased owing to the possible variations of the fundamental equations, illustrations of which will be given shortly; but, even so, this aggregate does not constitute the complete aggregate of relations among the variables of the manifoldness. Each of the bilinear products $\Theta_{m,p}, \Phi_{p,m}$ involves all the $2r$ sets of variables x, but one factor, as $\Theta_{m,p}$, involves only r of them, and it need not necessarily be associated with a factor which involves all the remaining r. And so we can from each relation deduce a number of sets of others.

We first make one of the integers $\psi, \chi, \phi, \dots, \mu, \lambda$—say λ—the same as one of the $2r$ integers other than λ, which can be done in $2r-1$ ways; the effect of this will be to make some of the terms Φ and Θ vanish, and we shall be left with a relation among the variables of the manifoldness involving in the aggregate only $2r-1$ suffixes.

We next make two of the integers $\psi, \chi, \phi, \dots, \mu, \lambda$—say λ and μ—the same as two of the $2r$ integers other than λ and μ, which can be done in $\frac{1}{2}(2r-2)(2r-3)$ ways; and there will result a relation among the variables of the manifoldness, involving in the aggregate only $2r-2$ suffixes.

And so on, for the several sets derived as special cases. The last set so derived is that in which only $r+2$ suffixes are involved; and the type of relation in this set is

$$p_{12\gamma\delta\dots\theta}\, p_{34\gamma\delta\dots\theta} + p_{31\gamma\delta\dots\theta}\, p_{24\gamma\delta\dots\theta} + p_{23\gamma\delta\dots\theta}\, p_{14\gamma\delta\dots\theta} = 0,$$

where $\gamma, \delta, \dots, \theta$ are $r-2$ integers other than 1, 2, 3, 4.

6. To illustrate the preceding statements there follow here examples of all the relations among the variables of a linear manifoldness of order 3 belonging to any general manifoldness of order $n-1$. First, there are the relations which involve 8 $[= 2(3+1)]$ suffixes. We take

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ t_1, & t_2, & t_3, & t_4 \end{vmatrix} = (1234), \qquad \begin{vmatrix} x_5, & x_6, & x_7, & x_8 \\ y_5, & y_6, & y_7, & y_8 \\ z_5, & z_6, & z_7, & z_8 \\ t_5, & t_6, & t_7, & t_8 \end{vmatrix} = (5678);$$

so that $r = 4$, and $s = 1$ and $s = 2$ are the admissible values of s; and

there is a group of relations for every one of the

$$\frac{n\,!}{8\,!\; n-8\,!}$$

groups of eight suffixes. The total number of variables belonging to the manifoldness is $\frac{n\,!}{4!\; n-4!}$,

and the number of variables involving four of the eight suffixes chosen is 70, giving 35 bilinear products. Then the relations for $s = 1$ are

$$\begin{aligned}
(1234)(5678) &= (5234)(1678) + (1534)(2678) + (1254)(3678) \\
&\qquad + (1235)(4678) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i.}) \\
&= (6234)(5178) + (1634)(5278) + (1264)(5378) \\
&\qquad + (1236)(5478) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ii.}) \\
&= (7234)(5618) + (1734)(5628) + (1274)(5638) \\
&\qquad + (1237)(5648) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{iii.}) \\
&= (8234)(5671) + (1834)(5672) + (1284)(5673) \\
&\qquad + (1238)(5674) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{iv.}),
\end{aligned}$$

$$\begin{aligned}
(5678)(1234) &= (1678)(5234) + (5178)(6234) + (5618)(7234) \\
&\qquad + (5671)(8234) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{v.}) \\
&= (2678)(1534) + (5278)(1634) + (5628)(1734) \\
&\qquad + (5672)(1834) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{vi.}) \\
&= (3678)(1254) + (5378)(1264) + (5638)(1274) \\
&\qquad + (5673)(1284) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{vii.}) \\
&= (4678)(1235) + (5478)(1236) + (5648)(1237) \\
&\qquad + (5674)(1238) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{viii.});
\end{aligned}$$

and for $s = 2$, the relations are

$$\begin{aligned}
(1234)(5678) &= (5634)(1278) + (5264)(1378) + (5236)(1478) \\
&\quad + (1564)(2378) + (1536)(2478) + (1256)(3478) \ldots (\text{ix.}) \\
&= (5834)(1672) + (5284)(1673) + (5238)(1674) \\
&\quad + (1584)(2673) + (1538)(2674) + (1258)(3674) \ldots (\text{x.}) \\
&= (5734)(1628) + (5274)(1638) + (5237)(1648) \\
&\quad + (1574)(2638) + (1537)(2648) + (1257)(3648) \ldots (\text{xi.}) \\
&= (7834)(5612) + (7284)(5613) + (7238)(5614) \\
&\quad + (1784)(5623) + (1738)(5624) + (1278)(5634) \ldots (\text{ix.}') \\
&= (6734)(5128) + (6274)(5138) + (6237)(5148) \\
&\quad + (1674)(5238) + (1637)(5248) + (1267)(5348) \ldots (\text{x.}') \\
&= (6834)(5172) + (6284)(5173) + (6238)(5174) \\
&\quad + (1684)(5273) + (1638)(5274) + (1268)(5374) \ldots (\text{xi.}'),
\end{aligned}$$

$$
\begin{aligned}
(5678)(1234) &= (1278)(5634)+(1672)(5834)+(1628)(5734) \\
&\quad +(5612)(7834)+(5128)(6734)+(5172)(6834) \ldots \text{(xii.)} \\
&= (1378)(5264)+(1673)(5284)+(1638)(5274) \\
&\quad +(5613)(7284)+(5138)(6274)+(5173)(6284) \ldots \text{(xiii.)} \\
&= (1478)(5236)+(1674)(5238)+(1648)(5237) \\
&\quad +(5614)(7238)+(5148)(6237)+(5174)(6238) \ldots \text{(xiv.)} \\
&= (2378)(1564)+(2673)(1584)+(2638)(1574) \\
&\quad +(5623)(1784)+(5238)(1674)+(5273)(1684) \ldots \text{(xiv.')} \\
&= (2478)(1536)+(2674)(1538)+(2648)(1537) \\
&\quad +(5624)(1738)+(5248)(1637)+(5274)(1638) \ldots \text{(xiii'.)} \\
&= (3478)(1256)+(3674)(1258)+(3648)(1257) \\
&\quad +(5634)(1278)+(5348)(1267)+(5374)(8268) \ldots \text{(xii.')}
\end{aligned}
$$

But, of the last twelve equations, one half are identical (as marked) with the other half, a coincidence to be expected, for in the present case $s = r - s$, and so each product-combination will occur twice. In general, for $s = \frac{1}{2}r$, only $r_s - 1$, and not $2r_s - 1$, linearly independent equations occur—a result already taken account of in assigning the general number of equations.

It will be noticed that in the equations (i.)—(iv.), (ix.)—(xi.) all the 35 products occur, as likewise in the equations (v.)—(viii.), (xii.)—(xiv.) One more point remains for a remark, which will be made later (§ 10).

7. To deduce from the foregoing equations the first sub-class, we make the symbol 8 everywhere the same as one of the other symbols in turn; so that the resulting relations will involve only the seven symbols 1, 2, 3, 4, 5, 6, 7, and will constitute a group in these seven symbols, just as the preceding section gives relations constituting a group in eight symbols. The number of groups of this sub-class is

$$\frac{n\,!}{7\,!\;\overline{n-7}\,!}.$$

The relations are:

Those derived from equations (i.) to (viii.); by making (8) the same as (7), we obtain an equation

$$0 = (7234)(5617)+(1734)(5627)+(1274)(5637)+(1237)(5647).$$

Those derived from equations (ix.) to (xiv.); one such equation is

$$
\begin{aligned}
0 = (5734)(1672)&+(5274)(1673)+(1574)(2673) \\
&+(5237)(1674)+(1537)(2674)+(1257)(3674).
\end{aligned}
$$

And it is not difficult to prove that the complete aggregate of both sets in this sub-class is linearly resoluble into the seven groups of equations, one of which is of the form:

$$0 = (1237)(4567)+(1274)(3567)+(1734)(2567)+(7234)(1567),$$
$$0 = (1237)(4567)+(2376)(1457)+(2735)(1467)+(7234)(1567),$$
$$0 = (1237)(4567)+(1375)(2467)+(1725)(3467)+(7235)(4167),$$
$$0 = (1237)(4567)+(1276)(3457)+(1736)(2457)+(7236)(1457),$$
$$0 = (1237)(4567)+(1375)(2467)+(1734)(2567)+(7245)(1637),$$
$$0 = (1237)(4567)+(1274)(3567)+(1725)(3467)+(7345)(1267);$$

and for the remaining groups the six symbols other than 7 severally occur twice. The six equations given are linearly equivalent to five; for the sum of the right-hand sides of the first, third, and fourth is the same as that of the second, fifth, and sixth.

8. To deduce the relations of the second sub-class, we make two of the symbols, 7 and 8, everywhere the same as two of the other symbols; the resulting non-evanescent relations are those which involve only six symbols, and there are therefore

$$\frac{n!}{6!\,n-6!}$$

groups of such relations. Those for the symbols 1, 2, 3, 4, 5, 6 are—

$$(1234)(1256)+(1245)(1236)+(1253)(1246) = 0,$$
$$(1324)(1356)+(1345)(1326)+(1352)(1346) = 0,$$

and so on, fifteen in number, there being only one equation of the group in which two symbols as 1, 2 occur in each factor of a term.

9. Similarly, for the variables of a linear manifoldness of order 2, the relations involving six symbols are

$$\begin{aligned}(123)(456) &= (423)(156)+(143)(256)+(124)(356),\\ &= (523)(416)+(153)(426)+(125)(436),\\ &= (623)(451)+(163)(452)+(126)(453),\\ (456)(123) &= (156)(423)+(416)(523)+(451)(623),\\ &= (256)(143)+(426)(153)+(452)(163),\\ &= (356)(124)+(436)(125)+(453)(126);\end{aligned}$$

while the relations involving only the five symbols 1, 2, 3, 4, 5 are those given in § 4.

10. It will be noticed that the equations of § 7, which involved all the eight symbols, were given in such a form as to express the product (1234)(5678) in terms of the remaining products; and similarly for the corresponding case with the equations of § 9.

But such relations are also obtained in the general case, and in special cases, if in the general theorem we take as the fundamental product some combination other than $\Theta\Phi$, say $\Theta_{m,p}\,\Phi_{p,m}$; and there are then derivable $2r_s$ equations of a new form which *inter se* are linearly independent, save as to one relation as before; so that, taking all the possible equations, the total number is

$$2r_s \cdot \frac{1}{2} \cdot \frac{2r!}{s!\,2r-s!},$$

in $\frac{1}{2} \cdot \frac{2r!}{s!\,2r-s!}$ groups. The combinations of forms which occur in these are scattered up and down in the r_s groups (for different values of s) as first taken; and I do not propose to discuss the question of the number of linearly independent equations to which the foregoing are reducible.*

It may however be remarked (and the verification of the remark is not difficult) that the retention of the full aggregate of these groups of relations for s dispenses with the necessity of retaining any of the groups of relations for $s+1$. But this is not all; it is sufficient to form first the equations for $s=1$, each being of the form

$$\Theta\Phi = \text{sum of } r \text{ bilinear products},$$

the factors of the products having only one column different from the Θ or Φ as the case may be; and the number of such equations is $2r$. Now the number of products entering on the right-hand side is r^2; and it will be found that the groups of $2r$ equations, derived from each of these r^2 products as a fundamental $\Theta\Phi$, will enable us to dispense with the equations for $s=2$; and so on. Thus, for instance, taking the four products on the right-hand side of equation (i.) of § 6, and writing down one of the associated equations for each of them, we have

$$(5234)(1678)$$
$$= (6234)(1578)+(5634)(1278)+(5264)(1378)+(5236)(1478),$$
$$(1534)(2678)$$
$$= (6534)(2178)+(1634)(2578)+(1564)(2378)+(1536)(2478),$$

* For the manifoldness of order 2, the set of 60 equations, so derived and involving the six symbols 1, 2, 3, 4, 5, 6, is linearly reducible to the equations of § 9, a result easy to verify.

$$(1254)(3678)$$
$$= (6254)(3178)+(1654)(3278)+(1264)(3578)+(1256)(3478),$$
$$(1235)(4678)$$
$$= (6235)(4178)+(1635)(4278)+(1265)(4378)+(1236)(4578).$$

When the corresponding members of these equations are added together and equations (i.) and (ii.) are used, equation (ix.) follows immediately.

This concludes the investigation of the algebraical relations, and therefore that of the differential equations of the second class satisfied by concomitants of n-ary quantics.

11. When we pass to the consideration of the differential equations satisfied by a concomitant on account of its invariantive nature, it is in the first place necessary to settle the formulæ of transformation. Still dealing with the general case of a manifoldness of order $n-1$, we take as the original variables $x_1, x_2, \ldots, x_n$, and as the transformed variables $X_1, X_2, \ldots, X_n$. The necessary formulæ of transformation may be written

$$x_r = l_{r,1}X_1 + l_{r,2}X_2 + \ldots + l_{r,n}X_n$$

$(r = 1, 2, \ldots, n)$; and we assign with the condition $l_{r,r} = 1$. The determinant of transformation Δ is

$$\Delta = \begin{vmatrix} 1, & l_{1,2}, & l_{1,3}, & l_{1,4}, \ldots\ldots \\ l_{2,1}, & 1, & l_{2,3}, & l_{2,4}, \ldots\ldots \\ l_{3,1}, & l_{3,2}, & 1, & l_{3,4}, \ldots\ldots \\ l_{4,1}, & l_{4,2}, & l_{4,3}, & 1, \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

which on expansion differs from unity by terms which are products of two or more of the coefficients l; and therefore, if we retain, in the equations which express the invariantive relation, only quantities independent of the l's and terms involving the l's linearly, we may take this determinant Δ as unity.

12. If we temporarily denote a variable of a linear manifoldness of order $r-1$ by p_r, being composed as a determinant of r^2 hyper-point variables, it is convenient to choose the transformed variables P, so that

$$\Sigma p_r p_{n-r} = \Sigma P_r P_{n-r}.$$

Writing

$$p_r = \Sigma \pm x_1 y_2 \ldots t_r,$$

we have, so far as dimensions in the coefficients of transformation are concerned,

$$p_r = \Delta^{r/n} \Sigma \pm X_1 Y_2 \ldots T_r;$$

and, if we write

$$P_r = \Delta^{\theta_r} \Sigma \pm X_1 Y_2 \ldots T_r,$$

the foregoing relation connecting the old and the new complementary variables requires that

$$\theta_r + \theta_{n-r} = 1.$$

We may evidently take $\theta_1 = 0$, and therefore we must have $\theta_{n-1} = 1$; the quantities θ_r may, conformably with the nature of the functions, be taken as an arithmetical progression, and so a suitable value of θ_r is given by

$$\theta_r = \frac{r-1}{n-2}.$$

Hence from the point of view of dimensions, we have

$$\frac{p_r}{P_r} = \Delta^{r/n - \theta_r} = \Delta^{[n-2r]/[n(n-2)]}.$$

13. Consider now any general n-ary quantic; let it be of order m_r in the variables p_r, and let a denote a general coefficient. Then, after the linear transformations have been effected and the new general coefficient has been denoted by A, we have, so far as regards dimensions,

$$\frac{A}{a} = (\Delta^{1/n})^{m_1 + 2m_2 + 3m_3 + \ldots + (n-1) m_{n-1}}.$$

To find the index μ of any concomitant of the quantic which is of degree c in the coefficients of that quantic, and of order s_r in its variables p_r, it is sufficient to count dimensions. The equation which expresses the invariance may be dimensionally written in the form

$$A^c P_1^{s_1} P_2^{s_2} \ldots P_{n-1}^{s_{n-1}} = \Delta^{\mu} . a^c p_1^{s_1} p_2^{s_2} \ldots p_{n-1}^{s_{n-1}};$$

and therefore, so far as dimensions in Δ are concerned, we have

$$\frac{c}{n} \sum_{r=1}^{r=n-1} r m_r = \mu + \sum_{r=1}^{r=n-1} s_r \frac{n-2r}{n(n-2)},$$

or

$$\mu n (n-2) = \sum_{r=1}^{r=n-1} \{ cr (n-2) m_r - s_r (n-2r) \}.$$

Thus, in the case of a ternary form of the m^{th} order in point variables, the index of a concomitant, of degree c in the coefficients, of order m'

in point variables, and of class n' in line variables, is

$$\tfrac{1}{2}(cm-m'+n');$$

and in the case of a biternary form, of order m in point variables and class n in line variables, the index of a similar concomitant is

$$\tfrac{1}{2}\left\{c\,(m+2n)-m'+n'\right\}.$$

14. When the general typical variables of the manifoldnesses of order 0, 1, 2, ... are respectively denoted by (r), (rs), (rst), ..., the general ground-form can be expressed in the symbolical form

$$U=\left\{\Sigma a_r\,(r)\right\}^{m_1}\left\{\Sigma b_{rs}\,(rs)\right\}^{m_2}\left\{\Sigma c_{rst}\,(rst)\right\}^{m_3}\left\{\Sigma d_{rstp}\,(rstp)\right\}^{m_4}\dots;$$

and when this is expanded an actual general term is

$$\frac{m_1!}{\rho_1!\,\rho_2!\,\dots\,\rho_n!}\cdot\frac{m_2!}{\rho_{1,2}!\,\rho_{1,3}!\dots\rho_{n-1,n}!}\cdot\frac{m_3!}{\rho_{1,2,3}!\,\rho_{1,2,4}!\,\dots\,\rho_{n-2,n-1,n}!}\dots$$

$$\left[(1)^{\rho_1}(2)^{\rho_2}\dots(n)^{\rho_n}\right]\left[(12)^{\rho_{1,2}}(13)^{\rho_{1,3}}\dots(n-1,n)^{\rho_{n-1,n}}\right]\dots$$

$$A_{\rho_1,\,\rho_2,\,\dots,\,\rho_n,\,\rho_{1,2},\,\rho_{1,3},\,\dots,\,\rho_{n-1,n},\,\rho_{1,2,3},\,\rho_{1,2,4},\,\dots},$$

where
$$A_{\rho_1,\,\rho_2,\,\dots,\,\rho_n,\,\rho_{1,2},\,\rho_{1,3},\,\dots}$$

$$=a_1^{\rho_1}a_2^{\rho_2}\dots a_n^{\rho_n}\,.\,b_{1,2}^{\rho_{1,2}}b_{1,3}^{\rho_{1,3}}\dots b_{n-1,n}^{\rho_{n-1,n}}\,.\,c_{1,2,3}^{\rho_{1,2,3}}\dots,$$

and
$$\begin{aligned}
m_1&=\rho_1+\rho_2+\dots+\rho_n,\\
m_2&=\rho_{1,2}+\rho_{1,3}+\dots+\rho_{n-1,n},\\
m_3&=\rho_{1,2,3}+\rho_{1,2,4}+\dots+\rho_{n-2,n-1,n},\\
&\dots\quad\dots\quad\dots\quad\dots\quad\dots
\end{aligned}$$

Moreover, the symbolical form of the quantic may be supposed to be the reduced normal form of Clebsch's memoir; in that case the constant coefficients associated in it with the variables of a manifoldness satisfy among themselves the same identical relations as exist among the variables, for the quantic satisfies all the differential equations of the second class lately considered.

15. Suppose now that the linear transformations of § 11 are applied to the ground-form and to its concomitants.

The effect on the variables of the manifoldness of order 0 is as given in § 11. The effect on the variables of the manifoldness of order 1 is given by

$$(rs)=\Sigma\Sigma\begin{vmatrix} l_{r\rho}, & l_{r\sigma}\\ l_{s\rho}, & l_{s\sigma}\end{vmatrix}(P\Sigma);$$

but with the restriction (§ 11, *fin.*) as to the rejection of all terms which are of the second and of higher dimensions in the quantities l,

we have $$(rs) = (RS) + \Sigma' \left\{ (l_{s,a}\,\delta_s + l_{r,a}\,\delta_r)(RS) \right\},$$

where Σ' implies that summation is to be taken for all the terms given by values of a in the series 1, 2, ..., n other than the co-subscript in l, and

$$l_{s,a}\,\delta_s\,(RS) = l_{s,a}\,(RA),$$

$$l_{r,a}\,\delta_r\,(RS) = l_{r,a}\,(AS).$$

Thus, for example,

$$(12) = (12) + \sum_{r=3}^{r=n} \left\{ l_{2,r}\,(1R) + l_{1,r}\,(R2) \right\}.$$

Similarly the effect of the transformation (with the restriction as to the rejection of terms of dimensions beyond the first in l) on the variables of the manifoldness of order 2 is

$$\begin{aligned}(rst) &= (RST) + \Sigma' \left\{ (l_{r,p}\,\delta_r + l_{s,p}\,\delta_s + l_{t,p}\,\delta_t)(RST) \right\} \\ &= (RST) + \Sigma' \left\{ l_{r,p}\,(PST) + l_{s,p}\,(RPT) + l_{t,p}\,(RSP) \right\},\end{aligned}$$

where in each case the summation is for the $n-1$ values of p in the series 1, 2, 3, ... , n other than the co-subscript in l.

The law of transformation for the variables of any manifoldness is now obvious.

16. These changes in the variables imply corresponding changes in the symbolical linear factors of U when it is transformed, and imply therefore transformed coefficients in these linear factors; let the new coefficient be denoted by the same symbol as the former corresponding coefficient, but with the literal part dashed. The coefficients, associated with the variables of a manifoldness of any order $r-1$, are cogredient with the variables of manifoldness of complementary order $n-r-2$; and thus we obtain as the necessary equations

$$a'_r = a_r + \sum_{s=1}^{s=n}{}' \; l_{s,r}\,a_s$$

(where Σ' has a meaning similar to that before used),

$$b'_{r,s} = b_{r,s} + \Sigma' (l_{p,s}\,b_{r,p} + l_{p,r}\,b_{p,s}),$$

$$c'_{r,s,t} = c_{r,s,t} + \Sigma' (l_{p,r}\,c_{p,s,t} + l_{p,s}\,c_{r,p,t} + l_{p,t}\,c_{r,s,p}),$$

and so on. And in these it must be borne in mind that every coefficient as written must be taken in connection with the associated

symbolical variable; so that $b_{t,s}$ occurring with (t, s) is $-b_{s,t}$, and one of the equations giving the transformations of the b's is thus

$$b'_{12} = b_{12} + \sum_{p=3}^{p=n} (l_{p,2}\, b_{1,p} - l_{p,1}\, b_{2,p}).$$

To find the changes in the transformed actual coefficient A', it is necessary to substitute these values for the transformed symbolical coefficients and reduce the result to an expression in terms of the original actual coefficients. The first term will be the original actual coefficient A; the remaining terms will be products of the quantities l with other coefficients A. It will be sufficient for our purpose to write down the part of the expression which has for its coefficient any assigned one of the quantities, say $l_{\lambda,\mu}$. Now $l_{\lambda,\mu}$ enters as the coefficient of

(i.) a_λ in a'_μ;

(ii.) $b_{r,\lambda}$ in $b'_{r,\mu}$, where r has the $n-2$ values of $1, 2, \ldots, n$ which are not λ or μ;

(iii.) $c_{\lambda,r,s}$ in $c'_{\mu,r,s}$, where r and s may have the $n-2$ values of $1, 2, \ldots n$ which are not λ and μ;

and so on, and therefore the coefficient of $l_{\lambda,\mu}$ in

$$A'_{\rho_1,\, \rho_2,\, \ldots,\, \rho_n,\, \rho_{1,2},\, \ldots,\, \rho_{n-1,n},\, \rho_{1,2,3}\, \ldots}$$

is

$$\rho_\mu A_{\rho_1,\, \rho_2,\, \ldots,\, \rho_\lambda+1,\, \ldots,\, \rho_\mu-1,\, \ldots\, \rho_n,\, \rho_{1,2},\, \rho_{1,3},\, \ldots}$$

$$+\sum^{r}{}' \rho_{\mu,r} A_{\rho_1,\, \rho_2,\, \ldots,\, \rho_n,\, \rho_{1,2},\, \ldots,\, \rho_{\lambda,r}+1,\, \ldots,\, \rho_{\mu,r}-1,\, \ldots,\, \rho_{n-1,n},\, \rho_{1,2,3},\, \ldots}$$

$$+\Sigma\sum^{s}{}' \rho_{\mu,r,s} A_{\rho_1,\, \rho_2,\, \ldots,\, \rho_n,\, \rho_{1,2},\, \rho_{1,3},\, \ldots,\, \rho_{n-1,n},\, \rho_{1,2,3}\, \ldots\, \rho_{\lambda,r,s}+1,\, \ldots,\, \rho_{\mu,r,s}-1,\, \ldots,\, \rho_{n-2,n-1,n},\, \rho_{1,2,3,4},\, \ldots}$$

$$+\ldots\ldots\ldots\ldots$$

Let this be denoted by (A, λ, μ); then we have

$$A' = A + \Sigma\Sigma' \left\{ l_{\lambda,\mu} (A, \lambda, \mu) \right\}.$$

17. The equation which expresses the invariance of the concomitant may be written

$$\Phi = \Delta^{\mu'} \phi.$$

When in it all the variables are expressed in terms of the transformed variables, and all the coefficients are expressed in terms of the untransformed coefficients, the equation comes to be an identity, and therefore the terms, multiplied by the same combinations of the

quantities l, are equal to one another. Selecting, then, the coefficient of $l_{\lambda,\mu}$, and, after the differentiations are performed, referring all the quantities which occur to the untransformed function ϕ, we have as the complete part contributed by the left-hand side

$$\Sigma \frac{\partial \phi}{\partial A}(A, \lambda, \mu).$$

This summation extends over all the coefficients A; for some of them, however, the quantity (A, λ, μ) will vanish and the corresponding term will be absent.

On the right-hand side $\Delta^{r'}$ is (§ 11) unity; the part arising from the variables of the manifoldness of order 0 is

$$x_\mu \frac{\partial \phi}{\partial x_\lambda};$$

the part arising from the variables of the manifoldness of order 1 is

$$\Sigma'(\mu r)\frac{\partial \phi}{\partial(\lambda r)},$$

where the summation is for the $n-2$ values of r other than λ and μ in the series $1, 2, \ldots, n$; the part arising from the variables of the manifoldness of order 2 is

$$\Sigma'\Sigma'(\mu rs)\frac{\partial \phi}{\partial(\lambda rs)},$$

with similar limits for the summations; and so on. Hence finally we have

$$\Sigma \frac{\partial \phi}{\partial A}(A, \lambda, \mu) = x_\mu \frac{\partial \phi}{\partial x_\lambda} + \Sigma'(\mu r)\frac{\partial \phi}{\partial(\lambda r)} + \Sigma'\Sigma'(\mu rs)\frac{\partial \phi}{\partial(\lambda rs)} + \ldots,$$

a differential equation satisfied by ϕ. This was derived by considering the coefficient of $l_{\lambda,\mu}$, one of the $n(n-1)$ quantities l; hence the function ϕ satisfies $n(n-1)$ differential equations of the foregoing type.

18. The following particular cases of this general result may be noticed:—

I. For *ternary* quantics there are two classes of variables; the first is constituted by x_1, x_2, x_3; the second by

$$u_1 = (23),$$
$$u_2 = (31),$$
$$u_3 = (12),$$

For the *uni-ternary n-tic* represented by

$$U = (a_1x_1+a_2x_2+a_3x_3)^n$$
$$= (\ldots, a_{r,s,t}, \ldots \between x_1, x_2, x_3)^n,$$

so that the term in $x_1^r x_2^s x_3^t$ $(r+s+t=n)$ has the coefficient

$$\frac{n!}{r!\,s!\,t!}\,a_{r,s,t},$$

the six differential equations to be satisfied by any concomitant are

$$\Sigma\Sigma r a_{r-1,s+1,t}\frac{\partial\phi}{\partial a_{r,s,t}} = x_1\frac{\partial\phi}{\partial x_2} - u_2\frac{\partial\phi}{\partial u_1},$$

$$\Sigma\Sigma r a_{r-1,s,t+1}\frac{\partial\phi}{\partial a_{r,s,t}} = x_1\frac{\partial\phi}{\partial x_3} - u_3\frac{\partial\phi}{\partial u_1},$$

$$\Sigma\Sigma s a_{r,s-1,t+1}\frac{\partial\phi}{\partial a_{r,s,t}} = x_2\frac{\partial\phi}{\partial x_3} - u_3\frac{\partial\phi}{\partial u_2},$$

$$\Sigma\Sigma s a_{r+1,s-1,t}\frac{\partial\phi}{\partial a_{r,s,t}} = x_2\frac{\partial\phi}{\partial x_1} - u_1\frac{\partial\phi}{\partial u_2},$$

$$\Sigma\Sigma t a_{r+1,s,t-1}\frac{\partial\phi}{\partial a_{r,s,t}} = x_3\frac{\partial\phi}{\partial x_1} - u_1\frac{\partial\phi}{\partial u_3},$$

$$\Sigma\Sigma t a_{r,s-1,t-1}\frac{\partial\phi}{\partial a_{r,s,t}} = x_3\frac{\partial\phi}{\partial x_2} - u_2\frac{\partial\phi}{\partial u_3}.$$

On the left-hand side $r+s+t=n$, and the double summation extends over all the values of r, s, t consistent with this condition, *i.e.*, over all the coefficients of the quantic.

For a pure covariant we have $\dfrac{\partial\phi}{\partial u_m} = 0$ $(m = 1, 2, 3)$;

for a pure contravariant we have $\dfrac{\partial\phi}{\partial x_m} = 0$ $(m = 1, 2, 3)$;

for an invariant we have $\dfrac{\partial\phi}{\partial x_m} = 0 = \dfrac{\partial\phi}{\partial u_m}$ $(m = 1, 2, 3)$.

For the *biternary n-tic* represented by

$$a_x^n u_a^m$$
$$= (a_1x_1+a_2x_2+a_3x_3)^n\,(u_1\alpha_1+u_2\alpha_2+u_3\alpha_3)^m$$
$$= \Sigma\,\frac{n!}{r!\,s!\,t!}\,\frac{m!}{\rho!\,\sigma!\,\tau!}\,a_{r,s,t,\rho,\sigma,\tau}\,x_1^r x_2^s x_3^t u_1^\rho u_2^\sigma u_3^\tau$$

(with the conditions $r+s+t=n$, $\rho+\sigma+\tau=m$) the six differential equations are

$$\Sigma\,(ra_{r-1,s+1,t,\rho,\sigma,\tau}+\sigma a_{r,s,t,\rho+1,\sigma-1,\tau})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_1\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_1},$$

$$\Sigma\,(sa_{r+1,s-1,t,\rho,\sigma,\tau}+\rho a_{r,s,t,\rho-1,\sigma+1,\tau})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_2\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_2},$$

$$\Sigma\,(ra_{r-1,s,t+1,\rho,\sigma,\tau}+\tau a_{r,s,t,\rho+1,\sigma,\tau-1})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_1\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_1},$$

$$\Sigma\,(ta_{r+1,s,t-1,\rho,\sigma,\tau}+\rho a_{r,s,t,\rho-1,\sigma,\tau+1})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_3\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_3},$$

$$\Sigma\,(sa_{r,s-1,t,\rho,\sigma,\tau}+\tau a_{r,s,t,\rho,\sigma+1,\tau-1})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_2\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_2},$$

$$\Sigma\,(ta_{r,s+1,t-1,\rho,\sigma,\tau}+\sigma a_{r,s,t,\rho,\sigma-1,\tau+1})\,\frac{\partial\phi}{\partial a_{r,s,t,\rho,\sigma,\tau}}=x_3\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_3}.$$

II. For *quaternary* quantics there are three classes of variables: the first is constituted by x_1, x_2, x_3, x_4; the second by

$$p_{12}=(12),\quad p_{14}=(14),$$
$$p_{23}=(23),\quad p_{24}=(24),$$
$$p_{31}=(31),\quad p_{34}=(34),$$

with the relation $\quad p_{12}p_{34}+p_{23}p_{14}+p_{31}p_{24}=0;$

and the third by

$$u_1=(234),\quad -u_2=(341),\quad u_3=(412),\quad -u_4=(123).$$

For the *uni-quaternary* n-tic represented by

$$U=(a_1x_1+a_2x_2+a_3x_3+a_4x_4)^n$$
$$=(\ldots,\ a_{q,r,s,t},\ \ldots\between x_1,\ x_2,\ x_3,\ x_4)^n,$$

there are twelve differential equations to be satisfied by any concomitant ϕ; they can be arranged in four triads, one of which is

$$\Sigma\Sigma\Sigma\, ra_{q+1,r-1,s,t}\,\frac{\partial\phi}{\partial a_{q,r,s,t}}=x_2\frac{\partial\phi}{\partial x_1}+p_{24}\frac{\partial\phi}{\partial p_{14}}-p_{23}\frac{\partial\phi}{\partial p_{31}}-u_1\frac{\partial\phi}{\partial u_2},$$

$$\Sigma\Sigma\Sigma\, sa_{q+1,r,s-1,t}\,\frac{\partial\phi}{\partial a_{q,r,s,t}}=x_3\frac{\partial\phi}{\partial x_1}+p_{34}\frac{\partial\phi}{\partial p_{14}}-p_{23}\frac{\partial\phi}{\partial p_{12}}-u_1\frac{\partial\phi}{\partial u_3},$$

$$\Sigma\Sigma\Sigma\, ta_{q+1,r,s,t-1}\,\frac{\partial\phi}{\partial a_{q,r,s,t}}=x_4\frac{\partial\phi}{\partial x_1}+p_{34}\frac{\partial\phi}{\partial p_{31}}-p_{24}\frac{\partial\phi}{\partial p_{12}}-u_1\frac{\partial\phi}{\partial u_4};$$

or in other four triads, one of which is

$$\Sigma\Sigma\Sigma\, q a_{q-1,r+1,s,t} \frac{\partial\phi}{\partial a_{q,r,s,t}} = x_1 \frac{\partial\phi}{\partial x_2} + p_{14} \frac{\partial\phi}{\partial p_{24}} - p_{31} \frac{\partial\phi}{\partial p_{32}} - u_2 \frac{\partial\phi}{\partial u_1},$$

$$\Sigma\Sigma\Sigma\, q a_{q-1,r,s+1,t} \frac{\partial\phi}{\partial a_{q,r,s,t}} = x_1 \frac{\partial\phi}{\partial x_3} + p_{14} \frac{\partial\phi}{\partial p_{34}} - p_{12} \frac{\partial\phi}{\partial p_{23}} - u_3 \frac{\partial\phi}{\partial u_1},$$

$$\Sigma\Sigma\Sigma\, q a_{q-1,r,s,t+1} \frac{\partial\phi}{\partial a_{q,r,s,t}} = x_1 \frac{\partial\phi}{\partial x_4} + p_{21} \frac{\partial\phi}{\partial p_{24}} - p_{13} \frac{\partial\phi}{\partial p_{34}} - u_4 \frac{\partial\phi}{\partial u};$$

the summation extending over the whole of the coefficients of the original quantic.

19. As an application of the use of these equations consider the proposition* relating to ternary quantics:

Every concomitant of a ternary quantic, which is a function of the variables x and u with pure numerical constants but is independent of the coefficients of the quantic, is a function of

$$u_x = u_1 x_1 + u_2 x_2 + u_3 x_3.$$

Let such a function be denoted by Π; thence by hypothesis

$$\frac{\partial\Pi}{\partial a} = 0,$$

and the equations determining Π are

$$\frac{1}{x_1}\frac{\partial\Pi}{\partial u_1} = \frac{1}{x_2}\frac{\partial\Pi}{\partial u_2} = \frac{1}{x_3}\frac{\partial\Pi}{\partial u_3} = \frac{1}{u_1}\frac{\partial\Pi}{\partial x_1} = \frac{1}{u_2}\frac{\partial\Pi}{\partial x_2} = \frac{1}{u_3}\frac{\partial\Pi}{\partial x_3}$$

$$= \Theta, \text{ say.}$$

Then $$d\Pi = \frac{\partial\Pi}{\partial x_1} dx_1 + \ldots + \frac{\partial\Pi}{\partial u_1} du_1 + \ldots$$

$$= \Theta d\,(x_1 u_1 + x_2 u_2 + x_3 u_3).$$

But $d\Pi$ is a perfect differential; hence Θ must be a function of u_x, and Π is therefore also a function of u_x.

As a second application, we can show that by means of these equations a covariant can be calculated when its leading term is known.

* Clebsch's *Geometrie*, p. 270.

Let $\qquad D_{1,2}$ denote the operator $\Sigma\Sigma r a_{r-1,s+1,t}\dfrac{\partial}{\partial a_{r,s,t}}$,

$\qquad D'_{1,2} \qquad$ " $\qquad$ " $\qquad \Sigma\Sigma s a_{r+1,s-1,t}\dfrac{\partial}{\partial a_{r,s,t}}$,

and so on; so that the equations to be satisfied are

$$\left.\begin{aligned} D_{1,2}\phi &= x_1\frac{\partial\phi}{\partial x_2} \\ D'_{1,2}\phi &= x_2\frac{\partial\phi}{\partial x_1}\end{aligned}\right\}\quad \left.\begin{aligned} D_{1,3}\phi &= x_1\frac{\partial\phi}{\partial x_3} \\ D'_{1,3}\phi &= x_3\frac{\partial\phi}{\partial x_1}\end{aligned}\right\}\quad \left.\begin{aligned} D_{2,3}\phi &= x_2\frac{\partial\phi}{\partial x_3} \\ D'_{2,3}\phi &= x_3\frac{\partial\phi}{\partial x_2}\end{aligned}\right\}$$

Then substituting as the value of ϕ,

$$C_{0,0}x_1^m + \ldots + \frac{1}{r!\,s!}C_{r,s}x_1^{m-r-s}x_2^r x_3^s + \ldots,$$

in the prior of the first two pairs of equations, we at once have

$$D_{1,2}\,.\,C_{r-1,s} = C_{r,s},$$

$$D_{1,3}\,.\,C_{r,s-1} = C_{r,s}.$$

From the former of these we have

$$C_{r,s} = D_{1,2}^r\,.\,C_{0,s};$$

from the latter we have $\quad C_{0,s} = D_{1,3}^s\,.\,C_{0,0};$

and therefore $\qquad C_{r,s} = D_{1,2}^r\,.\,D_{1,3}^s\,.\,C_{0,0}.$

Hence the knowledge of the leading term, $C_{0,0}\,x_1^m$, is sufficient to give the covariant.

It further appears from this analysis that the coefficient $C_{0,0}$ of the leading term of a covariant must satisfy the equations

$$D'_{1,2}\,C_{0,0} = 0,$$

$$D'_{1,3}\,C_{0,0} = 0,$$

as well as $\qquad D_{2,3}\,C_{0,0} = 0 = D'_{2,3}\,C_{0,0}\,.$

20. It is not however necessary to work out all the differentiations thus indicated in order to obtain the complete expression for the covariant; for by means of the following propositions—corresponding to that dealing with the symmetry, skew or direct, of covariants of binary forms—we are able to write down the remainder of the terms

when a certain number have been calculated. Since each of the determinants

$$\begin{vmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} 1, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0, & 1 \\ 0, & 1, & 0 \\ 1, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ 1, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & 0, & 1 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{vmatrix}$$

is -1, it follows that a covariant of a uni-ternary form is either symmetric or skew symmetric in those coefficients of the quantic which are—

(α) Similarly situated with regard to x_1 and x_2, or x_2 and x_3, or x_3 and x_1,—corresponding to the linear transformations represented by the first three of the above five determinants;

(β) Similarly situated with regard to the cycle x_1, x_2, x_3 or to the cycle x_1, x_3, x_2,—corresponding to the linear transformations represented by the last two of the above five determinants;

the symmetry in each case being associated with the corresponding compared variables.

And, in particular, an invariant is either symmetric or skew symmetric in the coefficients of terms of similar rank, *i.e.*, terms such that their aggregate, so far as variables are concerned, remains unaltered by any complete or incomplete interchange of the variables. Thus, in the case of the cubic

$$ax_1^3 + bx_2^3 + cx_3^3 + 6lx_1x_2x_3,$$

the first three are terms of similar rank; and the invariants are

$$abcl - l^4, \quad a^2b^2c^2 - 20abcl^3 - 8l^6.$$

By way of verification, there is the well-known indirect test* that zero is the sum of the numerical coefficients of the literal parts in the coefficients of the variable terms of any concomitant of an explicitly general quantic, with the literal coefficients of which are combined the corresponding multinomial coefficients.

* One very simple way of proving this is to deduce it immediately from the theorem that every concomitant of a quantic can be symbolically expressed as an aggregate of products of umbral real factors and umbral symbolical determinants.

On the Stability of a Liquid Ellipsoid which is rotating about a Principal Axis under the influence of its own attraction. By A. B. Basset, M.A.

[*Read Nov. 10th*, 1887.]

1. When a mass of liquid is rotating in a state of steady motion under the influence of its own attraction, the different ellipsoidal forms which its free surface can assume may be classified as follows:—

I. *Maclaurin's Spheroid*, in which the free surface is an oblate spheroid, and the liquid rotates as a rigid body about the axis of the spheroid. If ρ be the density of the liquid, ζ the angular velocity of the spheroid (which in this case is identical with the molecular rotation), it is known that $\zeta^2/4\pi\rho$ must not be greater than $\cdot 1123$, in order that steady motion may be possible, and in this case the free surface may be one or other of two oblate spheroids, which coalesce when $\zeta^2/4\pi\rho = \cdot 1123$.

II. *Jacobi's Ellipsoid*, in which the free surface is an ellipsoid with three unequal axes, and the liquid rotates as a rigid body about the least axis. In this case $\zeta^2/4\pi\rho$ must not be greater than $\cdot 0934$, in order that the ellipsoid may be a possible form of the free surface. Hence, if $\zeta^2/4\pi\rho < \cdot 0934$, there are three ellipsoidal forms, viz., the two Maclaurin's spheroids, and the Jacobian ellipsoid. When $\zeta^2/4\pi\rho = \cdot 0934$, Jacobi's ellipsoid coalesces with the most oblate of the two spheroids, and when $\zeta^2/4\pi\rho$ lies between $\cdot 0934$ and $\cdot 1123$, the ellipsoidal form is impossible.

III. *Dedekind's Ellipsoid*, in which the free surface remains stationary in space, but there is an internal motion of the particles of liquid, due to molecular rotation ζ about lines parallel to the least axis. In this case, if a and b are the greatest and mean axes respectively, $a^2b^2\zeta^2/(a^2+b^2)^2\pi\rho$ must not be greater than $\cdot 0934$; and when the former quantity is equal to $\cdot 0934$, we must have $a = b$, and Dedekind's ellipsoid coalesces with the most oblate of the two Maclaurin's spheroids.

IV. *The Irrotational Ellipsoid*, in which the axis of rotation is the *mean* axis, and the motion is irrotational. In this case the spheroidal form is not possible.

V. An ellipsoid in which there is molecular rotation ζ, and an *independent* angular velocity $\zeta+\Omega$ about the axis to which ζ refers.

In this case the axis will be the *mean* or *least* axis, according as

$$\frac{\zeta}{\Omega} < \text{ or } > \frac{a^2-b^2}{a^2+b^2}\left(1 \pm \frac{2a}{\sqrt{a^2-b^2}}\right).$$

When this inequality becomes an equality, the free surface will be a prolate spheroid rotating about an equatorial axis. This case includes the four preceding cases.

VI. *Riemann's Ellipsoid*, in which the rotation takes place about an instantaneous axis lying in a principal plane. This case includes all the preceding cases; moreover, if the instantaneous axis does not lie in a principal plane, steady motion is impossible.*

In the present paper, I propose to consider the stability of a liquid ellipsoid which in steady motion is rotating about a principal axis, and which is subjected to a disturbance such that the free surface in the beginning of the disturbed motion is an ellipsoid. A disturbance of this character may be communicated by enclosing the liquid ellipsoid in a case which is subjected to an impulsive couple about any diameter together with a deformation of its surface, and is therefore equivalent to a disturbance produced by an impulsive pressure communicated to the free surface of the liquid. The disturbed motion may therefore be investigated by means of Riemann's general equations of motion, a proof of which I have given in Vol. XVII. of the Society's *Proceedings*; and references to the equations of this paper will be denoted as follows: [E].

The method employed is founded upon Riemann's paper, and the present investigation is an amplification of his work upon this portion of the subject.

2. By [E. 21], the potential energy of an ellipsoidal mass of gravitating liquid of mass M and uniform density ρ is

$$U = D - \tfrac{3}{4}M\pi\rho\, abc\int_0^\infty \frac{d\lambda}{P},$$

where $P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, and D is a constant. Let R be the radius of a sphere of equal volume, then

$$U = 0 \text{ when } a = b = c = R,$$

therefore $$D = \tfrac{3}{2}M\pi\rho R^2,$$

and $$U = \tfrac{3}{2}M\pi\rho R^2 - \tfrac{3}{4}M\pi\rho\, abc\int_0^\infty \frac{d\lambda}{P} \quad\ldots\ldots\ldots\ldots(1).$$

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* For proofs of the foregoing theorems, see—Riemann, *Abhand. König. Wiss., Göttingen*, Vol. IX.; Greenhill, *Proc. Camb. Phil. Soc.*, Vols. III. and IV.

Now U is evidently positive; hence the integral must be a maximum when $a = b = c = R$, and will become indefinitely small when any one of the axes of the ellipsoid becomes infinitely small or infinitely large.

Let $2c$ be the axis of rotation, then, employing the notation of the preceding paper, let

$$E = \frac{M}{10}\left\{\frac{\omega_3^2(a^2-b^2)^2}{a^2+b^2}+\frac{4a^2b^2\zeta^2}{a^2+b^2}-4\pi\rho\, abc\int_0^\infty \frac{d\lambda}{P}\right\} \quad \text{.........(2).}$$

By [E. 20 and 21], E is the variable part of the energy of a mass of liquid whose free surface is constrained to maintain a fixed ellipsoidal form and which is rotating about the least axis. In steady motion ω_3 and ζ, and therefore E, are certain functions of a, b, c; let E_0 be the value of E in steady motion.

Let a disturbance (which for brevity will be called an ellipsoidal disturbance) be communicated to the liquid by means of an impulsive pressure applied to its free surface, which is such that in the beginning of the disturbed motion the free surface is a slightly different ellipsoid. Then, if $E_0+\delta E$ is the energy of the disturbed motion, we obtain by [E. 20 and 21],

$$\delta E = \frac{M}{10}\left[\dot{a}^2+\dot{b}^2+\dot{c}^2+\frac{\omega_1^2(b^2-c^2)^2}{b^2+c^2}+\frac{\omega_2^2(c^2-a^2)^2}{c^2+a^2}+\frac{4b^2c^2\xi^2}{b^2+c^2}\right.$$
$$\left.+\frac{4c^2a^2\eta^2}{c^2+a^2}\right]+E-E_0.$$

All the terms in square brackets are positive, and in the beginning of the disturbed motion are small quantities; hence, if $E > E_0$, these terms must remain small quantities and the free surface can never deviate far from its form in steady motion, and the motion is therefore stable. But, if $E < E_0$, the terms in square brackets may become a finite positive quantity, and the difference $E-E_0$ may become a finite negative quantity, such that the difference between the two sets of terms always remains equal to the infinitesimal quantity δE. When this is the case the free surface may deviate far from its form in steady motion, and the motion may be unstable.

Hence, for the particular kind of disturbance which we are considering, the condition of stability requires that the energy in steady motion should be a minimum. Or, in other words, if the steady motion is stable, it must be impossible by any kind of ellipsoidal disturbance to abstract energy from the system.

3. Let the disturbing pressure be divided into two parts p_1, p_2, the

former of which produces a variation of the axes and no change in the angular momentum, whilst the latter produces no instantaneous variation of the axis but changes the angular momentum. The resultant of p_2 will consist of a single force, which produces a translation of the whole mass of liquid, and which it is unnecessary to consider; and a couple G. If the axis of this couple lie in the principal plane, which is perpendicular to the axis of rotation in steady motion, the energy will be evidently increased by its application; but, if the axis of the couple does not lie in this principal plane, the component of the couple about the axis of rotation may diminish the energy if it acts in the opposite direction to that of rotation, in which case the motion will be unstable.

In Maclaurin's spheroid the component of the couple about the axis of rotation necessarily vanishes, since p_2 always passes through the axis of rotation; the case of Dedekind's ellipsoid, in which the free surface is stationary, will be considered later on.

Hence, so far as the action of p_2 is concerned, Jacobi's ellipsoid, the Irrotational ellipsoid, and the ellipsoids belonging to the general class V., including the prolate spheroid rotating about an equatorial axis, but excluding Dedekind's ellipsoid, are stable whenever the couple component about the axis of rotation of the disturbing pressure either vanishes or is in the same direction as the rotation; but when this is not the case the motion may be unstable.

In the case of Dedekind's ellipsoid, by (2),

$$E_0 = \frac{2M}{5}\left\{\frac{a^2b^2\zeta^2}{a^2+b^2} - \pi\rho abc\int_0^\infty \frac{d\lambda}{P}\right\},$$

where

$$\frac{4a^2b^2\zeta^2}{a^2+b^2} = \frac{Aa^2-Cc^2}{a^2} = \frac{Bb^2-Cc^2}{b^2},$$

and the effect of a disturbing couple about the axis of rotation will be to increase the energy by the quantity

$$\frac{M\omega_3^2(a^2-b^2)^2}{10(a^2+b^2)},$$

whence $E > E_0$, and therefore the motion so far as this kind of disturbance is concerned is stable.

4. We must now consider the disturbance p_1 which produces a variation of the axes. From the last two of [E. 16] we obtain

$$(a-b)^2 w = \text{const.} = r, \quad (a+b)^2 w' = \text{const.} = r' \quad \text{.........(3)},$$

whence, from [E. 9],

$$\frac{\zeta}{c} = \frac{r'-r}{2abc}, \quad h_3 = \frac{M}{5}(r'+r) \quad \text{.....................(4)}.$$

Also, from [E. 6],

$$h_3 = \frac{M}{5(a^2+b^2)}\left\{(a^2-b^2)^2\omega_3 + 4a^2b^2\zeta\right\},$$

whence
$$E = \frac{M}{5}\left\{\frac{r^2}{(a-b)^2} + \frac{r'^2}{(a+b)^2} - 2H\right\} \quad\text{..................(5)},$$

where
$$H = \pi\rho abc\int_0^\infty \frac{d\lambda}{P}.$$

We must now obtain the value of E_0. Putting $\ddot{a}$, $\ddot{b}$, $\ddot{c}$ each equal to zero in the first three of [E. 16], and taking account of (3), we obtain

$$\left.\begin{aligned} 0 &= \tfrac{1}{2}Cc - \frac{\sigma}{c} \\ \frac{r'^2}{(a+b)^3} + \frac{r^2}{(a-b)^3} &= \tfrac{1}{2}Aa - \frac{\sigma}{a} = \frac{1}{2a}(Aa^2 - Cc^2) \\ \frac{r'^2}{(a+b)^3} - \frac{r^2}{(a-b)^3} &= \tfrac{1}{2}Bb - \frac{\sigma}{b} = \frac{1}{2b}(Bb^2 - Cc^2) \end{aligned}\right\} \quad\text{......(6)}.$$

Whence (5) becomes

$$\begin{aligned} E_0 &= \tfrac{1}{5}(Aa^2 + Bb^2 - 2Cc^2) - 2H \\ &= -H - \tfrac{3}{5}Cc^2 \quad\text{....................................(7)}, \end{aligned}$$

since
$$Aa^2 + Bb^2 + Cc^2 = 2H.$$

Whence E_0 is a finite *negative* quantity.

The constants r, r' express the fact that the angular momentum and the vorticity* are unchanged during the motion; also since the disturbance p_1 does not change the angular momentum or vorticity, these constants must have the same values as in steady motion.

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* The equation [E. 17], which expresses the fact that the vorticity is constant, may be shortly proved thus :—

Since ξ, η, ζ are independent of x, y, z, the vortex lines must all be parallel to some diameter r of the ellipsoid. Let l, m, n be the direction cosines of r, dS an element of the plane conjugate to r, and ϵ the angle between r and S.

The condition that the vorticity should be constant requires that

$$\text{const.} = \iint \omega\sin\epsilon\, dS = \omega S\sin\epsilon = \omega S p r^{-1},$$

where p is the perpendicular from the centre on to the tangent plane parallel to the plane S. But, since the volume of the ellipsoid is constant, Sp = const., therefore ω/r = const., or

$$\omega^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = \text{const.},$$

i.e.,
$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = \text{const.}$$

Since the volume of the ellipsoid is constant, the conditions that E may be a minimum require that

$$\left.\begin{aligned} \frac{dE}{da}-\frac{c}{a}\frac{dE}{dc}=0 \\ \frac{dE}{db}-\frac{c}{b}\frac{dE}{dc}=0 \end{aligned}\right\} \qquad (8).$$

On performing the differentiations it will be found that (8) lead to (6); hence the first conditions are satisfied.

We must now enquire whether, in the general case, E has a minimum value when r and r' are unchanged by the disturbance.

Let $z = 5E/M$, $R^3 = abc$, $x = a$, $y = b$, then

$$z = \frac{r^2}{(x-y)^2}+\frac{r'^2}{(x+y)^2}-2\pi\rho R^3\int_0^\infty \frac{d\lambda}{\sqrt{(x^2+\lambda)(y^2+\lambda)(R^6/x^2y^2+\lambda)}} \quad \ldots(9)$$

Since a, b, c are positive, and a is never less than b, we have to examine the form of the surface (9) between the planes $y = 0$, $x-y = 0$.

First suppose r is not zero.

When $x=y$, $z=\infty$. If y has any finite value $<$ or $= x$, then, as x increases from y to infinity, z diminishes, and the value of E_0 in steady motion shows that z will vanish and become negative, and when x is very large z is very small. Moreover, z can never become equal to $-\infty$ for any values of x or y, and when x and y are *both* very large z is very small, unless $x-y$ is small.

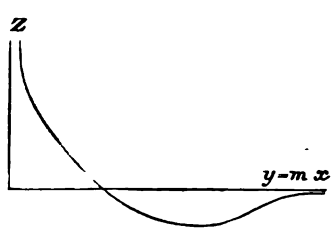

Fig. 1.

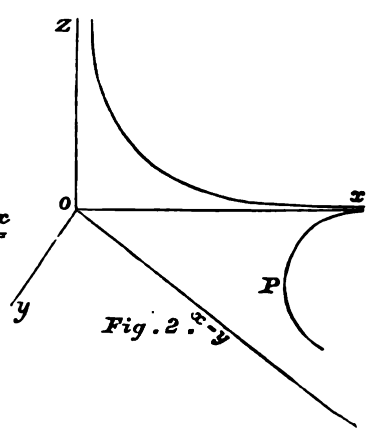

Fig. 2.

A general idea of the form of the surface may be obtained from the accompanying figures. Fig. 1 is the curve of section made by the plane $y = mx$, $m < 1$; and Fig. 2 shows the curves of section made by the planes xz and xy. The surface cuts the plane of xy along the curve xP, and the sheet underneath this plane gradually bends upwards towards the plane.

It therefore follows that in this case z must have a minimum value, which is given by (7).

By a similar course of reasoning it may be shown that z has also a minimum value when $\tau = 0$.

The conditions that τ should vanish require that in steady motion

$$w\,(a-b)^2 = 0,$$

which requires either that $a = b$, which is the case of Maclaurin's spheroid; or that $w = 0$, which by [E. 8] is the same as

$$\omega_3 + \frac{2ab\,\Omega_3}{a^2+b^2} = 0,$$

which is a special case of V.

5. The analytical difficulties of examining whether E is a minimum by means of the usual conditions that E_{aa}, E_{bb}, $E_{aa}E_{bb}-E_{ab}^2$ must all be positive, where $E_{aa} = d^2E/da^2$, &c., would be considerable; but in the case of Maclaurin's spheroid, this may be done without much trouble.

Since c is a function of the independent variables a and b, we have (omitting the factor $M/5$, and putting $Q = Aa^2 - Cc^2$, $R = Bb^2 - Cc^2$)

$$E_a = -\frac{2\tau^2}{(a-b)^3} - \frac{2\tau'^2}{(a+b)^3} + \frac{Q}{a},$$

$$E_b = \frac{2\tau^2}{(a-b)^3} - \frac{2\tau'^2}{(a+b)^3} + \frac{R}{b},$$

$$\left.\begin{aligned} E_{aa} &= \frac{6\tau^2}{(a-b)^4} + \frac{6\tau'^2}{(a+b)^4} + \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{Q}{a^2} \\ &= 6\,(w^2+w'^2) \qquad + \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{Q}{a^2} \\ E_{bb} &= 6\,(w^2+w'^2) \qquad + \frac{1}{b}\left(\frac{dR}{db} - \frac{c}{b}\frac{dR}{dc}\right) - \frac{R}{b^2} \\ E_{ab} &= 6(w'^2-w^2) \qquad + \frac{1}{a}\,\frac{dQ}{db} - \frac{c}{ab}\frac{dQ}{dc} \end{aligned}\right\} \quad \ldots\ldots(10).$$

Equations (10) are perfectly general, but in the case of Maclaurin's spheroid

$$w = w' = \tfrac{1}{2}\zeta = Q^{\frac{1}{2}}/2a,$$

and a must be put equal to b after the differentiations have been performed. Whence

$$\left.\begin{aligned} E_{aa} = E_{bb} &= \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc} + \frac{2Q}{a}\right) \\ E_{ab} &= \frac{1}{a}\frac{dQ}{db} - \frac{c}{a^2}\frac{dQ}{dc} \end{aligned}\right\} \quad \text{.............(11).}$$

Now

$$Q = 2\pi\rho\, abc \int_0^\infty \left(\frac{1}{c^2+\lambda} - \frac{1}{a^2+\lambda}\right)\frac{\lambda\, d\lambda}{P},$$

therefore (omitting $2\pi\rho\, abc$)

$$\frac{dQ}{db} = \int_0^\infty \left(\frac{1}{a^2+\lambda} - \frac{1}{c^2+\lambda}\right)\frac{\lambda b\, d\lambda}{(b^2+\lambda)\, P},$$

$$\frac{dQ}{dc} = \int_0^\infty \left(\frac{1}{a^2+\lambda} - \frac{3}{c^2+\lambda}\right)\frac{\lambda c\, d\lambda}{(c^2+\lambda)\, P}.$$

Therefore, when $a = b$,

$$a\frac{dQ}{db} - c\frac{dQ}{dc} = \int_0^\infty \frac{\left[2\lambda^2 c^2 + \lambda\left\{8a^2c^2 - (a^2+c^2)^2\right\} + 2a^4c^2\right]\lambda\, d\lambda}{(a^2+\lambda)^2\,(c^2+\lambda)^2\, P}.$$

The numerator of this expression can never become negative, since it is positive when $\lambda = 0$, and the roots of the equation for λ, obtained by equating it to zero, are imaginary. Hence

$$a\frac{dQ}{db} > c\frac{dQ}{dc}.$$

Now dQ/db, and therefore dQ/dc, are negative; it therefore follows from (11) that E_{aa} will be positive if $E_{aa} - E_{ab}$ is positive. Now

$$a(E_{aa} - E_{ab}) = \frac{dQ}{da} - \frac{dQ}{db} + \frac{2Q}{a},$$

$$\frac{dQ}{da} = 2Aa + a^2\frac{dA}{da} - c^2\frac{dC}{da},$$

$$\frac{dQ}{db} = a^2\frac{dA}{db} - c^2\frac{dC}{da};$$

and

$$\frac{dA}{da} = -3a\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3\,(c^2+\lambda)^{\frac{1}{2}}} = 3\frac{dQ}{d}.$$

Therefore the condition becomes

$$2Aa^2 - Cc^2 - a^4 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3 (c^2+\lambda)^{\frac{1}{2}}} > 0.$$

Now, if e be the excentricity of the meridian section of the spheroid (the factor $2\pi\rho a^2 c$ being omitted),

$$A = \frac{1}{a^3 e^3} \{\sin^{-1} e - e\sqrt{1-e^2}\},$$

$$C = \frac{2}{a^3 e^3} \left\{ \frac{e}{\sqrt{1-e^2}} - \sin^{-1} e \right\},$$

$$\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3 (c^2+\lambda)^{\frac{1}{2}}} = \frac{1}{a^5 e^5} \left[\tfrac{3}{4} \{\sin^{-1} e - e\sqrt{1-e^2}\} - \tfrac{1}{2} e^3 \sqrt{1-e^2} \right],$$

whence the condition becomes

$$\sin^{-1} e \left\{ 2 - e^2 - \frac{3}{8e^2} \right\} + \sqrt{1-e^2} \left(\frac{3}{8e} - \frac{5e}{8} \right) > 0.$$

The above expression is positive for all values of e lying between 0 and 1, both inclusive, whence Maclaurin's spheroid is stable.

6. In the last edition of Thomson and Tait's "Natural Philosophy," Vol. I. Part II., pages 329 and 333, it is stated that Maclaurin's spheroid is unstable if the excentricity exceeds that of the spheroid which coalesces with the limiting Jacobian ellipsoid; that is, when

$$e > \sin 54°21'27'' \text{ or } \cdot 8127.$$

Unfortunately, no proof of this statement is given, and if the analysis of the present paper be correct, it follows that the disturbance which produces instability, cannot be what I have called an ellipsoidal disturbance, but must be such that the boundary in the beginning of the disturbed motion must be a surface which is not an ellipsoid but one slightly differing therefrom.

Apparently, however, Poincaré does not agree with Sir W. Thomson's result, for on p. 379 of an elaborate memoir* he says :—"Les ellipsoïdes de révolution, qui sont plus aplatis que celui qui est en même temps un ellipsoïde de Jacobi, mais dont l'aplatissement reste inférieur à une certaine limite, sont stables si le fluide est parfaitement dépourvu de viscosité ; ils ne sont plus si le fluide est visqueux et si

* "Sur l'Équilibre d'une Masse Fluide animée d'un mouvement de Rotation. *Acta Mathematica*, Vol. vii., p. 259.

peu qu'il soit." It should be noticed that Poincaré considers a disturbance of a much more general character than I have done.

Independently of any mathematical analysis, it seems almost certain that Maclaurin's spheroid must become unstable when the excentricity exceeds a certain limit. For, suppose the spheroid is shaped like an orange, and let a small jet of air be directed for a short time to some point on its surface. The effect of this will be to cause waves to diverge from the point of application of the jet, which will travel over the surface, but the motion will not be otherwise affected. But, if the shape of the spheroid resembles that of a thin disc, the probable effect of the jet will be to cause the liquid to curl itself up, or possibly to break up into two or more detached portions, and the motion will be thoroughly unstable. It appears to me that the disturbed motion might be investigated by a more simple process than has been employed by Poincaré, by assuming that in the beginning of the disturbed motion the equation of the free surface is of the form

$$\nu = \gamma + \Sigma A_{mn} P_n(\mu) \cos m\phi,$$

where ν, μ are elliptic coordinates of a point, and γ is the value of ν in steady motion, and proceeding upon the lines of my former paper; but any investigation of this kind must form the subject of a future communication.

7. The motion of a liquid spheroid which rotates about its axis of figure has been fully discussed by Dirichlet, whose equations have been deduced on p. 261 of my former paper, the density of the liquid being there taken as unity. From [E. 22] it follows that, if the rotating liquid is inclosed in a case (which may be either a prolate or an oblate spheroid), and the case is removed, it will be impossible for the free surface to retain the spheroidal form unless *initially* $\zeta^2/2\pi\rho < 1$, where ρ is the density; and that, if this condition is not satisfied, the free surface during the subsequent motion will assume some other revolutional form. Also, if $2c$ be the length of the axis of figure, and the free surface is initially spheroidal, it will cease to be so, if at any period of the subsequent motion

$$\frac{\zeta^2}{2\pi} > 1 + \frac{3\dot{c}^2}{8\pi\rho c^2}.$$

The following errata in my former paper should be noticed :—

In *equation (1), the second member should be* = 0.

Page 259, line 9, the following expression should be added to the right-hand side, viz.:

$$\dot{a}^2/a^2-(w-w')\left\{(a-b)\,w-(a+b)\,w'\right\}/a+(v-v')\left\{(c-a)\,v+(c+a)\,v'\right\}/a.$$

Page 261, line 6, *read* $w=w'=\frac{1}{2}\zeta$.

,, ,, equation (23), *read* $3\dot{a}^2/a^4$ *instead of* $3\dot{a}^2/4\dot{a}^4$.

Geometry of the Quartic. By R. Russell, M.A.

[*Read Nov. 10th*, 1887.]

The system of points, the properties of which I intend to discuss, arose from an attempt to interpret geometrically the sextic covariant of a quartic.

I consider a quartic whose coefficients may be any whatever, real or imaginary. Its roots are of the form a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4. These are represented as follows:—Assume any two rectangular axes and take the point whose coordinates are a_1, b_1; that point may be considered to represent the complex quantity a_1+ib_1. We see, therefore, that the four roots of a quartic may be represented by four points in a plane.

I. If α, β, γ, δ be the four roots of the quartic, the factors of the sextic covariant are the numerators of

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta};\quad \frac{1}{z-\gamma}+\frac{1}{z-\alpha}-\frac{1}{z-\beta}-\frac{1}{z-\delta};$$
$$\frac{1}{z-\alpha}+\frac{1}{z-\beta}-\frac{1}{z-\gamma}-\frac{1}{z-\delta};$$

z of course denoting a quantity $x+iy$.

Let us consider the roots of the quadratic

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}=0;$$

and let z represent a root of it. Now $z-\alpha$ defines the length and direction of the line joining z and α, and therefore $\dfrac{1}{z-\alpha}$ defines a line whose length is the reciprocal of that line, and whose direction is the re-

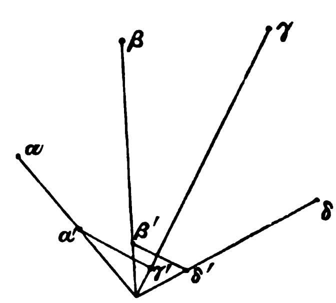

Fig. 1.

flexion of it with respect to axis of x. Consequently, any property which holds for the points

$$\frac{1}{z-\alpha}\cdot\frac{1}{z-\beta}\cdot\frac{1}{z-\gamma}\cdot\frac{1}{z-\delta},$$

will equally hold if these points are taken in the directions $z\alpha$, $z\beta$, $z\gamma$, $z\delta$ respectively. We may, therefore, represent the four quantities

$$\frac{1}{z-\alpha},\ \frac{1}{z-\beta},\ \frac{1}{z-\gamma},\ \frac{1}{z-\delta}$$

by the points α', β', γ', δ'. The above quadratic reduces to $\gamma'-\alpha'=\delta'-\beta'$, showing that the lines γ', α' and δ', β' are equal and parallel. Hence z is such a point that the quadrilateral has inverted into a parallelogram. *If therefore the roots of a quartic be represented by four points in a plane, the roots of the sextic covariant are those six points (all real) from which as origin the quadrilateral inverts into a parallelogram.*

The six points are arranged as follows:—I_1 and J_1 are the points from which the quadrilateral inverts into a parallelogram, the extremities of whose diagonals are the inverses of β, γ and α, δ respectively; I_2, J_2 those from which the extremities of the diagonals are the inverses of γ, α and β, δ; and I_3, J_3 those from which the extremities of the diagonals are the inverses of α, β and γ, δ.

II. If two quadratics $a_1z^2+2b_1z+c_1$, $a_2z^2+2b_2z+c_2$ be connected harmonically (*i.e.*, $a_1c_2+a_2c_1-2b_1b_2=0$), then the two pairs of points representing their roots are concyclic and harmonic. The proof is obvious.

The following properties of the quadratic factors of the sextic covariant are well known.

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}$$

is connected harmonically with $z-\beta\,.\,z-\gamma$, and $z-\alpha\,.\,z-\delta$. Similarly with respect to the other two quadratic factors. Besides, the three factors themselves are, two and two, harmonic. Hence $\beta\gamma I_1J_1$, $\alpha\delta I_1J_1$ are both concyclic and harmonic, and similar properties hold with regard to I_2J_2 and I_3J_3. Also $I_1J_1I_2J_2$, $I_2J_2I_3J_3$, and $I_3J_3I_1J_1$ are concyclic and harmonic. Three circles so related are of course orthogonal, two and two. *Hence the six points "I J" are the points of intersection of three mutually orthogonal circles,* which we shall call the "IJ" circles.

Before giving a geometrical construction for these points, I shall establish the following proposition, and give the interpretation of it.

III. There are three homographic transformations which leave the roots of a quartic unchanged.

The most general homographic relation connecting z and ζ is

$$lz\zeta + mz + n\zeta + p = 0.$$

If, when $z = \beta,\ \gamma,\ \alpha,\ \delta,\ \zeta = \gamma,\ \beta,\ \delta,\ \alpha$, respectively, then we have

$m = n$, and
$$l\beta\gamma + m(\beta+\gamma) + p = 0,$$
$$l\alpha\delta + m(\alpha+\delta) + p = 0,$$

therefore
$$\frac{l}{\beta+\gamma-\alpha-\delta} = \frac{m \text{ or } n}{\alpha\delta-\beta\gamma} = \frac{p}{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)},$$

thus determining the homographic relation.

If in the above we put $z = \zeta$, we get a quadratic $lz^2 + 2mz + p = 0$, which gives two quantities absolutely unaltered by the transformation. This quadratic is equivalent to

$$\frac{1}{z-\beta} + \frac{1}{z-\gamma} - \frac{1}{z-\alpha} - \frac{1}{z-\delta} = 0.$$

IV. What is the geometrical significance of this?

Since $m = n$, the homographic relation is obviously equivalent to $(z-\theta)(\zeta-\theta) = \phi^2$, where θ and ϕ are constants. Now, denoting z, ζ, and θ by three points in the plane, it is obvious that the product of the distances $z\theta$ and $\zeta\theta$ is equal to the modulus of ϕ^2, and that their directions are equally inclined to direction of ϕ, and therefore the double points IJ are situated on a line through θ in the direction of ϕ, and such that $\theta I^2 = \theta J^2 = z\theta \,.\, \zeta\theta$.

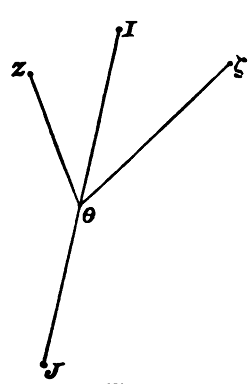

Fig. 2.

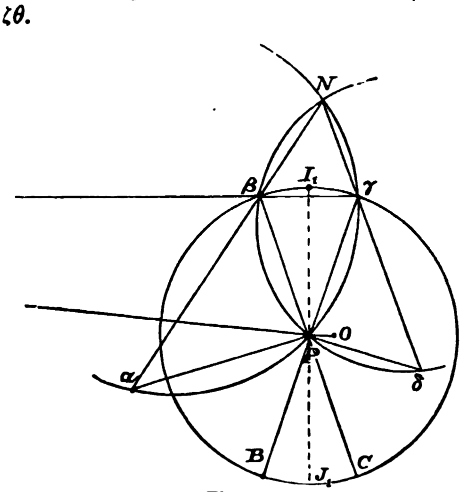

Fig. 3.

We are now in a position to give a geometrical construction for the "IJ" points. Let P be the point denoting the quantity θ in the last section, and I_1J_1 the direction of ϕ. When $z = \beta.\gamma.\alpha.\delta$, then $\zeta = \gamma.\beta.\delta.\alpha$; therefore, from the last section, the lines $P\beta$ and $P\gamma$ are equally inclined to I_1J_1, and $P\beta.P\gamma = PI^2$. Exactly the same statement holds with regard to α and δ. We see, therefore, that the triangles $\alpha P\beta$ and $\gamma P\delta$ are similar, and therefore the angle $P\gamma\delta = \beta\alpha P$, and $\alpha\beta P = \gamma\delta P$. *Hence P is determined as follows*:—Produce $\alpha\beta$ and $\gamma\delta$ to meet at N, and describe circles round $\alpha\gamma N$ and $\delta\beta N$; then P is the intersection of these circles.

I_1, J_1 lie on the internal bisector of the angle $\beta P\gamma$, and PI_1 or PJ_1 is a mean proportional between $P\beta$ and $P\gamma$ or between $P\alpha$ and $P\delta$.

I shall in what follows discuss the problem from a geometrical point of view.

V. It is obvious, from Fig. 3, that if we invert with respect to P as origin, and a circle whose radius is PI_1 or PJ_1, we obtain a quadrilateral similar and equal to the given one, and they are reflexions with respect to the line I_1J_1. There are, of course, two other points, Q and R, which possess the same property, and PQ and PR are equally inclined to I_1J_1, and such that $PQ.PR = PI_1^2$.

We shall now see that I_1, J_1 possess the property of inverting the quadrilateral into a parallelogram.

Produce βP and γP so that $PB = P\beta$ and $PC = P\gamma$; then, since

$$PI^2 = PJ_1^2 = P\beta.P\gamma = P\beta.PC = P\gamma.PB,$$

a circle passes through β, γ, I_1, J_1, B, C. Draw PO perpendicular to I_1J_1. Since PI_1 and PO are the internal and external bisectors of the angle $\beta P\gamma$, therefore they meet the line $\beta\gamma$ in two points harmonic conjugates with respect to β and γ, and therefore the lines I_1J_1 and $\beta\gamma$ are conjugates; *i.e.*, each passes through the pole of the other, and therefore the four points β, γ, I_1, J_1 are concyclic and harmonic Similarly α, δ, I_1, J_1 are concyclic and harmonic. Now invert with respect to J_1.

The two circles invert into two lines, and in each line I_1' is the harmonic conjugate of a point at infinity (inverse of J_1) with respect to $\beta'\gamma'$ and $\alpha'\delta'$ respectively, that is, I_1' is simultaneously the middle point of $\beta'\gamma'$ and $\alpha'\delta'$. We thus have a parallelogram.

VI. The triangle PQR is a new *fundamental triangle* related to the quadrilateral. The "IJ" points lie on the internal bisectors of the

vertical angles and at distances from the vertices in each case, which are mean proportionals between the conterminous sides.

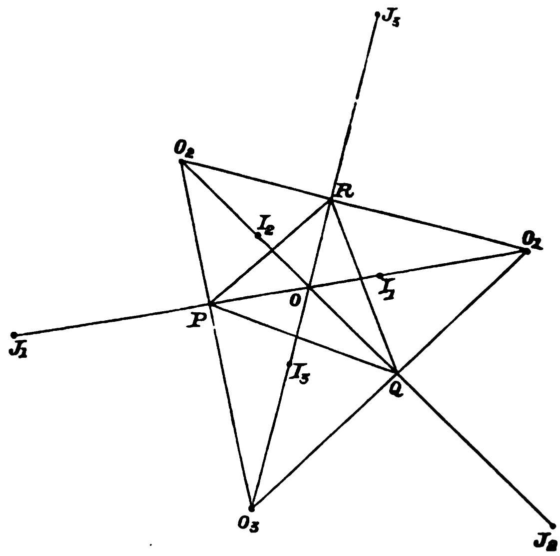

Fig. 4.

Now, having constructed the triangle PQR, let O_1, O_2, O_3, O be the centres of circles touching the sides; then, if we describe circles on O_2O_3, O_3O_1, O_1O_2 as diameters, these circles meet the internal bisectors in the "IJ" points.

VII. The circles $I_2J_2I_3J_3$, $I_3J_3I_1J_1$, $I_1J_1I_2J_2$ are mutually orthogonal, and their centres are at O_1, O_2, O_3,

$$I_1O\,.\,J_1O = PI_1^2 - PO^2 = PQ\,.\,PR - PO^2$$

$$= PO_2\,.\,PO_3 - PO^2 = PO_1\,.\,PO - PO^2 = PO\,.\,OO_1.$$

Hence $$I_1O\,.\,J_1O = I_2O\,.\,J_2O = I_3O\,.\,J_3O,$$

therefore $I_2J_2I_3J_3$ lie on a circle, and, since I_3J_3 and I_2J_2 are bisected at right angles by RO_1 and QO_1 respectively, therefore O_1 is the centre of that circle.

The "IJ" circles having O_1 and O_2 as centres are orthogonal, for their radii are O_1J_3 and O_2J_3, and the angle $O_1J_3O_2$ is right; therefore, etc.

VIII. If we invert the four original points α, β, γ, δ and the "IJ" points with respect to any circle, the points still retain their property.

For, since $\beta\gamma I_1 J_1$ and $\alpha\delta I_1 J_1$ are concyclic and harmonic, their inverses will be so also, and therefore the inverses of the "IJ" points will be "IJ" points of the inverses of the original points.

IX. Given the "IJ" and any one of the four original points—δ, suppose—find the remaining points.

$\alpha\delta I_1 J_1$, $\beta\delta I_2 J_2$, $\gamma\delta I_3 J_3$ are concyclic and harmonic, and therefore, when δ is given, α, β, and γ are *singly determinate.*

I may remark that this shows that, if we are given a sextic which is the sextic covariant of some quartic unknown, then that quartic is of the form $lU+mV$, where U and V are any two quartics satisfying the condition.

X. We shall now consider the properties of another system of *four points* related to the original system in a most remarkable manner. *These are the points from which if we invert, the original four invert into a triangle and its orthocentre.*

The three "IJ" circles have a common orthogonal circle whose centre is at O, and the negative square of whose radius is the value of

$$OP\,.\,OO_1 = OQ\,.\,OO_2 = OR\,.\,OO_3.$$

Let D be the inverse of one of the original points δ with respect to this circle, and let us invert the whole figure with respect to a circle having D as its centre.

Let α', β', γ', δ' be the inverses of α, β, γ, δ with respect to D; then, since D and δ are inverse points with respect to the circle, orthogonal to the "IJ" circles, therefore, from (VIII.), in the inverted figure, δ' will be the centre of the common orthogonal circle; and since $\alpha'\delta' I_1' J_1'$, $\beta'\delta' I_2' J_2'$, $\gamma'\delta' I_3' J_3'$ are collinear and harmonic, therefore α', β', γ' are situated at the centres of the "$I'J'$" circles. This will be obvious if we consider Figure 4. The quadrilateral has therefore inverted into a triangle and its orthocentre.

If therefore, we take the inverses of the four original points with respect to the circle, orthogonal to the "IJ" circles, we obtain four new points from which as origins the quadrilateral inverts into a triangle and its ortho-centre. Denote these four points by the letters A, B, C, D.

The four points A, B, C, D have the same "IJ" points as α, β, γ, δ. This needs no proof.

XI. I shall next prove that A, B, C, D are the inverses of δ, γ, β, a respectively with respect to circle going through $I_2J_2I_3J_3$. I use the same letters as in Fig. 4.

a, δ, I_1, J_1 are concyclic and harmonic. Invert with respect to circle $I_2J_2I_3J_3$ (centre O_1), then A', D', I_1, J_1 are harmonic. Hence I_1J_1 is common segment of harmonic section of $a\delta$, $A'D'$; therefore $A'a$, $D'\delta$ must intersect on that line at a point O which is the harmonic conjugate of O_1 with respect to I_1J_1. But that is exactly the point O in Fig. 4. Hence the points A', D', in Fig. 5, are the same as the points A, D in (X.).

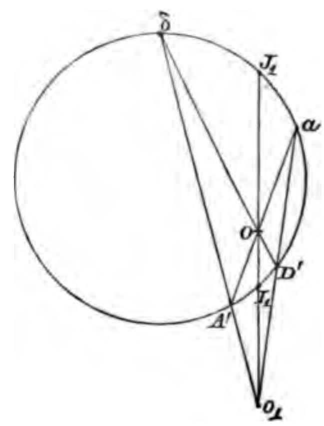

Fig. 5.

Thus we see that A, B, C, D are the inverses of

δ, γ, β, a with respect to circle $I_2J_2I_3J_3$,

γ, δ, a, β with respect to circle $I_3J_3I_1J_1$,

β, a, δ, γ with respect to circle $I_1J_1I_2J_2$,

a, β, γ, δ with respect to common orthogonal circle;

and thus they form two quadrilaterals simultaneously in perspective from four different centres.

I never recollect having come across two quadrilaterals so related before.

XII. If the roots of a cubic be represented by three points in a plane, the roots of its Hessian may be represented by the two points from which the triangle by inversion becomes equilateral.

Let a, β, γ be the roots of the cubic, then

$$\frac{1}{z-a}+\frac{\omega}{z-\beta}+\frac{\omega^2}{z-\gamma} \quad \text{and} \quad \frac{1}{z-a}+\frac{\omega^2}{z-\beta}+\frac{\omega}{z-\gamma}$$

are the factors of its Hessian. We have, therefore, to determine the property of a point in the plane satisfying the condition

$$\frac{1}{z-a}+\frac{\omega}{z-\beta}+\frac{\omega^2}{z-\gamma}=0, \text{ where } \omega^3=1.$$

This may be written

$$\frac{1}{z-a}-\frac{1}{z-\gamma}+\omega\left(\frac{1}{z-\beta}-\frac{1}{z-\gamma}\right)=0,$$

showing that the lines joining $\dfrac{1}{z-\alpha}$, $\dfrac{1}{z-\beta}$, $\dfrac{1}{z-\gamma}$ are inclined at angles of 60°.

XIII. If the PQR triangle be equilateral, then, P, Q, R denoting the vertices (considered as complex quantities), we have

$$\overline{Q-R}^2+\overline{R-P}^2+\overline{P-Q}^2=0.$$

But $$P=\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\quad Q=\frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\quad R=\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta},$$

and therefore the above condition amounts to

$$(\beta-\gamma)^2(\alpha-\delta)^2(\beta+\gamma-\alpha-\delta)^4+(\gamma-\alpha)^2(\beta-\delta)^2(\gamma+\alpha-\beta-\delta)^4$$
$$+(\alpha-\beta)^2(\gamma-\delta)^2(\alpha+\beta-\gamma-\delta)^4=0$$

or denoting the roots of $4a^3t^3-Iat+J=0$, by λ,μ,ν,

$$\left(\lambda-\frac{H}{a^2}\right)^2(\mu-\nu)^2+\left(\mu-\frac{H}{a^2}\right)^2(\nu-\lambda)^2+\left(\nu-\frac{H}{a^2}\right)^2(\lambda-\mu)^2=0,$$

or $$a^2I^2+36aJH+12H^2I=0,$$

where $$H\equiv ac-b^2.$$

We see therefore that, if we invert from any of the eight-point roots of $12H^2I-36HUJ+U^2I^2=0$, the "PQR triangle" of the new quartic is equilateral. We see also, from the way in which the "IJ" points are obtained from the "PQR triangle," that $I_1I_2I_3$ and $J_1J_2J_3$ form two equilateral triangles symmetrically arranged, the centre of perspective being their common centre. The arrangement is as in Fig. 6. We may obtain these eight points as follows:— Taking one root or point from each factor of the sextic covariant, we obtain two triangles or cubics. These two cubics have a common Hessian whose roots are two roots of the above equation

$$12H^2I-36HUJ+U^2I^2=0.$$

There are obviously four ways of thus choosing the cubics or triangles, and in each case they will have a common Hessian, thus giving rise to the eight points mentioned. Or, again:—

In Fig. 6 the circles round $I_1I_2I_3$ and $J_1J_2J_3$ are concentric. We must therefore, in the original figure, have inverted from one of the limiting points of the circles round $I_1I_2I_3$ and $J_1J_2J_3$. But there are four ways of thus describing circles, and we arrive, as before, at these same eight points.

The two quartics represented by the expression

$$12H^2I-36HUJ+U^2I^2$$

are the quartics of the system $\alpha U+\beta H$ whose "I" vanishes. In Fig. 6 they are represented by the points P, Q, R, ∞, and $O_1O_2O_3O$.

There is no difficulty in proving that O_1 and P are the limiting points of the circles $I_2I_3J_1$ and $J_2J_3I_1$.

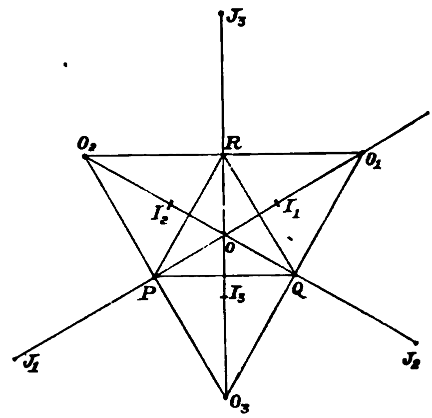

Fig. 6.

In the general figure, since the "I" of each of the above quartics vanishes, the four points denoting the roots of either quartic are such that, if from one point we invert, the remaining three become vertices of an equilateral triangle.

An inspection of Fig. 6 will also show that the four points denoting the roots of either quartic are the inverses with respect to the "IJ circles" of the points denoting roots of the other.

XIV. This again leads to a *new canonical form for the quartic and sextic covariant*, which we proceed to find.

Taking origin at O and axis of x along I_1J_1 we have, on denoting the length of the side of triangle PQR by A,

$$OI_1 = a\left(1-\frac{1}{\sqrt{3}}\right),\quad OJ_1 = a\left(1+\frac{1}{\sqrt{3}}\right);$$

therefore

$$I_1 = a\left(1-\frac{1}{\sqrt{3}}\right),\qquad J_1 = -a\left(1+\frac{1}{\sqrt{3}}\right),$$

$$I_2 = a\left(1-\frac{1}{\sqrt{3}}\right)e^{\frac{2}{3}i\pi},\quad J_2 = -a\left(1+\frac{1}{\sqrt{3}}\right)e^{\frac{2}{3}i\pi},$$

$$I_3 = a\left(1-\frac{1}{\sqrt{3}}\right)e^{\frac{4}{3}i\pi},\quad J_3 = -a\left(1+\frac{1}{\sqrt{3}}\right)e^{\frac{4}{3}i\pi};$$

therefore

$$(z-I_1)(z-J_1) = z^2+\frac{2az}{\sqrt{3}}-\frac{2a^2}{3},$$

$$(z-I_2)(z-J_2) = z^2+\frac{2az}{\sqrt{3}}\,\omega-\frac{2a^2}{3}\,\omega^2,\ \text{ where } \omega^3=1,$$

$$(z-I_3)(z-J_3) = z^2+\frac{2az}{\sqrt{3}}\,\omega^2-\frac{2a^2}{3}\,\omega.$$

These are the factors of the sextic covariant. We may obviously put $a = \dfrac{\sqrt{3}}{\sqrt{2}}$, when the factors become

$$z^2+z\sqrt{2}-1,\quad z^2+z\sqrt{2}\,\omega-\omega^2,\quad z^2+z\sqrt{2}\,\omega^2-\omega;$$

and therefore the sextic covariant is

$$z^6+5\sqrt{2}\,z^3-1=0.$$

The most general form for the quartic corresponding to this form is

$$\lambda\omega\,(z^2+z\sqrt{2}\,\omega-\omega^2)^2+\mu\omega^2\,(z^2+z\sqrt{2}\,\omega^2-\omega)^2,$$

or

$$U\equiv(\lambda\omega+\mu\omega^2)\,z^4+2\sqrt{2}\,(\lambda\omega^2+\mu\omega)\,z^3-2\sqrt{2}(\lambda\omega+\mu\omega^2)\,z+(\lambda\omega^2+\mu\omega)$$

$$-2H\equiv(\lambda\omega^2+\mu\omega)^2z^4+2\sqrt{2}(\lambda\omega+\mu\omega^2)^2z^3-2\sqrt{2}(\lambda\omega^2+\mu\omega)^2z+(\lambda\omega+\mu\omega^2)^2.$$

XV. I have not, as yet, succeeded in obtaining as neat a geometrical interpretation of the Hessian as might be desired. The following is the simplest method I can think of for determining the points denoting the Hessian, being given those which represent the quartic.

1. Given the "PQR triangle" and the centre of gravity of the four points representing roots of a quartic, determine the quartic.

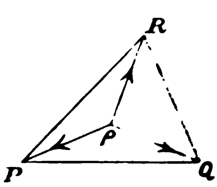

Fig. 7.

$$P=\frac{\beta\gamma-a\delta}{\beta+\gamma-a-\delta},\quad Q=\frac{\gamma a-\beta\delta}{\gamma+a-\beta-\delta},$$

$$R=\frac{a\beta-\gamma\delta}{a+\beta-\gamma-\delta},$$

and $$\rho=\tfrac{1}{4}\,(a+\beta+\gamma+\delta).$$

There is no difficulty in proving that

$$\frac{\beta+\gamma-a-\delta}{4}=\sqrt{Q-\rho\,.\,R-\rho},$$

$$\frac{\gamma+a-\beta-\delta}{4}=\sqrt{R-\rho\,.\,P-\rho},$$

$$\frac{a+\beta-\gamma-\delta}{4}=\sqrt{P-\rho\,.\,Q-\rho}.$$

But $\dfrac{\beta+\gamma-a-\delta}{4},\quad \dfrac{\gamma+a-\beta-\delta}{4},\quad \dfrac{a+\beta-\gamma-\delta}{4}$

represent the directions and half the lengths of the lines joining the middle points of the opposite connectors of the points α, β, γ, δ. These pass through the centre of gravity.

Hence, if from ρ (Fig. 7) we draw lines bisecting internally the angles $Q\rho R$, $R\rho P$, $P\rho Q$ and on each side of ρ take off mean proportionals between the conterminous sides, we have the six middle points of the opposite connectors of the points α, β, γ, δ. The points α, β, γ, δ are then found by completing the parallelograms.

This leads at once to a construction for the Hessian. For, if ρ' denote centre of gravity of points representing the Hessian, we have

$$\rho'-\rho = -\frac{ad-bc}{2(ac-b^2)}+\frac{2b}{a} = -\frac{a^2d-3abc+2b^3}{2a(ac-b^2)}.$$

Expressing the quantity on right-hand side in terms of $P-\rho$, $Q-\rho$, $R-\rho$, this easily reduces to

$$\frac{3}{\rho'-\rho} = \frac{1}{P-\rho}+\frac{1}{Q-\rho}+\frac{1}{R-\rho}, \text{ or } \frac{P-\rho'}{P-\rho}+\frac{Q-\rho'}{Q-\rho}+\frac{R-\rho'}{R-\rho} = 0,$$

showing that, if we invert from ρ (centre of gravity of original quartic), then ρ' inverts into centre of gravity of the triangle $P'Q'R'$. Hence, if we invert round ρ, the centre of gravity of the Hessian is the inverse of the centre of gravity of the triangle $P'Q'R'$. The Hessian is then constructed as in the preceding.

XVI. I shall finish the present paper by proving another property of the "IJ" points. It is (denoting $z-\alpha$, $z-\beta$, $z-\gamma$, $z-\delta$ by U_z) that

$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\gamma} \frac{\partial z}{\sqrt{U_z}}, \qquad \int_\alpha^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\delta} \frac{\partial z}{\sqrt{U_z}},$$

so that I or J is the place at which the result of integration between two of the singular points (roots of quartic) is bisected.

Transforming the element $\frac{\partial z}{\sqrt{U_z}}$ by means of the homographic relation in (III.), we obtain without any difficulty

$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = -\int_\gamma^{I_1 \text{ or } J_1} \frac{\partial \zeta}{\sqrt{U_\zeta}},$$

and
$$\int_\alpha^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = -\int_\delta^{I_1 \text{ or } J_1} \frac{\partial \zeta}{\sqrt{U_\zeta}};$$

therefore
$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\gamma} \frac{\partial \zeta}{\sqrt{U_\zeta}},$$

and $$\int_0^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{1} \frac{\partial \zeta}{\sqrt{U_\zeta}}, \text{ proving the proposition.}$$

Of course, similar properties hold with regard to I_2, J_2, I_3, J_3.

On the Stability or Instability of certain Fluid Motions, II. *By* Lord RAYLEIGH, Professor of Natural Philosophy in the Royal Institution.

[*Read Nov.* 10*th*, 1887.]

As the question of the stability, or otherwise, of fluid motions is attracting attention in consequence of Sir W. Thomson's recent work, I think it advisable to point out an error in the solution which I gave some years ago* of one of the problems relating to this subject; and I will take the opportunity to treat the problem with greater generality.

In the steady laminated motion, the velocity (U) is a function of y only. In the disturbed motion $U+u$, v, the small quantities u, v are supposed to be periodic functions of x, proportional to e^{ikx}, and, as dependent upon the time, to be proportional to e^{int}, where n is a constant, real or imaginary. Under these circumstances the equation determining v (51) is

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right) - \frac{d^2U}{dy^2}v = 0 \quad\text{..................(1).}$$

The vorticity (Z) of the steady motion is $\frac{1}{2}\frac{dU}{dy}$. If throughout any layer Z be constant, d^2U/dy^2 vanishes, and wherever $n+kU$ does not also vanish

$$\frac{d^2v}{dy^2}-k^2v = 0 \quad\text{..............................(2),}$$

or

$$v = Ae^{ky}+Be^{-ky} \quad\text{............................ (3).}$$

If there are several layers in each of which Z is constant, the various solutions of the form (3) are to be fitted together, the arbitrary constants being so chosen as to satisfy certain boundary conditions. The first of these conditions is evidently

$$\Delta v = 0 \quad\text{..................................(4).}$$

* *Math. Soc. Proc.* XI., p. 57, 1880.

The second may be obtained by integrating (1) across the boundary. Thus

$$\left(\frac{n}{k}+U\right).\Delta\left(\frac{dv}{dy}\right)-\Delta\left(\frac{dU}{dy}\right).v=0 \quad\text{..............(5).}$$

At a fixed wall $v = 0$.

In the special problem to which attention is here directed, the laminated motion is supposed to take place between two fixed walls, at $y = 0$ and $y = b_1+b'+b_2$; and the vorticity is supposed to be constant throughout each of the three layers bounded by

$$y = 0, \qquad y = b_1;$$
$$y = b_1, \qquad y = b_1+b';$$
$$y = b_1+b', \quad y = b_1+b'+b_2.$$

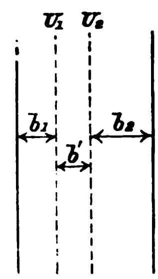

Fig. 1.

There are thus two internal surfaces at $y = b_1$, $y = b_1+b'$, where the vorticity changes. The values of U at these surfaces may be denoted by U_1, U_2.

In conformity with (4) and with the condition that $v = 0$ when $y = 0$, we may take in the first layer

$$v = v_1 = \sinh ky \quad\text{.................(6)};$$

in the second layer

$$v = v_2 = v_1+M_1 \sinh k(y-b_1) \quad\text{........................(7)};$$

in the third layer

$$v = v_3 = v_2+M_2 \sinh k(y-b'-b_1) \quad\text{..................(8).}$$

The condition that $v = 0$, when $y = b_1+b'+b_2$, now gives

$$0 = M_2 \sinh kb_2+M_1 \sinh k(b_2+b')+\sinh k(b_2+b'+b_1) \quad\text{.........(9).}$$

We have still to express the two other conditions (5) at the surfaces of transition. At the first surface,

$$v = \sinh kb_1, \qquad \Delta\left(\frac{dv}{dy}\right) = kM_1;$$

at the second surface,

$$v = M_1 \sinh kb'+\sinh k(b_1+b'), \qquad \Delta\left(\frac{dv}{dy}\right) = kM_2.$$

If we denote the values of $\Delta\left(\dfrac{dU}{dy}\right)$ at the two surfaces respectively

by Δ_1, Δ_2, our conditions become

$$\left.\begin{aligned}(n+kU_1)\,M_1-\Delta_1\sinh kb_1&=0\\(n+kU_2)\,M_2-\Delta_2\left\{M_1\sinh kb'+\sinh k\,(b_1+b')\right\}&=0\end{aligned}\right\}\text{......(10)}.$$

By (9) and (10) the values of M_1, M_2, n are determined.

The equation for n is found by equating to zero the determinant

$$\begin{vmatrix}\sinh kb_2, & \sinh k\,(b_2+b'), & \sinh k\,(b_2+b'+b_1)\\ n+kU_2, & -\Delta_2\sinh kb', & -\Delta_2\sinh k\,(b_1+b')\\ 0, & n+kU_1, & -\Delta_1\sinh kb_1\end{vmatrix};$$

so that n has the values determined by the quadratic

$$An^2+Bn+C=0\ \text{..........................(11)},$$

where

$$A=\sinh k\,(b_2+b'+b_1)\ \text{..(12)},$$

$$\begin{aligned}B=k\,(U_1+U_2)\sinh k\,(b_2+b'+b_1)&+\Delta_2\sinh kb_2\sinh k\,(b_1+b')\\&+\Delta_1\sinh kb_1\sinh k\,(b_2+b')\ \text{..................(13)},\end{aligned}$$

$$\begin{aligned}C=k^2U_1U_2\sinh k\,(b_2+b'+b_1)&+k\,U_1\Delta_2\sinh kb_2\sinh k\,(b_1+b')\\&+k\,U_2\Delta_1\sinh kb_1\sinh k\,(b_2+b')\\&+\Delta_1\Delta_2\sinh kb_1\sinh kb_2\sinh kb'\text{..............(14)}.\end{aligned}$$

To find the character of the roots, we have to form the expression for B^2-4AC. Having regard to

$$\begin{aligned}&\sinh k\,(b_2+b')\sinh k\,(b_1+b')\\&\qquad-\sinh k\,(b_2+b'+b_1)\sinh kb'=\sinh kb_1\sinh kb_2,\end{aligned}$$

we find

$$\begin{aligned}B^2-4AC=&\left\{k\,(U_1-U_2)\sinh k\,(b_2+b'+b_1)\right.\\&\left.+\Delta_1\sinh kb_1\sinh k\,(b_2+b')-\Delta_2\sinh kb_2\sinh k\,(b_1+b')\right\}^2\\&+4\Delta_1\Delta_2\sinh^2 kb_1\sinh^2 kb_2\ \text{......................................(15)}.\end{aligned}$$

Hence, if Δ_1, Δ_2 have the same sign, that is, if the curve expressing U as a function of y be of one curvature throughout, B^2-4AC is positive, and the two values of n are real. Under these circumstances the disturbance is stable.

We will now suppose that the surfaces at which the vorticity changes are symmetrically situated, so that

$$b_1=b_2=b.$$

In this case we find

$$A = \sinh k\,(2b+b') \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

$$B = k\,(U_1+U_2)\sinh k\,(2b+b') + (\Delta_1+\Delta_2)\sinh kb \sinh k(b+b') \ldots(17),$$

$$C = k^2 U_1 U_2 \sinh k(2b+b') + k\,(U_1\Delta_2 + U_2\Delta_1)\sinh kb \sinh k(b+b')$$
$$+ \Delta_1\Delta_2 \sinh^2 kb \sinh kb' \quad\ldots\ldots\ldots\ldots\ldots(18),$$

$$B^2 - 4AC = \{k\,(U_1-U_2)\sinh k\,(2b+b')$$
$$+ (\Delta_1-\Delta_2)\sinh kb \sinh k\,(b+b')\}^2 + 4\Delta_1\Delta_2 \sinh^4 b \ldots\ldots(19).$$

Under this head there are two sub-cases which may be especially noted. The first is that in which the values of U are the same on both sides of the median plane, so that the middle layer is a region of constant velocity without vorticity, and the velocity curve is that shown in Fig. 2. We may suppose that $U = V$ in the middle layer, and that $U = 0$ at the walls, without loss of generality, since any constant velocity (U_0) superposed upon this system merely alters n by the corresponding quantity $-kU_0$, as is evident from (1). Thus

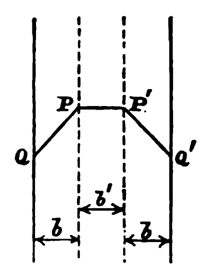

Fig. 2.

$$U_1 = U_2 = V, \quad \Delta_2 = \Delta_1 = \Delta = -\frac{V}{b};$$

and $$B^2 - 4AC = 4\Delta^2 \sinh^4 kb.$$

Hence $$n + kV = \frac{V}{b}\,\frac{\sinh kb \sinh k\,(b+b') \pm \sinh^2 kb}{\sinh k\,(2b+b')} \quad\ldots\ldots(20).$$

If the middle layer be absent $b' = 0$, and

$$n + kV = \frac{V}{b}\,\frac{2\sinh^2 kb}{\sinh 2kb} = \frac{V}{b}\tanh kb \quad\ldots\ldots\ldots\ldots(21),$$

in conformity with (44) of the former paper; but the more general result (20) does not agree with (46).

The other case which we shall consider is that in which the velocities U on the two sides of the median plane are opposite to one another; so that

$$U_1 = -U_2 = V, \quad \Delta_2 = -\Delta_1 = -\mu V \quad\ldots\ldots\ldots\ldots(22).$$

Here $B = 0$, and

$$C = -k^2V^2 \sinh k\,(2b+b') - 2k\mu V^2 \sinh kb \sinh k\,(b+b')$$
$$-\mu^2 V^2 \sinh^2 kb \sinh kb'.$$

Thus

$$\frac{n^2}{k^2V^2} = \frac{k^2 \sinh k\,(2b+b') + 2k\mu \sinh kb \sinh k\,(b+b') + \mu^2 \sinh^2 kb \sinh kb'}{k^2 \sinh k\,(2b+b')} \quad \ldots\ldots\ldots\ldots\ldots(23).$$

Here the two values of n are equal and opposite; and, since Δ_1, Δ_2 are of opposite signs, the question is open as to whether n is real or imaginary.

It is at once evident that n is real if μ be positive, that is, if Δ_1 and V are of the same sign, as in Fig. 3.

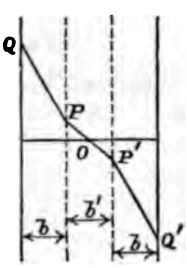

Fig. 3.

Even when μ is negative, n^2 is necessarily positive for great values of k, that is, for small wave-lengths. For we have ultimately, from (23),

$$n = \pm kV.$$

We will now inquire for what values of μ n^2 may be negative when k is very small, that is, when the wave-length is very great. Equating the numerator of (23) to zero, and expanding the hyperbolic sines, we get as a quadratic in μ,

$$\mu^2 b^2 b' + 2\mu b\,(b+b') + 2b + b' = 0,$$

whence

$$\mu = -\frac{1}{b}, \text{ or } -\frac{1}{b} - \frac{2}{b'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

When μ lies between these limits (and then only), n^2 is negative, and the disturbance (of great wave-length) increases exponentially with the time.

We may express these results by means of the velocity V_0 at the wall where $y = 0$. We have

$$V_0 = V\frac{b+\frac{1}{2}b'}{\frac{1}{2}b'} + \Delta_1 b = V\left(\frac{b+\frac{1}{2}b'}{\frac{1}{2}b'} + \mu b\right).$$

The limiting values of V_0 are therefore

$$\frac{bV}{\frac{1}{2}b'} \text{ and } 0.$$

The velocity curve corresponding to the first limit is shown in Fig. 4 by the line $QPOP'Q'$, the point Q being found by drawing a line AQ parallel to OP to meet the wall in Q. If $b' = 2b$, QP is parallel to OA, or the velocity is constant in each of the extreme layers.

At the second limit $V_0 = 0$, and the velocity-curve is that shown in Fig. 5.

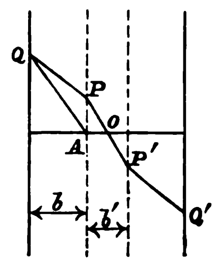

Fig. 4.

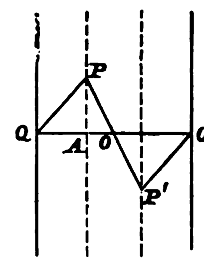

Fig. 5.

It is important to notice that motions represented by velocity-curves intermediate between these limits are unstable in a manner not possible to motions in which the velocity-curve, as in Fig. 2, is of one curvature throughout.

According to the first approximation, the motion of Fig. 5 is on the border-land between stability and instability for disturbances of great wave-length; but, if we pursue the calculation, we find that it is really unstable. Taking, in (23),

$$\mu = -\frac{1}{b} - \frac{2}{b'}$$

and writing for brevity $kb = x$, $kb' = x'$, we get

$$\frac{n^2}{k^2V^2} = \frac{\left\{\begin{array}{r} x^2x'^2 \sinh(2x+x') - 2xx'(2x+x') \sinh x \sinh(x+x') \\ +(2x+x')^2 \sinh^2 x \sinh x' \end{array}\right\}}{x^2x'^2 \sinh(2x+x')};$$

from which, on expanding the hyperbolic sines and retaining two terms, we get, after reduction,

$$\frac{n^2}{k^2V^2} = -\frac{x^2}{3} = -\frac{k^2b^2}{3} \quad\text{......................}(25),$$

indicating instability.

[January, 1888.* According to (23), we may always, with a prescribed wave-length, determine two values of μ (or V_0), V being regarded as given, between which n^2 will be negative, and the motion unstable. But, if these values of μ were imaginary, the result would

* This paragraph is re-written, and embodies an improvement suggested in a *report* communicated to me by the Secretary.

be of no significance in the present problem. We may, however, write (23) in the form

$$\frac{n^2}{k^2V^2} = \frac{\{\mu \sinh kb \sinh kb' + k \sinh k\,(b+b')\}^2 - k^2 \sinh^2 kb}{k^2 \sinh kb' \sinh k\,(2b+b')},$$

from which we see that, whatever be the value of k, it is possible so to determine μ that the disturbance shall be unstable. The condition is simply that μ must lie between the limits

$$-k\,\frac{\sinh k\,(b+b') \pm \sinh kb}{\sinh kb \sinh kb'},$$

or

$$-k\left[\coth kb + \begin{matrix}\coth\\ \tanh\end{matrix}\Big\} \frac{kb'}{2}\right] \quad\text{....................(26),}$$

in which the upper alternative corresponds to the superior limit to the *numerical value* of μ.

When k is very large, the limits are very great and very close. When k is small, they become

$$-\frac{1}{b} - \frac{2}{b'} \text{ and } -\frac{1}{b},$$

as has already been proved. As k increases from 0 to ∞, the numerical value of the upper limit increases continuously from $1/b+2/b'$ to ∞, and in like manner that of the inferior limit from $1/b$ to ∞. The motion therefore cannot be stable *for all values of* k, if μ (being negative) exceed numerically $1/b$. The final condition of complete stability is therefore that algebraically

$$\mu > \frac{-1}{b}.$$

In the transition case

$$V_0 = \left(\mu + \frac{1}{b} + \frac{2}{b'}\right) Vb = \frac{2Vb}{b'};$$

it is that represented in Fig. 4. If PQ be bent more downwards than is there shown, as for example in Fig. 5, the steady motion is certainly unstable.

It would be of interest, in some particular case of instability (such as that of Fig. 5), to calculate for what value of k the instability, measured by in, is greatest, and to ascertain the degree of this instability.]

Reverting to the general equations (11), (12), (13), (14), (15), let

us suppose that $\Delta_2 = 0$, amounting to the abolition of the corresponding surface of discontinuity. We get

$$B = k(U_1 + U_2)\sinh k(b_2 + b' + b_1) + \Delta_1 \sinh kb_1 \sinh k(b_2 + b'),$$

$$B^2 - 4AC = \{k(U_1 - U_2)\sinh k(b_2 + b' + b_1) + \Delta_1 \sinh kb_1 \sinh k(b_2 + b')\}^2;$$

so that

$$n = -kU_2 \quad \text{.............................}(27),$$

or

$$n = -kU_1 - \frac{\Delta_1 \sinh kb_1 \sinh k(b_2 + b')}{\sinh k(b_1 + b' + b_2)} \quad \text{...........}(28).$$

The latter is the general solution for two layers of constant vorticity of breadths b_1 and $b' + b_2$. An equivalent result may be obtained by supposing in (11), &c., that $b' = 0$, or that $b_1 = 0$.

The occurrence of (27) suggests that any value of $-kU$ is admissible as a value of n, and the meaning of this is apparent from (1). For, at the place where $n + kU = 0$, (2) need not be satisfied, or the arbitrary constants in (3) may change their values. It is evident that, with the prescribed values of n and k, a solution may be found satisfying the required conditions at the walls and at the surfaces where dU/dy changes value, as well as equation (4) at the plane where $n + kU = 0$. Equation (5) is there satisfied independently of the value of v. In this motion an additional vorticity is supposed to be communicated at the plane in question, and moves with the fluid at velocity U.

December 8*th*, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Captain Ignacio Beyens, Cadiz; Mr. W. B. Allcock, M.A., Fellow of Emmanuel College, Cambridge; and Mr. J. W. Mulcaster, Head-master of the High School, Consett, Durham, were elected members.

The Auditor (Mr. Basset) made his report. Upon the motion of Dr. Glaisher, seconded by Mr. Heppel, the Treasurer's report was then adopted.

The following communications were made:—

The Algebra of Linear Partial Differential Operators: Captain P. A. MacMahon, R.A.

On a Method in the Analysis of Ternary Forms: Mr. J. J. Walker, F.R.S.

Confocal Paraboloids: Mr. A. G. Greenhill, M.A.

Harmonic Decomposition of Functions and some Allied Expansions: Mr. A. R. Johnson, M.A.

Uni-Brocardal Triangles and their Inscribed Triangles: Mr. R. Tucker, M.A.

The following presents were received:—

"Educational Times," for December, 1887.

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Part II.

"Proceedings of the Canadian Institute," Third series, Vol. V., Fasc. No. 1, 1887.

"Acta Mathematica," X., 4.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 4 and 5.

"Journal für die reine und angewandte Mathematik," Band 102, Heft II.

"Journal für die reine und angewandte Mathematik," Inhalt und Namen-Verzeichniss der Bände 1—100 (1826—1887), 4to; Berlin, 1887.

"Beiblätter zu den Annalen der Physik und Chemie," Band XI., Stück 10.

"Nieuw Archief voor Wiskunde," Deel XIV., Stuk 1; Amsterdam, 1887.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXII., 2 et 3 Livraisons; Harlem, 1887.

"Bollettino delle Pubblicazioni Italiane, ricevute per diritto di Stampa," Num. 45 & 46.

"Memorias de la Sociedad Cientifica 'Antonio Alzate,'" Tomo I., Cuaderno, Núm. 4; Mexico, 1887.

"Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Neerlanden," door D. Bierens de Haan, Tweede Verzameling, 8vo (Niet in den Handel), 1887.

"Annales de la Faculté des Sciences de Toulouse," Tome I., Fasc. 1—4; Paris, 1887.

"The Journal of the College of Science," Imperial University, Japan, Vol. I., Part IV.; Tokio.

Harmonic Decomposition of Functions and some Allied Expansions

By A. R. Johnson, M.A.

[*Read December 8th*, 1887.]

The general method of procedure for the solution of Green's problem in the case of the sphere necessitates integrations over the surface of the sphere, or else, when the function of the coordinates that expresses the given surface potential is rational and integral, requires the decomposition of the function into surface harmonics by

the method of indeterminate coefficients. It is the object of this paper to furnish methods of solution that shall be free from these operations, when the expression for the surface potential is a rational and integral function of the coordinates. The solution is effected, firstly in a determinantal form, secondly in the form of a finite series. When the function that expresses the surface potential is unrestricted in form, the analogous method of solution requires an infinite series, which the paper gives if certain conditions are fulfilled. One of the chief results of the paper is a formula that enables us to separate at once any rational and integral function into its harmonic components at the surface of the sphere of reference.

1. Let $f_n(x, y, z)$ denote a rational integral and homogeneous function of degree n, and let V_s denote a solid harmonic of order s. Then it is well known that $f_n(x, y, z)$ can be expressed in the form

$$f_n(x, y, z) = V_n + \left(\frac{r}{a}\right)^2 V_{n-2} + \left(\frac{r}{a}\right)^4 V_{n-4} + \dots \qquad \dots\dots\dots(1),$$

where the last term is

$$\left(\frac{r}{a}\right)^n V_0 \quad \text{or} \quad \left(\frac{r}{a}\right)^{n-1} V_1,$$

according as n is even or odd. Performing the operation ∇^2 on this identity, once, twice, thrice, etc., and remarking that

$$\nabla^2 r^m V_n = m(m+2n+1)\, r^{m-2} V_n,$$

there result the relations

$$\left.\begin{aligned}
&a^2\nabla^2 f_n(x, y, z) \\
&= 2(2n-1)V_{n-2} + 4(2n-3)\left(\frac{r}{a}\right)^2 V_{n-4} + 6(2n-5)\left(\frac{r}{a}\right)^4 V_{n-6} + \dots, \\
&a^4\nabla^4 f_n(x, y, z) \\
&= 4.2(2n-3)(2n-5)V_{n-4} + 6.4(2n-5)(2n-7)\left(\frac{r}{a}\right)^2 V_{n-6} + \dots, \\
&a^6\nabla^6 f_n(x, y, z) \\
&= 6.4.2(2n-5)(2n-7)(2n-9)V_{n-6} \\
&\qquad + 8.6.4(2n-7)(2n-9)(2n-11)\left(\frac{r}{a}\right)^2 V_{n-8} + \dots,
\end{aligned}\right\} \qquad \dots\dots\dots\dots(2),$$

and so on, the last relation being

$$a^n\nabla^n f_n(x, y, z) = (n+1)\,n\,(n-1)(n-2)\dots V_0 = (n+1)!\,V_0,$$

or
$$a^{n-1}\nabla^{n-1}f_n(x, y, z)$$
$$= (n+2)(n-1)\,n\,(n-3)\ldots 5.2.V_1 = \frac{(n+2)\,n!}{3}\,V_1,$$
according as n is even or odd.

If the given potential at the surface of the sphere whose centre is the origin and whose radius is a, be $f_n(x, y, z)$, then putting $r = a$, in the relations (1) and (2), we have a system of relations which hold true at the surface of the sphere. Let the radius vector ρ to the point $\xi\eta\zeta$ meet the sphere at the point xyz. Then the potential at $\xi\eta\zeta$ within the sphere is

$$V = \left(\frac{\rho}{a}\right)^n V_n + \left(\frac{\rho}{a}\right)^{n-2} V_{n-2} + \ldots,$$

where in V_n, V_{n-2}, etc., $\quad r = a.$

Eliminating V_n, V_{n-2}, etc., between this relation and the results of putting $r = a$, in (1) and (2), we have V given by the determinantal equation

$$0 = \begin{vmatrix} V, & \left(\frac{\rho}{a}\right)^n, & \left(\frac{\rho}{a}\right)^{n-2}, & \left(\frac{\rho}{a}\right)^{n-4}, & \left(\frac{\rho}{a}\right)^{n-6}, & \ldots \\ f_n, & 1, & 1, & 1, & 1, & \ldots \\ a^2\nabla^2 f_n, & . & (2,1)(2n-1,1), & (4,1)(2n-3,1), & (6,1)(2n-5,1), & \ldots \\ a^4\nabla^4 f_n, & . & . & (4,2)(2n-3,2), & (6,2)(2n-5,2), & \ldots \\ a^6\nabla^6 f_n, & . & . & . & (6,3)(2n-5,3), & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

................(A),

the last row being

$$a^n\nabla^n f_n, ., ., \ldots, (n+1)! \quad\text{or}\quad a^{n-1}\nabla^{n-1} f_n, ., ., \ldots \frac{(n+2)\,n!}{3},$$

according as n is even or odd; and the last column

$$1,\ 1,\ (n+1)\,n,\ \ (n+1)\,n\,(n-1)(n-2),\ \text{etc.};$$

or $$\frac{\rho}{a},\ 1,\ (n+2)(n-1),\ \ (n+2)(n-1)\,n\,(n-3),\ \text{etc.};$$

according as n is even or odd.

In the determinant (A), (p, s) denotes $p\,(p-2)(p-4)\ldots$ to s factors. The expression for the potential in space external to the sphere

is obtained by writing $\left(\frac{a}{\rho}\right)^{s+1}$ for $\left(\frac{\rho}{a}\right)^s$ in (A). Consequently the required distribution is obtained by replacing V by $4\pi a\sigma$, and $\left(\frac{\rho}{a}\right)^s$ by $2s+1$, in (A), when σ will be the required density of distribution.

If the potential at the surface of the sphere be given as a rational and integral function of x, y, z of degree n, it may be expressed in the form f_n+f_{n-1} by inserting powers of $\frac{x^2+y^2+z^2}{a^2}$, and then the internal and external potentials and the surface density may be at once written down by a double application of the preceding results.

This method is useful for resolving $f_n(x, y, z)$ into surface harmonics, for the coefficient of $\left(\frac{\rho}{a}\right)^s$ in the expression obtained for V from (A) is V_s.

E.g., To resolve f_4 into surface harmonics ($r = 1$).

From (A),
$$0 = \begin{vmatrix} V, & \rho^4, & \rho^2, & 1 \\ f_4, & 1, & 1, & 1 \\ \nabla^2 f_4, & 0, & 14, & 20 \\ \nabla^4 f_4, & 0, & 0, & 120 \end{vmatrix}$$

whence we have
$$f_4 = S_4+S_2+S_0,$$

where
$$S_4 = f_4-\tfrac{1}{14}\nabla^2 f_4+\tfrac{1}{280}\nabla^4 f_4,$$
$$S_2 = \tfrac{1}{14}\nabla^2 f_4-\tfrac{1}{84}\nabla^4 f_4,$$
$$S_0 = \tfrac{1}{120}\nabla^4 f_4.$$

The same method is applicable to p variables instead of only three; care being taken to write, in (A), $(2s, q)(2n-2s+p-2, q)$ instead of $(2s, q)(2n-2s+1, q)$ in the element belonging to the $(q+2)^{\text{th}}$ row and $(s+2)^{\text{th}}$ column.

2. But the solution may be effected in another manner, which will in general be found more convenient than the foregoing method. Put
$$V = \phi_0 f_n+\phi_2\nabla^2 f_n+\phi_4\nabla^4 f_n+\dots \quad \dots\dots\dots\dots\dots\dots\text{(B)},$$
where the ϕ's represent functions of r at present undetermined.

Then

$$\nabla^2 V = \phi_0 \nabla^2 f_n + \phi_2 \nabla^4 f_n + \phi_4 \nabla^6 f_n + \dots$$
$$+ \frac{1}{r}\frac{d^2(r\phi_0)}{dr^2} f_n + \frac{1}{r}\frac{d^2(r\phi_2)}{dr^2}\nabla^2 f_n + \frac{1}{r}\frac{d^2(r\phi_4)}{dr^2}\nabla^4 f_n + \dots$$
$$+ \frac{2n}{r}\frac{d\phi_0}{dr} f_n + \frac{2\,\overline{n-2}}{r}\frac{d\phi_2}{dr}\nabla^2 f_n + \frac{2\,\overline{n-4}}{r}\frac{d\phi_4}{dr}\nabla^4 f_n + \dots .$$

Consequently, if we determine the ϕ's so that

$$0 = \frac{d^2(r\phi_0)}{dr^2} + 2n\frac{d\phi}{dr} \qquad \text{.........................(5),}$$

and $$0 = \frac{d^2}{dr^2}(r\phi_{2s}) + 2(n-2s)\frac{d\phi_{2s}}{dr} + r\phi_{2s-2} \quad (s = 1, 2, 3, \text{etc.}) \text{...(6),}$$

the condition $\nabla^2 V = 0$ is then satisfied.

From equation (5), we have

$$\phi_0 = Cr^{-2n-1} + C',$$

and from (6), $$\phi_{2s} = -\int^r dr\, r^{-2n+4s-2} \int^r dr\, r^{2n-4s+2}\phi_{2s-2}.$$

The constants of integration remain still at our disposal. We choose them so that the ϕ's are finite when $r = 0$, and each ϕ except ϕ_0 vanishes when $r = a$, and $\phi_0 = 1$, when $r = a$.

Thus $\phi_0 = 1$,

$$\phi_2 = -\int_a^r dr\, r^{-2n+2}\int_0^r dr\, r^{2n-2},$$

$$\phi_4 = \int_a^r dr\, r^{-2n+6}\int_0^r dr\, r^{2n-6}\int_a^r dr\, r^{-2n+2}\int_0^r dr\, r^{2n-2},$$

and in general

$$\phi_{2s} = (-1)^s \int_a^r dr\, r^{-2n+4s-2}\int_0^r dr\, r^{2n-4s+2}\int_a^r dr\, r^{-2n+4s-6}\int_0^r dr\, r^{2n-4s+6}\dots$$
$$\dots\int_a^r dr\, r^{-2n+6}\int_0^r dr\, r^{2n-6}\int_a^r dr\, r^{-2n+2}\int_0^r dr\, r^{2n-2} \qquad \text{...........(7).}$$

Substituting the values of ϕ found from (7) in (B), it is seen that V satisfies Laplace's equation, is finite within the sphere, and is equal to $f_n(x, y, z)$ at the surface. V, therefore, is the potential within the sphere.

The following are the calculated developments of the first few ϕ's :—

$$\phi_0 = 1,$$

$$\phi_2 = \frac{1}{2\,(2n-1)}\{a^2 - r^2\},$$

$$\phi_4 = \frac{1}{2^3\,(2n-1)(2n-3)(2n-5)}\{a^4(2n-1) - 2a^2r^2(2n-3) + r^4(2n-5)\},$$

$$\phi_6 = \frac{2^2\,(n-1)!\,(2n-11)!}{3!\,(2n-1)!\,(n-6)!}\{a^6(2n-1)(2n-3) - 3a^4r^2(2n-1)(2n-7)$$
$$+ 3a^2r^4(2n-3)(2n-9) - r^6(2n-7)(2n-9)\},$$

$$\phi_8 = \frac{2^3\,(n-1)!\,(2n-15)!}{4!\,(2n-1)!\,(n-8)!}\{a^8(2n-1)(2n-3)(2n-5)$$
$$-4a^6r^2(2n-1)(2n-3)(2n-11) + 6a^4r^4(2n-1)(2n-7)(2n-13)$$
$$-4a^2r^6(2n-3)(2n-11)(2n-13) + r^8(2n-9)(2n-11)(2n-13)\},$$

$$\phi_{10} = \frac{2^4\,(n-1)!\,(2n-19)!}{5!\,(2n-1)!\,(n-10)!}\{a^{10}(2n-1)(2n-3)(2n-5)(2n-7)$$
$$-5a^8r^2(2n-1)(2n-3)(2n-5)(2n-15)$$
$$+10a^6r^4(2n-1)(2n-3)(2n-11)(2n-17)$$
$$-10a^4r^6(2n-1)(2n-7)(2n-15)(2n-17)$$
$$+5a^2r^8(2n-3)(2n-13)(2n-15)(2n-17)$$
$$-r^{10}(2n-11)(2n-13)(2n-15)(2n-17)\},$$

$$\phi_{12} = \frac{2^5\,(n-1)!\,(2n-23)!}{6!\,(2n-1)!\,(n-12)!}\{a^{12}(2n-1)(2n-3)(2n-5)(2n-7)(2n-9)$$
$$-6a^{10}r^2(2n-1)(2n-3)(2n-5)(2n-7)(2n-19)$$
$$+15a^8r^4(2n-1)(2n-3)(2n-5)(2n-15)(2n-21)$$
$$-20a^6r^6(2n-1)(2n-3)(2n-11)(2n-19)(2n-21)$$
$$+15a^4r^8(2n-1)(2n-7)(2n-17)(2n-19)(2n-21)$$
$$-6a^2r^{10}(2n-3)(2n-15)(2n-17)(2n-19)(2n-21)$$
$$+r^{12}(2n-13)(2n-15)(2n-17)(2n-19)(2n-21)\}.$$

From the preceding values that of ϕ_{2s} is inferred to be the product of

$$\frac{2^{s-1}\,(n-1)!\,(2n-4s+1)!}{s!\,(2n-1)!\,(n-2s)!}$$

and a factor formed in the following manner :—Write down in order the terms of the expansion of $(a^2-r^2)^s$. To the first term apply the multiplier

$$(2n-1)(2n-3)\dots(2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{2s-3});$$

to the second term apply a multiplier obtained from the preceding one by doubling the part $\overline{2s-3}$ of the last factor and adding unity, thus—

$$(2n-1)(2n-3)\dots(2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{4s-5});$$

to the third term apply a multiplier obtained by changing the last but one factor in the same way and subtracting 2 from the last factor, thus—

$$(2n-1)(2n-3)\dots(2n-\overline{2s-7})(2n-\overline{4s-9})(2n-\overline{4s-3});$$

to the fourth term apply a multiplier obtained from the preceding one by changing the part $\overline{2s-7}$ into twice itself + unity, and by augmenting the last but one factor so as to exceed the last factor by 2, thus—

$$(2n-1)(2n-3)\dots(2n-\overline{4s-13})(2n-\overline{4s-5})(2n-\overline{4s-3}).$$

Repeat this process and find multipliers to the other terms. Thus the next multiplier and the last two will be

$$(2n-1)(2n-3)\dots(2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}),$$

$$\text{and } (2n-3)(2n-\overline{2s+3})\dots(2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}),$$

$$(2n-\overline{2s+1})(2n-\overline{2s+3})\dots(2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3}).$$

We may also write ϕ_{2s} in the form

$$\phi_{2s}=\sum_{p=0}^{p=s}(-1)^{s-p}\frac{2(2n-4s+4p+1)}{p!\,(s-p)!}\,\frac{(2n-4s+2p-1)!\,(n-s+p)!}{(2n-2s+2p+1)!\,(n-2s+p-1)!}$$

$$\times a^{2s-2p}r^{2p}\dots\dots\dots\dots(7').$$

The correctness of this inferred form can be established by means of the form given in (7), by the method of mathematical induction.

3. We might proceed to find the potential in the space external to the sphere by starting with the solution

$$\phi_0=\left(\frac{a}{r}\right)^{2n+1}.$$

But the solution by the aid of an elementary principle in the theory of the potential, may be written down at once from (B) thus:—

Let ϕ'_{2s} denote the result of interchanging a and r in ϕ_{2s}. Then, in external space,

$$V=\left(\frac{a}{r}\right)^{2n+1}f_n+\left(\frac{a}{r}\right)^{2n-1}\phi'_2\nabla^2 f_n+\left(\frac{a}{r}\right)^{2n-3}\phi'_4\nabla^4 f_n+\text{etc.} \quad \dots\text{(B')};$$

for to the term $Ka^{2s-2p}r^{2p}\nabla^{2s}f_n$ of $\phi_{2s}\nabla^{2s}f_n$ corresponds in the external potential a term

$$\left(\frac{a}{r}\right)^{2(2p+n-2s)+1}Ka^{2s-2p}r^{2p}\nabla^{2s}f_n$$

$$=\left(\frac{a}{r}\right)^{2n-2s+1}K\left(\frac{a}{r}\right)^{4p-2s}a^{2s-2p}r^{2p}\nabla^{2s}f_n$$

$$=\left(\frac{a}{r}\right)^{2n-2s+1}Kr^{2s-2p}a^{2p}\nabla^{2s}f_n.$$

From (B) and (B′) we deduce the density of the distribution whose potential over the surface of the sphere $r=a$ is f, to be

$$\sigma=\frac{1}{4\pi a}\{\psi_0 f_n+\psi_2\nabla^2 f_n+\psi_4\nabla^4 f_n+\dots\},$$

where $$\psi_{2s}=a\left[\frac{d\phi_{2s}}{dr}-\frac{d\phi'_{2s}}{dr}\right]_{r=a}.$$

Hence, we calculate

$$\psi_2=-\frac{2}{2n-1}a^2,$$

$$\psi_4=-\frac{2^3}{(2n-1)(2n-3)(2n-5)}a^4,$$

$$\psi_6=-\frac{2^7\,(n-1)!\,(2n-11)!}{(2n-1)!\,(n-6)!}a^6,$$

$$\psi_8=-\frac{2^8.5\,(n-1)!\,(2n-15)!}{(2n-1)!\,(n-8)!}a^8,$$

$$\psi_{10}=-\frac{2^{10}.8!\,(n-1)!\,(2n-19)!}{4!\,5!\,(2n-1)!\,(n-10)!}a^{10},$$

$$\psi_{12}=-\frac{2^{12}.10!\,(n-1)!\,(2n-23)!}{5!\,6!\,(2n-1)!\,(n-12)!}a^{12}.$$

And generally

$$\psi_{2s} = \frac{2^{s-1}(n-1)!\,(2n-4s+1)!}{s!\,(2n-1)!\,(n-2s)!}$$

$$\times \{ -2s\,(2n-1)(2n-3)\dots(2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{2s-3})$$

$$+s\,(2s-4)\,(2n-1)\,(2n-3)\dots(2n-\overline{2s-7})(2n-\overline{2s-5})(2n-\overline{4s-5})+\text{etc.}$$

$$+(-1)^{s-1}s\,(2s-4)(2n-3)(2n-\overline{2s+3})\dots(2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3})$$

$$+(-1)^{s}\,2s\,(2n-\overline{2s+1})(2n-\overline{2s+3})\dots(2n-\overline{4s-7})(2n-\overline{4s-5})(2n-\overline{4s-3})\}.$$

From the calculations of $\psi_2, \psi_4, \psi_6 \dots \psi_{12}$ we infer that the part within parentheses is independent of $2n$. Put, therefore, in that part $2n=1$. Thus it is seen that

$$\psi_{2s} = -\frac{2^{2s}(2s-2)!\,(n-1)!\,(2n-4s+1)!}{s!\,(s-1)!\,(2n-1)!\,(n-2s)!}\,a^{2s}.$$

Hence, summarily,

$$\sigma = \frac{1}{4\pi a}\left\{(2n+1)f_n - \frac{(n-1)!}{(2n-1)!}\sum_{s=1}\frac{2^{2s}(2s-2)!\,(2n-4s+1)!}{s!\,(s-1)!\,(n-2s)!}\,a^{2s}\nabla^{2s}f_n\right\} \qquad \dots\dots\dots\dots\dots\dots(C.).$$

If then the potential at the surface of the sphere be given as a rational and integral function of the coordinates, we may express it in the form f_n+f_{n-1}, and a double application of (B), (B′), and (C) gives at once the internal and external potentials, and the required density of distribution.

It is noteworthy that, by grouping the terms of the same dimensions in (B), we are enabled to resolve f_n into its component surface harmonics. To do this, we note that the coefficient of

$$r^{2p}\nabla^{2s}f_n \text{ in } \phi_{2s}\nabla^{2s}f_n$$

is

$$(-1)^{s-p}\,\frac{2\,(2n-4s+4p+1)}{p!\,(s-p)!}\;\frac{(2n-4s+2p-1)!\,(n-s+p)!}{(2n-2s+2p+1)!\,(n-2s+p-1)!}\,a^{2s-2p}.$$

To find, therefore, the terms of dimensions $n-2m$ in (B), we must put $n-2m=n-2s+2p$, *i.e.*, $p=s-m$, and sum with respect to s. Hence the harmonic component of f_n of order $n-2m$, relative to the sphere $r=a$, is

$$\left(-\frac{a^2}{r^2}\right)^m \frac{2\,(2n-4m+1)(n-m)!}{m!\,(2n-2m+1)!}\sum_{s=m}\frac{(2n-2s-2m-1)!}{(s-m)!\,(n-s-m-1)!}(-r^2)^s\nabla^{2s}f \qquad \dots\dots\dots(a).$$

This result enables us to decompose any rational and integral function into its *harmonic components* at the surface of the sphere of reference.

When n is even, we must except the constant component which comes out as

$$\frac{1}{(n+1)!}a^n\nabla^n f_n.$$

5. We now propose to determine the solution of Laplace's equation that satisfies the conditions

$$V=f_n,\quad (r=a);\quad V=f_m,\quad (r=b).$$

Let V_1 and V_2 be solutions of Laplace's equation, such that

$$V_1=f_n,\quad (r=a);\quad V_1=0,\quad (r=b);$$

$$V_2=0,\quad (r=a);\quad V_2=f_m,\quad (r=b).$$

Then $$V=V_1+V_2.$$

Put $$V=\chi_0 f_n+\chi_2\nabla^2 f_n+\chi_4\nabla^4 f_n+\chi_6\nabla^6 f_n+\ \dots\dots\dots\dots\dots\text{(D)},$$

where the χ's are determined by relations analogous to (5) and (6). Take for the value of χ_0,

$$\chi_0=\frac{r^{-2n-1}-b^{-2n-1}}{a^{-2n-1}-b^{-2n-1}},$$

so that $$\chi_0=1,\quad (r=a);\quad \chi_0=0,\quad (r=b).$$

The constants of integration of the remaining χ's must be determined so as to make them vanish when $r=a$, $r=b$. We then find, employing the notation

$$u_s\equiv\frac{r^{-s}-b^{-s}}{a^{-s}-b^{-s}},$$

$$\chi_0=u_{2n+1},$$

$$\chi_2=\frac{1}{2(2n-1)}\left\{u_{2n-3}a^2-u_{2n+1}r^2\right\},$$

$$\chi_4=\frac{1}{2^3(2n-1)(2n-3)(2n-5)}$$

$$\times\left\{u_{2n-7}a^4(2n-1)-u_{2n-3}2a^2r^2(2n-3)+u_{2n+1}r^4(2n-5)\right\}.$$

And, in general, the value of χ_{2s} is found by prefixing the several terms of ϕ_{2s} as found by the law of formation in § 2, with $u_{2n-4s+1}$, $u_{2n-4s-3}$, $u_{2n-4s-7}$, ... u_{2n+1} taken in order. Thus V_1 is completely determined; and V_2 may be written down in the same way by making the necessary changes from a to b and from n to m. Consequently *$V=V_1+V_2$ is completely determined.*

The solution of the problem of finding the distributions on two spherical surfaces, that shall produce a potential f'_n at the outer $(r = a)$ and a potential f_m at the inner $(r = b)$, may now be written down. The required density of distribution on the surface $r = a$ is $\sigma + \sigma' + \sigma''$; where σ is given by (C), σ' is the value of $\frac{1}{4\pi a} V_1$ found by interpreting u_s as $\frac{-sa^{-s}}{a^{-s}-b^{-s}}$, and σ'' is the result of substituting $\frac{-sa^{-s}}{a^{-s}-b^{-s}}$ for v_s in the expression for $\frac{V_2}{4\pi a}$ $\left(v_s \text{ in the first place standing for } \frac{r^{-s}-a^{-s}}{b^{-s}-a^{-s}}\right)$. The required density of distribution over $(r=b)$ is $\sigma_1' + \sigma_1''$; where σ_1' is found by interpreting the values of u_s and v_s in the expressions for $\frac{V_1}{4\pi b}$, $\frac{V_2}{4\pi b}$ to mean $\frac{-sb^{-s}}{a^{-s}-b^{-s}}$ and $\frac{sb^{-s}}{a^{-s}-b^{-s}}$, respectively.

6. The series given above are inapplicable when the surface potential is not expressible as the sum of a finite number of rational, integral and homogeneous functions. When this impossibility exists, suppose that the surface potential is put into the form $V = f(x, y, z)$, where f denotes a homogeneous function of degree zero, such that neither it nor any of the functions $\nabla^{2s} f$ become infinite at any real point on the sphere.

We shall first investigate a series for the external potential. As before, assume

$$V = \phi_0 f + \phi_2 \nabla^2 f + \phi_4 \nabla^4 f + \text{etc.} \qquad \text{(E)}.$$

Then we have, as before, the relations

$$0 = \frac{1}{r}\frac{d^2}{dr^2}(r\phi_{2s}) - \frac{4s}{r}\frac{d\phi_{2s}}{dr} + \phi_{2s-2} \qquad \text{(8)},$$

when $s = 1, 2, 3$, etc.; and

$$0 = \frac{1}{r}\frac{d^2}{dr^2}(r\phi_0) \qquad \text{(9)}$$

From (9), we get $\phi_0 = \frac{a}{r}$,

and from (8)

$$\phi_{2s} = a\int_a^r dr\, r^{4s-2}\int_r^\infty dr\, r^{-4s+2}\int_a^r dr\, r^{4s-6}\int_r^\infty dr\, r^{-4s+6}\dots$$

$$\dots\int_a^r dr\, r^6\int_r^\infty dr\, r^{-6}\int_a^r dr\, r^2\int_r^\infty dr\, r^{-2} \qquad \text{(10)}.$$

The following are the developments found for the first few ϕ's:—

$$\phi_0 = \frac{a}{r},$$

$$\phi_2 = -\frac{1}{2}(a^2 - ar),$$

$$\phi_4 = \frac{1}{2^3 . 3 . 5}(a^4 - 6a^2r^2 + 5ar^3),$$

$$\phi_6 = -\frac{1}{2^4 . 3^3 . 5 . 7}(a^6 - 7a^4r^2 + 27a^2r^4 - 21ar^5),$$

$$\phi_8 = \frac{2^2 . 6!}{4!\, 13!}(a^8.1.3.5 - 4a^6r^2.1.3.11 + 6a^4r^4.1.7.13 - 4a^2r^6.3.11.13 \\ + ar^7.9.11.13),$$

$$\phi_{10} = -\frac{2^3 . 8!}{5!\, 17!}(a^{10}.1.3.5.7 - 5a^8r^2.1.3.5.15 + 10a^6r^4.1.3.11.17 \\ - 10a^4r^6.1.7.15.17 + 5a^2r^8.3.13.15.17 - ar^9.11.13.15.17),$$

$$\phi_{12} = \frac{2^4 . 10!}{6!\, 21!}(a^{12}.1.3.5.7.9 - 6a^{10}r^2.1.3.5.7.19 + 15a^8r^4.1.3.5.13.21 \\ - 20a^6r^6.1.3.11.19.21 + 15a^4r^8.1.7.17.19.21 \\ - 6a^2r^{10}.3.15.17.19.21 + ar^{11}.13.15.17.19.21).$$

Hence we infer the following rule for the formation of ϕ_{2s}:—ϕ_{2s} is the product of $\dfrac{(-1)^s 2^{s-2}(2s-2)!}{s!\,(4s-3)!}$ and a factor which is found by writing down the terms of $(a^2 - r^2)^s$ taken in order, replacing the last by $(-1)^s ar^{2s-1}$, and applying to each term a multiplier whose law of formation is easily seen by inspection of the following terms of the factor, viz.,—

$$\begin{aligned}
& a^{2s}.1.3.5 \ldots\ldots (2s-7)(2s-5)(2s-3) \\
& -sa^{2s-2}r^2.1.3.5 \ldots\ldots (2s-7)(2s-5)(4s-5) \\
& +\frac{s(s-1)}{1.2}a^{2s-4}r^4.1.3.5 \ldots\ldots (2s-7)(4s-9)(4s-3) \\
& -\frac{s(s-1)(s-2)}{1.2.3}a^{2s-6}r^6.1.3.5 \ldots\ldots (4s-13)(4s-5)(4s-3) \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
& +(-1)^{s-2}\frac{s(s-1)}{1.2}a^4r^{2s-4}.1.7(2s+5) \ldots\ldots (4s-7)(4s-5)(4s-3) \\
& +(-1)^{s-1}s.a^2r^{2s-2}\,3(2s+3)(2s+5) \ldots\ldots (4s-7)(4s-5)(4s-3) \\
& +(-1)^s a\, r^{2s-1}(2s+1)(2s+3)(2s+5) \ldots\ldots (4s-7)(4s-5)(4s-3)
\end{aligned} \quad \ldots\ldots (11).$$

Or we may say at once that

$$\phi_{2s} = \sum_{p=0}^{p=s-1} (-1)^{s+p} \frac{(2s-p-1)!\,(2s-2p-2)!}{s!\,(s-p-1)!\,(4s-2p-1)!} (4s-4p-1)\, a^{2s-2p} r^{2p}$$
$$+ \frac{1}{(2s)!} a r^{2s-1} \quad \text{.............................(11}'\text{)}.$$

If we start with the solution $\phi_0 = 1$, and determine the constants of integration of the other ϕ's, so that every term of the series shall be of positive dimensions in the variables, and, except the first, shall vanish when $r = a$, we get the expression for the internal potential,

viz.,
$$V = f + \sum_{s=1} \left(\frac{r}{a}\right)^{2s} \phi'_{2s} \nabla^{2s} f \quad \text{.....................(E}'\text{)};$$

where ϕ'_{2s} denotes the result of interchanging a and r in ϕ_{2s}. For the density of distribution, we get

$$\sigma = \frac{1}{4\pi a}\left[f - \sum_{s=1} \left\{ \frac{2}{(2s)!} - \frac{2^{2s-1}\{(2s-2)!\}^2}{s!\,(s-1)!\,(4s-3)!} \right\} a^{2s} \nabla^{2s} f\right].$$

7. Lastly, for the solution of Laplace's equation, which satisfies the conditions $V = f$, $(r = a)$; $V = 0$, $(r = b)$, we get

$$V = \chi_0 f + \chi_2 \nabla^2 f + \chi_4 \nabla^4 f + \dots \quad \text{..................(F)};$$

where we find

$$\chi_0 = u_1,$$

$$\chi_2 = -\frac{1}{1.2}\,\frac{r}{a}\,\{r^2 u_3 - ar\,.\,u_1\},$$

$$\chi_4 = \frac{1}{2^3.3.5}\left(\frac{r}{a}\right)^3 \{r^4 u_7 - 6a^2 r^2 u_3 + 5a^3 r u_1\},$$

$$\chi_6 = -\frac{1}{2^4.3^3.5.7}\left(\frac{r}{a}\right)^5 \{r^6 u_{11} - 7a^2 r^4 u_7 + 27 a^4 r^2 u_3 - 21 a^5 r u_1\};$$

where
$$u_s = \frac{r^{-s} - b^{-s}}{a^{-s} - b^{-s}}.$$

Hence we infer that

$$\chi_{2s} = \sum_{p=0}^{p=s-1} (-1)^{s+p} \frac{(2s-p-1)!\,(2s-2p-2)!}{s!\,(s-p-1)!\,(4s-2p-1)!} (4s-4p-1)\, a^{2p} r^{2s-2p} u_{4s-4p-1}$$
$$+ \frac{1}{(2s)!} a r^{2s-1} u_1.$$

In the same way we may write down the solution of $\nabla^2 V = 0$, which satisfies $V = F$, $(r = b)$; $V = 0$, $(r = a)$. Adding this to the first solution of this article, we get the expression for the solution of Laplace's equation that satisfies the conditions $V = f$, $(r = a)$; $V = F$, $(r = b)$. A series for the density of distribution may then be readily derived, but the result is not very elegant.

8. The legitimate employment of the series (E) requires the negligibility of $\phi_{2s}\nabla^{2s+2}f$ when s is infinite, and, secondly, the convergency of the series. If, however, for some finite value of s, $\nabla^{2s}f = 0$, the series becomes a finite one and may be safely employed. On splitting (E) into groups of terms of the same dimensions, there occur groups of order $-2m$, $(m = 1, 2, 3$, etc.), viz.,

$$(-1)^m \frac{(4m-1)(2m-2)!}{(m-1)!}\left(\frac{a^2}{r^2}\right)^m \sum_{s=m}^{s=\infty} \frac{(-1)^s (s+m-1)!}{s!\,(2s+2m-1)!} r^{2s}\nabla^{2s}f;$$

and but one group of odd dimensions, viz.,

$$\frac{a}{r}\sum_{s=0}^{s=\infty}\frac{(-1)^s}{(2s)!}r^{2s}\nabla^{2s}f.$$

If we start with the solution $\phi_0 = 1$, and determine the constants of integration of the other ϕ's, so that every term of the series shall be of positive dimensions in the variables and, except the first, shall vanish when $r = a$, we get the expression for the internal potential. The result is

$$V = f + \sum_{s=1}\left(\frac{r}{a}\right)^{2s}\phi'_{2s}\nabla^{2s}f \qquad\qquad (\mathrm{F}');$$

where ϕ'_{2s} denotes the result of interchanging a and r in (11), and multiplying by $\dfrac{(-1)^s 2^{s-2}(2s-2)!}{s!\,(4s-3)!}$.

9. By writing $2n+d-3$ in place of $2n$, we shall adapt our series to the case of d variables.

E.g., when $\quad r^2 \equiv x_1^2 + x_2^2 + \ldots + x_d^2 = 1,$

$$f_n(x_1, x_2, \ldots x_n) = S_n + S_{n-2} + \ldots + S_{n-2m} + \text{etc.};$$

where $S_{n-2m} = \left(-\dfrac{1}{r^2}\right)^m \dfrac{2\,(2n-4m+d-2)\,(n-m+\frac{1}{2}[d-3])!}{m!\,(2n-2m+d-2)!}$

$$\times \sum_{s=m} \frac{(2n-2s-2m+d-4)!}{(s-m)!\,(n-s-m+\frac{1}{2}[d-5])!}(-r^2)^s\nabla^{2s}f_n$$

when d is odd, and

$$=\left(-\frac{1}{r^2}\right)^m \frac{1}{m!}\sum_{s=m}\frac{1}{2^{2s+2}}\,\frac{(n-s-m+\frac{1}{2}d-2)!}{(s-m)!\,(n-m+\frac{1}{2}d-1)!}\,(-r^2)^s\,\nabla^{2s}f_n$$

when d is even; each S satisfying the condition

$$\nabla^2 S\equiv\left(\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}+\ldots+\frac{d^2}{dx_d^2}\right)S=0.$$

[ADDENDUM.—It may be noticed that the series (E) is convergent if the series $\sum_{s=1}^{s=\infty}\frac{1}{(2s-1)!}\,r^{2s-2}\,\nabla^{2s}f$ is so, a series which is the development of the mean value of $\nabla^2 f$ over the surface of a sphere of radius r described round xyz as centre. For, comparison of (11) and (11′) shows that each term of the development of ϕ_{2s} is numerically less than $\frac{1}{(2s)!}\,ar^{2s-1}$, and the sum of the first two terms when s is large is less than $\frac{1}{(2s)!}\,ar^{2s-1}$. Hence ϕ_{2s} is numerically less than

$$\frac{ar}{2}\,\frac{1}{(2s-1)!}\,r^{2s-2}$$

when s is large, and the s^{th} term of the series (E) is numerically less than the s^{th} term of the series

$$\frac{ar}{2}\sum_{s=1}^{s=\infty}\frac{1}{(2s-1)!}\,r^{2s-2}\nabla^{2s}f,$$

when s is large.

Moreover, since from (E),

$$\nabla^2 V=Lt_{s=\infty}\,\phi_{2s}\nabla^{2s+2}f,$$

and therefore

$$<\frac{a}{2r}\,Lt_{s=\infty}\,\frac{r^{2s}}{(2s-1)!}\,\nabla^{2s+2}f,$$

we see that $\nabla^2 V=0$ will be satisfied, if the series

$$\sum_{s=1}^{s=\infty}\frac{1}{(2s-1)!}\,r^{2s-2}\nabla^{2s}f$$

is as convergent as a series $\sum_{s=1}^{s=\infty}\frac{1}{s^{\alpha}}$, where α is some positive quantity greater than 2. Therefore, when this is the case, the series (E) may always be legitimately employed.]

On $\kappa\lambda-\kappa'\lambda'$ Modular Equations. By ROBERT RUSSELL, M.A.

[*Read Nov. 10th,* 1887.]

The object of the following paper is to give a general method of obtaining the Modular Equation corresponding to any of the Jacobian transformations when the order of that transformation is prime.

1. Using Jacobi's notation, I shall begin by proving a few well-known formulæ which will be found necessary afterwards.

To express $\sqrt[4]{\kappa}$ and $\sqrt[4]{\kappa'}$ in terms of q, where $q=e^{-\pi K'/K}$, I shall assume that

$$\Theta(K)=\sqrt{\frac{2K}{\pi}},\quad \Theta(0)=\sqrt{\frac{2\kappa' K}{\pi}},\quad \mathrm{H}(K)=\sqrt{\frac{2\kappa K}{\pi}},$$

where $$\Theta(u)=1-2q\cos\frac{\pi u}{K}+2q^4\cos\frac{2\pi u}{K}-2q^9\cos\frac{3\pi u}{K}+\ldots \quad \ldots(1),$$

$$\mathrm{H}(u)=2\sqrt[4]{q}\sin\frac{\pi u}{2K}-2\sqrt[4]{q^9}\sin\frac{3\pi u}{2K}+2\sqrt[4]{q^{25}}\sin\frac{5\pi u}{2K}-\ldots \quad \ldots(2).$$

If we transform $\dfrac{dx}{\sqrt{1-x^2\,.\,1-\kappa^2x^2}}$ by the quadric substitution

$$y=\frac{(1+\kappa)\,x}{1+\kappa x^2},$$

we obtain $$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}}=(1+\kappa)\,\frac{dx}{\sqrt{1-x^2\,.\,1-\kappa^2x^2}},$$

where $$\lambda=\frac{2\sqrt{\kappa}}{1+\kappa},$$

and therefore $$\lambda'=\frac{1-\kappa}{1+\kappa},\quad\text{or}\quad \kappa'\lambda=2\sqrt{\kappa\lambda'};$$

and if in this we put $$h'=\frac{2\sqrt{\kappa}}{1+\kappa},$$

we have $$\kappa=\frac{1-h}{1+h},$$

therefore $$\frac{dy}{\sqrt{1-y^2\,.\,1-h'^2y^2}}=\frac{2}{1+h}\,\frac{dx}{\sqrt{1-x^2\,.\,1-\left(\frac{1-h}{1+h}\right)^2x^2}}.$$

Hence we have the following formulæ:—

If $\lambda = \dfrac{2\sqrt{\kappa}}{1+\kappa}$, then

$$\lambda' = \frac{1-\kappa}{1+\kappa},\ \ \Lambda = (1+\kappa)\,K,\ \ \Lambda' = \frac{1+\kappa}{2}\,K',\ \text{ and }\ q' = q^{\frac{1}{2}} \ldots\ldots (3).$$

If $\lambda = \dfrac{1-\kappa'}{1+\kappa'}$, then

$$\lambda' = \frac{2\sqrt{\kappa'}}{1+\kappa'},\ \ \Lambda = \frac{1+\kappa'}{2}\,K,\ \ \Lambda' = \overline{1+\kappa'}\,K',\ \text{ and }\ q' = q^2 \ldots\ldots (4).$$

Now $\mathrm{H}(\Lambda) = \sqrt{\dfrac{2\lambda\Lambda}{\pi}} = \sqrt{\dfrac{2\sqrt{\kappa}}{1+\kappa}\,\dfrac{2\Lambda}{\pi}} = \sqrt{\dfrac{2\sqrt{\kappa}}{1}\,\dfrac{2K}{\pi}}$, from (3),

therefore $\qquad \mathrm{H}(\Lambda) = \sqrt{2}\,\sqrt[4]{\kappa}\sqrt{\dfrac{2K}{\pi}} = \dfrac{\sqrt{2}}{\sqrt[4]{\kappa}}\sqrt{\dfrac{2\kappa K}{\pi}}$,

therefore $\qquad \sqrt[4]{\kappa} = \dfrac{\mathrm{H}(\Lambda)}{\sqrt{2}\,\Theta(K,\kappa)}$ or $= \sqrt{2}\,\dfrac{\mathrm{H}(K,\kappa)}{\mathrm{H}(\Lambda,\lambda)}$,

therefore $\qquad \sqrt[4]{\kappa} = \sqrt{2}\,\sqrt[8]{q}\,\dfrac{1+q+q^3+q^6+\ldots}{1+2q+2q^4+2q^9+\ldots}$ .... (5),

or $\qquad = \sqrt{2}\,\sqrt[8]{q}\,\dfrac{1+q^2+q^6+q^{12}+\ldots}{1+q+q^3+q^6+\ldots}$.

Again, making use of the values of λ, λ', Λ, Λ' in (4), we have

$$\Theta(0,\lambda) = \sqrt{\frac{2\lambda'\Lambda}{\pi}} = \sqrt{\frac{2\sqrt{\kappa'}}{1+\kappa'}\,\frac{2\Lambda}{\pi}} = \sqrt{\sqrt{\kappa'}\,\frac{2K}{\pi}} = \frac{1}{\sqrt[4]{\kappa'}}\sqrt{\frac{2\kappa' K}{\pi}}$$

$$= \sqrt{\kappa'}\sqrt{\frac{2K}{\pi}},$$

therefore $\qquad \sqrt[4]{\kappa'} = \dfrac{\Theta(0,\lambda)}{\Theta(K,\kappa)} = \dfrac{\Theta(0,\kappa)}{\Theta(0,\lambda)}$.

Hence $\qquad \sqrt[4]{\kappa'} = \dfrac{1-2q+2q^4-2q^9+\ldots}{1-2q^2+2q^8-2q^{18}+\ldots}$ ....(6).

In (5), changing q into $-q$, we have

$$\sqrt{\frac{\kappa}{\kappa'}} = \sqrt{2}\,\sqrt[8]{q}\,\frac{1-q-q^3+q^6\ldots}{1-2q+2q^4-2q^9+\ldots} \ \ \ldots\ldots\ldots\ldots (7).$$

Hence, multiplying (6) and (7), we have

$$\sqrt[4]{\kappa} = \sqrt{2}\,\sqrt[8]{q}\,\frac{1-q-q^3+q^6+q^{10}\ldots}{1-2q^2+2q^8-2q^{18}+\ldots} \ \ \ldots\ldots\ldots\ldots (8);$$

by changing q into $-q$ in (6), we have also

$$\sqrt[4]{\kappa'} = \frac{1-2q^2+2q^8-2q^{18}\ldots}{1+2q+2q^4+2q^9\ldots}.$$

2. Expanding (6) and (8) by the method of synthetic division, we have

$$\begin{aligned}\sqrt[4]{\kappa} = \sqrt{2}\,\sqrt[8]{q}\,(1-q+2q^2-3q^3+4q^4-6q^5+9q^6-12q^7+16q^8-22q^9 \\ +29q^{10}-38q^{11}+50q^{12}-64q^{13}+82q^{14}-105q^{15}+132q^{16} \\ -166q^{17}+208q^{18}-258q^{19}+320q^{20}-395q^{21}+484q^{22} \\ -592q^{23}+722q^{24}-876q^{25}\ldots) \quad \ldots\ldots\ldots\ldots(11),\end{aligned}$$

$$\begin{aligned}\sqrt[4]{\kappa'} = (1-2q+2q^2-4q^3+6q^4-8q^5+12q^6-16q^7+22q^8-30q^9+40q^{10} \\ -52q^{11}+68q^{12}-88q^{13}+112q^{14}-144q^{15}+182q^{16}-228q^{17} \\ +286q^{18}-356q^{19}+440q^{20}-544q^{21}+668q^{22}-816q^{23} \\ +976q^{24}-1208q^{25}\ldots) \quad \ldots\ldots\ldots\ldots(12).\end{aligned}$$

We shall now put $\sqrt[4]{\kappa} = \phi(\omega)$ and $\sqrt[4]{\kappa'} = \psi(\omega)$,

where $$\omega = \frac{iK'}{K}.$$

It is obvious from this definition that $\phi(\omega)$ and $\psi(\omega)$ satisfy the following conditions:—

$$\phi(\omega+1) = e^{\frac{1}{4}i\pi}\frac{\phi(\omega)}{\psi(\omega)} \text{ and } \psi(\omega+1) = \frac{1}{\psi(\omega)} \ \ldots\ldots\ldots(13).$$

From the expressions for $\sqrt[4]{\kappa}$ and $\sqrt[4]{\kappa'}$ in terms of q, I shall now show how to derive the medular equation for any prime value of n.

The several values of $\sqrt[4]{\lambda}$ and $\sqrt[4]{\lambda'}$, corresponding to the $\overline{n+1}$ transformations, are

$$\phi(n\omega),\ \phi\left(\frac{\omega}{n}\right),\ \phi\left(\frac{\omega+16}{n}\right),\ \phi\left(\frac{\omega+2.16}{n}\right) \ldots\ldots \phi\left(\frac{\omega+\overline{n-1}.16}{n}\right);$$

$$\psi(n\omega),\ \psi\left(\frac{\omega}{n}\right),\ \psi\left(\frac{\omega+16}{n}\right),\ \psi\left(\frac{\omega+2.16}{n}\right) \ldots\ldots \psi\left(\frac{\omega+\overline{n-1}.16}{n}\right).$$

Choose any pair of these, say

$$\sqrt[4]{\lambda} = \phi\left(\frac{\omega+16r}{n}\right),$$

therefore $$\sqrt[4]{\lambda'} = \psi\left(\frac{\omega+16r}{n}\right),$$

therefore $\sqrt{\kappa\lambda} = 2\sqrt[8]{q}\,\sqrt[8]{q^{1/n}e^{(16i\pi r)/n}}\,(1-q+2q^3-3q^6\ldots)$
$$\times\,(1-q^{1/n}e^{(16i\pi r)/n}+2q^{3/n}e^{(32i\pi r)/n}-\ldots),$$

$$\sqrt[4]{\kappa'\lambda'} = (1-2q+2q^4-4q^9\ldots)(1-2q^{1/n}e^{(16i\pi r)/n}+2q^{4/n}e^{(32i\pi r)/n}-\ldots).$$

Putting $q^{1/n}e^{(16i\pi r)/n} = a$, we have

$$\sqrt[4]{\kappa\lambda} = 2\sqrt[8]{a}\,\sqrt[8]{a^n}\,(1-a^n+2a^{3n}\ldots)(1-a+2a^3\ldots),$$

$$\sqrt[4]{\kappa'\lambda'} = 2\,(1-2a^n+2a^{4n}\ldots)(1-2a+2a^4-\ldots),$$

that is, we have exactly the same form as if we had taken

$$\sqrt[4]{\lambda} = \phi\,(n\omega) \quad\text{and}\quad \sqrt[4]{\lambda'} = \psi\,(n\omega).$$

3. In order to find the relation connecting $\sqrt[4]{\kappa\lambda}$ and $\sqrt[4]{\kappa'\lambda'}$, we have to eliminate q between the equations

$$\left.\begin{aligned}\sqrt[4]{\kappa\lambda} &= 2q^{\frac{1}{8}(n+1)}\,(1-q+2q^3-3q^6+4q^{10}\ldots)(1-q^n+2q^{3n}-3q^{6n}+\ldots)\\ \sqrt[4]{\kappa'\lambda'} &= (1-2q+2q^4-4q^9+6q^{16}\ldots)\,(1-2q^n+2q^{4n}-4q^{9n}+\ldots)\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots(14).$$

Now $n+1$ must be divisible by either 2, 4, or 8, when reduced to its lowest terms. Let

$$\frac{n+1}{8} = \frac{p}{s},$$

where s is some one of the numbers 1, 2, 4.

Hence, raising the above formulæ (14) to the power s, we have

$$(\sqrt[4]{\kappa\lambda})^s = q^p f_1\,(q), \quad (\sqrt[4]{\kappa'\lambda'})^s = f_2\,(q).$$

Denoting $(\sqrt[4]{\kappa\lambda})^s$ and $(\sqrt[4]{\kappa'\lambda'})^s$ by x and y respectively, we see that corresponding to any value of x we have p values of q; for

$$\theta x^{1/p} = qF_1\,(q), \quad\text{where}\quad \theta^p = 1,$$

and therefore also we have p values of y.

Hence the relation between x and y must be of the form

$$A_x y^p + B_x y^{p-1} + C_x y^{p-2} + \ldots + P_x = 0,$$

where the quantities A, B, C ... P must be of the degree p *at most* in x.

Obviously, the relation existing between x and y must be unaltered by interchanging these quantities, so that we may equally write the

modular equation in the form

$$A_y x^p + B_y x^{p-1} + C_y x^{p-2} + \ldots + L_y x + P_y = 0 \ldots\ldots\ldots\ldots (15),$$

where $A_y, B_y, C_y \ldots P_y$ are the same functions of y that $A_x, B_x, C_x \ldots P_x$ were of x.

4. I now proceed to show how to determine the functions $A, B, C \ldots P$.

I. Substituting in (15) for x and y their values in terms of q, we see that the term $L_y x$ contains no powers of q less than q^p, and the previous terms none so low; hence P_y must be such a rational function of y that when expanded in powers of q it shall contain no powers of q less than p.

But $$y = 1 - aq + bq^2 \ldots,$$

therefore $$P_y = (y-1)^p.$$

The form of the modular equation must therefore in general be

$$A_y x^p + B_y x^{p-1} + C_y x^{p-2} + \ldots + L_y x + (y-1)^p = 0 \ldots\ldots\ldots (15').$$

II. If $\Omega = \dfrac{i\Lambda'}{\Lambda}$, $\qquad \omega = \dfrac{iK'}{K}$,

then one value of Ω is given by

$$\Omega = \frac{\omega}{n},$$

therefore $$\Omega + 1 = \frac{\omega+n}{n} = \frac{(\omega+n-16)+16}{n} \quad \ldots\ldots\ldots\ldots\ldots (16).$$

Therefore $\phi(\Omega+1)$ is the value of $\sqrt[4]{\lambda}$ corresponding to

$$\sqrt[4]{\kappa} = \phi(\omega - n + 16).$$

But $$\phi(\Omega+1) = e^{\frac{1}{8} i\pi} \sqrt[4]{\frac{\lambda}{\lambda'}}, \quad \psi(\Omega+1) = \frac{1}{\sqrt[4]{\lambda'}};$$

and, since $n-16$ is odd, we have

$$\phi(\omega+n-16) = e^{\frac{1}{8}(n-16) i\pi} \sqrt[4]{\frac{\kappa}{\kappa'}}, \quad \psi(\omega+n-16) = \frac{1}{\sqrt[4]{\kappa'}};$$

therefore

$$\phi(\Omega+1)\,\phi(\omega+n-16) = e^{\frac{1}{8}(n+1-16) i\pi} \sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}} = e^{\frac{1}{8}(n+1) i\pi} \sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}} = \sqrt[4]{\kappa_1\lambda_1},$$

and $$\psi(\Omega+1)\,\psi(\omega+n-16) = \frac{1}{\sqrt[4]{\kappa'\lambda'}} = \frac{1}{\sqrt[4]{\kappa_1'\lambda_1'}}.$$

But $\frac{n+1}{8}$ when reduced to its lowest terms is $\frac{p}{s}$ (p of course being odd), therefore

$$\sqrt[4]{\kappa_1\lambda_1}=e^{(p/s)\,i\pi}\sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}},\ \text{ and }\ \sqrt[4]{\kappa_1'\lambda_1'}=\frac{1}{\sqrt[4]{\kappa'\lambda'}},$$

therefore $(\sqrt[4]{\kappa_1\lambda_1})^s=-\left(\sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}}\right)^s$, and $(\sqrt[4]{\kappa_1'\lambda_1'})^s=\frac{1}{(\sqrt[4]{\kappa'\lambda'})^s}$.

Hence $$x_1=-\frac{x}{y}\ \text{ and }\ y_1=\frac{1}{y}\ \ \dots\dots\dots\dots(17).$$

But, from (16), x_1 and y_1 must satisfy the modular equation corresponding to n; hence we have the following theorem,—

If in the modular equation (when of an odd degree) we replace x and y by $-\frac{x}{y}$ and $\frac{1}{y}$ respectively, the equation must remain unaltered.

5. Applying the principle to the modular equation (15′), we have

$$\frac{x^p}{y^p}A_{1/y}-\frac{x^{p-1}}{y^{p-1}}B_{1/y}+\frac{x^{p-2}}{y^{p-2}}C_{1/y}-\dots-\frac{(1-y)^p}{y^p}=0,$$

which must be exactly the same equation as before. Therefore

$$x^pA_{1/y}-x^{p-1}yB_{1/y}+x^{p-2}y^2C_{1/y}\dots+(y-1)^p\equiv x^pA_y+x^{p-1}B_y+\dots+(y-1)^p.$$

Hence $$A_y=A_{1/y},\ B_y=-yB_{1/y},\ C_y=+y^2C_{1/y}\ \dots\dots\dots(18),$$

and therefore the coefficients of x^{p-r} must be, if r is even, of the form

$$a\,(y^r+1)+b\,(y^{r-1}+y)+c\,(y^{r-2}+y^2)+\dots,$$

and, if r is odd, of the form

$$a\,(y^r-1)+b\,(y^{r-1}-y)+c\,(y^{r-2}-y^2)+\dots.$$

If, however, p is even, we must have $s=1$, and therefore we have, as before,

$$\sqrt[4]{\kappa_1\lambda_1}=e^{p i\pi}\sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}}=\sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}},$$

and $$\sqrt[4]{\kappa_1'\lambda_1'}=\frac{1}{\sqrt[4]{\kappa'\lambda'}}.$$

Hence, when p is even, if we replace x and y by $\frac{x}{y}$ and $\frac{1}{y}$, the equation must remain unaltered; hence

$$x^pA_{1/y}+x^{p-1}yB_{1/y}+x^{p-2}y^2C_{1/y}+\dots+(y-1)^p$$
$$\equiv x^pA_y+x^{p-1}B_y+x^{p-2}C_y+\dots+(y-1)^p,$$

therefore $\quad A_y = 1, \quad yB_{1/y} = B_y, \quad y^2C_{1/y} = C_y \dots,$

and the coefficient of x^{p-r} must be of the form

$$a(y^r+1)+b(y^{r-1}+y)+\dots \qquad \text{(19)}.$$

6. From what we have just proved it is at once evident that, if the modular equation is of an odd degree, then it must be a homogeneous symmetric function of x, y and -1^*; and, if of an even degree, it must be a homogeneous symmetric function of x, y, and 1.

First, if it is of an odd degree, it is unaltered by interchanging x and y, and it is also unaltered by changing x, y, and -1 into $-\dfrac{x}{y}$, $\dfrac{1}{y}$, and -1, that is, into $-x$, 1, and $-y$, that is, into x, -1, y; therefore x, y, and -1 are interchangeable.

The same reasoning applies to those cases where the degree of the equation is even, only that it is a symmetric function of x, y and $+1$.

The form can therefore be conveniently written down in terms of the quantities P, Q, R, where

$$\begin{aligned} P &= x+y+1, &&\text{or}\quad x+y-1,\\ Q &= xy+x+y, &&\text{or}\quad xy-x-y,\\ R &= +xy, &&\text{or}\quad -xy, \end{aligned}$$

according as the degree is even or odd.

7. I shall now illustrate the preceding theory by means of some examples, the results of which are already known, from the researches of Gutzlaff, Schröter, Hurwitz, Fiedler, and others.

For the case $n=3$, we have

$$\begin{aligned} \sqrt[4]{\kappa\lambda} &= 2q^{\frac{1}{2}}(1-q+2q^2-3q^3\dots)(1-q^3+2q^6\dots),\\ &= 2q^{\frac{1}{2}}(1-q+2q^2-4q^3\dots),\\ \sqrt[4]{\kappa'\lambda'} &= (1-2q+2q^4-4q^9\dots)(1-2q^3+2q^{12}-4q^{27}\dots)\\ &= (1-2q+2q^4-6q^3+\dots). \end{aligned}$$

In this case $p=1$ and $s=2$; and then the equation must be linear in x and y, where

$$x=\sqrt{\kappa\lambda}, \quad y=\sqrt{\kappa'\lambda'},$$

and must therefore be $\quad \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1=0.$

In exactly a similar manner, it follows that for $n=7$, we have

Gutzlaff's form

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1=0.$$

For $n=5$, we have $\frac{n+1}{8}=\frac{3}{4}$; hence the equation must be of the third degree in $\kappa\lambda$ and $\kappa'\lambda'$. Denoting these by x and y, we have

$$x^3+ax^2(y-1)+x(by^2+cy+b)+\overline{y-1}^3=0;$$

and, since it must be unaltered by interchanging x and y, it reduces to

$$x^3+3x^2(y-1)+x(3y^2+cy+3)+\overline{y-1}^3=0.$$

But $$\sqrt[4]{\kappa\lambda}=2q^{\frac{3}{4}}(1-q+2q^2\ldots)(1-q^5+2q^6\ldots)$$
$$=2q^{\frac{3}{4}}(1-q+2q^2\ldots),$$
$$\sqrt[4]{\kappa'\lambda'}=1-2q+2q^2\ldots;$$

hence $$x=16q^3(1-4q+14q^2\ldots),$$
$$y=1-8q+32q^2\ldots.$$

Substituting and equating to zero the coefficient of q^3, we have at once

$$16(6+c)-8^3=0,$$

therefore $$c=26,$$

therefore $$x^3+3x^2(y-1)+x(3\,\overline{y-1}^2+32y)+\overline{y-1}^3=0,$$

therefore $$\overline{x+y-1}^3=-32xy,$$
$$x+y-1=-2\sqrt[3]{4xy},$$

or $$\kappa\lambda+\kappa'\lambda'+2\sqrt[3]{4\kappa\lambda\kappa'\lambda'}=1,$$

Schröter's form of the modular equation.

8. The next cases I shall consider are those for $n=11$, and $n=23$, inasmuch as the degree of the modular equation in each case is 3.

For $n=11$, we have $\frac{n+1}{8}=\frac{3}{2}$; that is, $p=3$, $s=2$; hence the modular equation is a cubic in $\sqrt{\kappa\lambda}$ and $\sqrt{\kappa'\lambda'}$. Denoting these by x and y respectively, we have, as in the case of $n=5$,

$$x^3+3x^2(y-1)+x(3y^2+cy+3)+\overline{y-1}^3=0,$$
$$x=\sqrt{\kappa\lambda}=4q^3(1-q+2q^3\ldots)^2(1-q^{11}+2q^{22}\ldots)^2$$
$$=4q^3(1-2q\ldots),$$
$$y=\sqrt{\kappa'\lambda'}=(1-2q+2q^2\ldots)^2(1-2q^{11}+2q^{2}\ldots^{2})^2$$
$$=1-4q+\ldots.$$

Substituting in the modular equation, and equating to zero the coefficient of q^3, we have

$$4(c+6)-4^3=0,$$

$$c=10.$$

Therefore $\quad x^3+3x^2(y-1)+x(3\,\overline{y-1}^2+16y)+\overline{y-1}^3=0,$

therefore $\quad \overline{x+y-1}^3+16xy=0,$

therefore $\quad \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{4\kappa\lambda\kappa'\lambda'}=1;$

and in an exactly similar way, we have, for $n=23$,

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\,\sqrt[12]{\kappa\lambda\kappa'\lambda'}=1,$$

Schröter's forms of these modular equations.

9. The next case in order of simplicity is that of $n=31$; here we have $\dfrac{n+1}{8}=4$, hence the relation between $\sqrt[4]{\kappa\lambda}$ and $\sqrt[4]{\kappa'\lambda'}$ must be of the fourth degree, and must fulfil the condition (19), and must also remain unchanged by interchanging x and y. These conditions reduce the equation at once to the form

$$x^4-4x^3(y+1)+x^2(6y^2+ay+6)-x(4y^3-ay^2-ay+4)+(y-1)^4=0.$$

But $\quad x=2q^4(1-q+2q^3\ldots)(1-q^{31}+2q^{93}\ldots),$

$$y=(1-2q+2q^3\ldots)(1-2q^{31}+2q^{93}\ldots).$$

Hence, substituting and equating to zero the coefficient of y^4, we have

$$-2(8-2a)+2^4=0,$$

therefore $\quad a=0;$

and the equation becomes

$$x^4-4x^3(y+1)+6x^2(y^2+1)-4x(y^3+1)+(y-1)^4=0,$$

which may be written

$$(x^2+y^2+1-2xy-2x-2y)^2=4xy(x+y+1),$$

or $\quad x^2+y^2+1-2xy-2x-2y=2\sqrt{xy}\,\sqrt{x+y+1},$

or $\quad (\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1-2\sqrt[4]{\kappa\lambda\kappa'\lambda'}-2\sqrt[4]{\kappa\lambda}-2\sqrt[4]{\kappa'\lambda'})^2$

$$=4\sqrt[4]{\kappa\lambda\kappa'\lambda'}\,(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1),$$

a form of the modular equation different to Schröter's form.

10. For $n=19$, we have $\dfrac{n+1}{8}=\dfrac{5}{2}$; hence the equation is of the fifth degree in $\sqrt{\kappa\lambda}$ and $\sqrt{\kappa'\lambda'}$, which, as usual, reduces to

$$x^5+5x^4(y-1)+x^3(10y^2+ay+10)+x^2(10y^3+by^2-by-10)$$
$$+x(5y^4+ay^3-by^2+ay+5)+(y-1)^5=0,$$

where $x=\sqrt{\kappa\lambda}=4q^5(1-q+2q^2-3q^3)^2=4q^5(1-2q+5q^2-10q^3\ldots)$,

$$y=\sqrt{\kappa'\lambda'}=\quad(1-2q+2q^2-4q^3)^2=\quad(1-4q+8q^2-16q^3\ldots).$$

We have now to substitute for x and y in the above equation, and equate to zero the coefficients of the powers of q. It will be found necessary to expand only as far as q^7, and the last two terms only need be considered.

They may be written

$$x\left[5\,\overline{y-1}^4+(a+20)(y^3+y)-(b+30)\,y^2\right]+(y-1)^5,$$

or $\quad 4q^5(1-2q+5q^2\ldots)\left[(a+20)(2-16q+80q^2\ldots)\right.$
$$\left.-(b+30)(1-8q+32q^2)\right]-4^5q^5(1-10q+60q^2\ldots).$$

The coefficients of q^5 and q^6 when equated to zero give

$$2(a+20)-(b+30)=4^4,$$

and that of q^7 gives

$$122(a+20)-53(b+30)=4^4\,.\,60,$$

therefore
$$16(a+20)=7\,.\,4^4,$$
$$a+20=112,$$
$$b+30=-32.$$

The assumed equation may be written in such a way as to facilitate the substitution of these numbers, viz.,

$$x^5+5x^4\,\overline{y-1}+x^3\left[10\,\overline{y-1}^2+\overline{a+20}\,.\,y\right]+x^2\left[10\,\overline{y-1}^3+\overline{b+30}\;\overline{y^2-y}\right]$$
$$+x\left[5\,\overline{y-1}^4+\overline{a+20}\;\overline{y^3+y}-\overline{b+30}\,y^2\right]+\overline{y-1}^5=0;$$

which, on substituting for $a+20$ and $b+30$ their values, becomes

$$(x+y-1)^5+112\,(x+y-1)^2xy-256xy\,(xy-x-y)=0,$$

or $\quad(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1)^5+112\,(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1)^2\,\sqrt{\kappa\lambda\kappa'\lambda'}$
$$-256\sqrt{\kappa\lambda\kappa'\lambda'}\,(\sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'})=0,$$

for $n=19$, agreeing with Fiedler's form.

11. We now come to the case of $n=47$, for which $\frac{n+1}{8}=6$, $p=6$, $s=1$. The relation between $\sqrt[4]{\kappa\lambda}$ and $\sqrt[4]{\kappa'\lambda'}$ or x and y must be, according to the principles previously proved, of the form

$$\begin{aligned}x^6-6x^5(y+1)+x^4(15y^2+ay+15)-x^3(20y^3+by^2+by+20)\\ +x^2(15y^4-by^3+cy^2-by+15)\\ -x(6y^5-ay^4+by^3+by^2-ay+6)+(y-1)^6=0,\end{aligned}$$

where
$$\begin{aligned}x&=2q^6(1-q+2q^2-3q^3+4q^4-6q^5+9q^6\ldots),\\ y&=\quad(1-2q+2q^2-4q^3+6q^4-8q^5+12q^6-16q^7\ldots).\end{aligned}$$

This is the first case we have considered in which the calculation presents the slightest difficulty. The process in general is the same, so that, having explained the method in this example, I shall in future simply proceed with the calculation. I may here anticipate that I shall hereafter give another method of finding the constants which will not involve the expansions of $\sqrt[4]{\kappa\lambda}$ and $\sqrt[4]{\kappa'\lambda'}$ according to powers of q.

In order to calculate the values of a and b, we see that the last two terms are the only terms in the equation which contain any terms below q^{12}. Hence we take the coefficients of any two of the terms $q^7, q^8, q^9, \ldots q^{11}$, and equate them to zero. It will always be found that q^7 and q^8 give the same conditions for finding a and b, so that we shall have to proceed as far as q^8.

Now $y-1=-2q(1-q+2q^2-3q^3+4q^4-6q^5+8q^6\ldots)=-2q\rho$,

and
$$x=2q^6(\rho+q^6) \text{ as far as } q^{12};$$

substituting, we have, as far as q^{12},

$$4q^{12}(30-2b+c)-2q^6(\rho+q^6)(6\,\overline{y^5+1}-a\,\overline{y^4+y}+b\,\overline{y^3+y^2})+2^6q^6\rho^6,$$

or, dividing by $\rho+q^6$,

$$4q^{12}(30-2b+c)-2q^6(6\,\overline{y^5+1}-a\,\overline{y^4+y}+b\,\overline{y^3+y^2})+2^6q^6\rho^5(1-q^6)$$
$$\text{as far as } q^{12}.$$

The coefficient of q^6 and q^8, vanishing, give as conditions

$$6-a+b=2^4,$$

and
$$300-34a+26b=2^4.40;$$

therefore
$$a=-10,$$
$$b=0.$$

The coefficient of q^6 in y^5+1 is 3820, and in y^4+y is 1420, and in $\rho^8(1-q^6)$ is 1159; hence we have $c=-14$; and the modular equation reduces to

$$x^6-6x^5(y+1)+x^4(15y^2-10y+15)-20x^3(y^3+1)+x^2(15y^4-14y^2+15)$$
$$-x(6y^5+10y^4+10y+6)+(y-1)^6=0.$$

This equation may be put in a neater form as follows. Put

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1\equiv\ x+y+1\equiv P,$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[4]{\kappa\lambda\kappa'\lambda'}\equiv xy+x+y\equiv Q,$$
$$+\sqrt[4]{\kappa\lambda\kappa'\lambda'}\equiv\quad +xy\quad\equiv R;$$

then the above equation is a symmetric function of x, y, 1, and, when expressed in terms of P, Q, and R, becomes

$$128R^2+R(28P^3-96PQ)-(P^2-4Q)^3=0,$$

or
$$128R^2+4RP^3-(P^2-4Q)^3+24RP(P^2-4Q)=0.$$

$$(4\sqrt[3]{2}\,R^{\frac{2}{3}})^3+(\sqrt[3]{4}\,PR^{\frac{1}{3}})^3-(P^2-4Q)^3$$
$$+3(4\sqrt[3]{2}\,R^{\frac{2}{3}})(\sqrt[3]{4}\,PR^{\frac{1}{3}})(P^2-4Q)=0,$$

therefore
$$4\sqrt[3]{2}\,R^{\frac{2}{3}}+\sqrt[3]{4}\,PR^{\frac{1}{3}}-(P^2-4Q)=0.$$

The modular equation for $n=47$ consequently reduces to

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1-2\sqrt[4]{\kappa\lambda\kappa'\lambda'}-2\sqrt[4]{\kappa\lambda}-2\sqrt[4]{\kappa'\lambda'}$$
$$=\sqrt[3]{4}\,(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1)\sqrt[12]{\kappa\lambda\kappa'\lambda'}+4\sqrt[3]{2}\,\sqrt[6]{\kappa\lambda\kappa'\lambda'},$$

which may be compared with the form given by Hurwitz, *Math. Ann.* 17, page 69.

12. The next case in order of simplicity is that for $n=13$. In this case $\frac{n+1}{8}=\frac{7}{4}$, $p=7$, $s=4$; and therefore the modular equation is of the 7th degree in $\kappa\lambda$, $\kappa'\lambda'$; and, denoting these quantities by x and y, its form is

$$x^7+7x^6(y-1)+x^5(21y^2+ay+21)+x^4(35y^3+by^2-by-35)$$
$$+x^3(35y^4+cy^3+dy^2+cy+35)+x^2(21y^5+by^4+dy^3-dy^2-by-21)$$
$$+x(7y^6+ay^5-by^4+cy^3-by^2+ay+7)+(y-1)^7=0,$$

where
$$x=\kappa\lambda=16q^7(1-4q+14q^2-40q^3+101q^4-236q^5+\dots),$$
$$y=\kappa'\lambda'=(1-8q+32q^2-96q^3+256q^4-624q^5+1408q^6-\dots).$$

If we substitute these values in the modular equation, we see at

once that, in order to determine a, b and c, we only require the last two terms, viz.,

$$x\,(7x^6+ay^5-by^4+cy^3-by^2+ay+7)+\overline{y-1}^7.$$

Equating to zero the coefficients of q^7 and q^9, we obtain immediately two relations between a, b, and c.

They are $2a-2b+c+14 = 8^5.4$, and $4a-b+63 = 8^4.11$.

A third relation is obtained by equating to zero the coefficient of q^{11}; this involves very little labour, but, in order to find d, we require to expand as far as q^{14}, and the labour of doing so is rather unpleasant. I shall therefore explain another method of determining the constants which is practically independent of the expansions according to powers of q.

Since $\Omega = \dfrac{\omega}{7}$, therefore $\dfrac{\Omega}{2} = \dfrac{1}{7}\,\dfrac{\omega}{2}$.

Hence any relation connecting Ω and ω will still remain true if Ω and ω are replaced by $\dfrac{\Omega}{2}$ and $\dfrac{\omega}{2}$.

But if $$\sqrt[4]{\lambda} = \phi\,(\Omega) \quad\text{and}\quad \sqrt[4]{\lambda_1} = \phi\left(\frac{\Omega}{2}\right),$$

we know, from (3), that

$$\lambda_1 = \frac{2\sqrt{\lambda}}{1+\lambda},\quad \lambda_1' = \frac{1-\lambda}{1+\lambda}.$$

Hence $$\kappa_1\lambda_1 = \frac{4\sqrt{\kappa\lambda}}{1+\kappa\,.\,1+\lambda},\quad\text{and}\quad \kappa_1'\lambda_1' = \frac{1-\kappa\,.\,1-\lambda}{1+\kappa\,.\,1+\lambda},$$

therefore $$x_1 = \frac{4\sqrt{x}}{1+x+\sqrt{R}},\quad\text{and}\quad y_1 = \frac{1+x-\sqrt{R}}{1+x+\sqrt{R}},$$

where $$R \equiv \overline{x+1}^2-y^2.$$

Hence, if in the modular equation we replace x and y by

$$\frac{4\sqrt{x}}{1+x+\sqrt{R}}\quad\text{and}\quad\frac{1+x-\sqrt{R}}{1+x+\sqrt{R}},$$

respectively, the relation must still be true. Since x and y are interchangeable, we may equally replace x and y by

$$\frac{4\sqrt{y}}{1+y+\sqrt{R}}\quad\text{and}\quad\frac{1+y-\sqrt{R}}{1+y+\sqrt{R}},$$

respectively, where $$R \equiv \overline{y+1}^2-x^2.$$

I shall make use of this last form. Substituting in the given equation, we have, by putting odd terms on one side, and even terms on the other—

$$\begin{aligned} y^4 \Big[4^7 y^3 &+ 4^5 y^2 \{42(\overline{1+y}^2+R)+ax^2\} + 4^3 y \{70(\overline{1+y}^4+6\,\overline{1+y}^2 R+R^2) \\ &\qquad + 2cx^2(\overline{1+y}^2+R)+dx^4\} \\ &+ 4\,\{14(\overline{1+y}^6+15\,\overline{1+y}^4 R+15\,\overline{1+y}^2 R^2+R^3) \\ &\qquad + 2ax^2(\overline{1+y}^4+6\,\overline{1+y}^2 R+R^2)-2bx^4(\overline{1+y}^2+R)+cx^6\} \Big] \\ = R^{\frac{1}{2}} \Big[4^6 14 y^3 &+ 4^4 y^2 \{70(3\,\overline{1+y}^2+R)+2bx^2\} \\ &+ 4^2 y \{42(5\,\overline{1+y}^4+10\,\overline{1+y}^2 R+R^2) \\ &\qquad + 2bx^2(3\,\overline{1+y}^2+R)+2dx^4\} + 2^7 R^3 \Big]. \end{aligned}$$

Rationalizing, we have an equation of the 14th degree in x, in which there are no odd powers of x. The equation may therefore be considered of the 7th in x^2, and its roots, taking x^2 as the unknown, will be the squares of the roots of the assumed modular equation. The coefficient of x^{12} in the above is easily seen to be

$$\frac{2^{13}.\,14\,(1+y)^2+\{2^{13}(d-b+21)-2^4(2a+2b+c-14)\}\,y}{2^{14}},$$

and equating this to the sum of the squares of the roots of the modular equation, we have

$$2^9(d-b+21) = (2a+2b+c-14)^2-2^{11}(a+56),$$

which determines d in terms of a, b, c.

The coefficient of x^2 is

$$\begin{aligned} 7(1+y)^{12} &+ (14a-2b+210)\,\overline{1+y}^{10} y + (56c+280a-176a+2800)\,\overline{1+y}^8 y^2 \\ &+ 4^2.7\,(8a+10c-16b-24)\,\overline{1+y}^6 y^3 \\ &+ 4^3\,(72a-84b+42c+504)\,\overline{1+y}^4 y^4 \\ &+ 4^4\,(42a-14b+2c+406)\,\overline{1+y}^2 y^5 + 2\,.\,4^5\,(a+56)\,y^6; \end{aligned}$$

equating this to

$$\begin{aligned} (7y^6+ay^5-by^4+cy^3-by^2+ay+7)^2 \\ -2\,(y-1)^7\,(21y^5+by^4+dy^3-dy^2-by-21), \end{aligned}$$

or

$$\begin{aligned} \big[7\,\overline{y+1}^6 &+ \overline{a-42}\,\overline{y+1}^4.\,y-(4a+b-63)\,\overline{1+y}^2 y^2+(2a+2b+c-14)y^3\big]^2 \\ &-2\,\big[\overline{y+1}^8-4^2\,\overline{1+y}^6 y+6\,.\,4^2\,\overline{1+y}^4 y^2-4^4\,\overline{y+1}^2 y^3+4^4\big] \\ &\quad\times\big[21\,\overline{y+1}^4+\overline{b-63}\,\overline{y+1}^2 y+(d-b+21)\,y^2\big], \end{aligned}$$

we see that we get five equations to determine a, b, c, d, from any four of which a, b, c, d can be determined; but, the equations not being linear, the 5th equation determines which solution is to be retained. Thus we see that it is possible to determine completely the modular equation in this case without the assistance of the "q-formulæ," and the same reasoning evidently applies to every case where $\frac{n+1}{8} = \frac{p}{4}$ (p being odd).

In writing down the equations just referred to, I have used a slightly different set of constants, viz., I put

$$a+42 = \alpha, \quad b+105 = \beta, \quad c+140 = \gamma, \quad d-210 = \delta.$$

These constants were suggested by noticing that the modular equation may be written in the form

$$(x+y-1)^7 + xy\,f(x, y) = 0,$$

and when in this form the constants a, b, c, d are always associated with the numbers 42, 105, 140, and 210. The equations for determining the constants are

$$\alpha^2 = 504\alpha - 196\beta + 56\gamma + 2\delta \quad \text{...............(1)},$$

$$\alpha(4\alpha+\beta) = 238\alpha + 882\beta - 553\gamma + 16\delta \quad \text{............(2)},$$

$$(2\alpha+2\beta+\gamma)^2 = 3840\alpha + 1280\beta + 896\gamma + 512\delta \quad \text{.........(3)},$$

$$(4\alpha+\beta)(2\alpha+2\beta+\gamma) = -2912\alpha + 2400\beta + 80\gamma + 256\delta \quad \text{......(4)},$$

and a fifth which I have not written down.

It is rather remarkable that in these equations the absolute terms disappear. From these four equations we can determine $\alpha, \beta, \gamma, \delta$. Inasmuch, however, as two equations are so easily found from the "q-formulæ" in practice, it will be found convenient to utilise them. In the present case they were shown to be

$$2\alpha - 2\beta + \gamma = 8^5 . 4 \quad \text{...........................(5)},$$

and

$$4\alpha - \beta = 8^4 . 11 \quad \text{..............................(6)}.$$

Eliminating δ from (1) and (2) above, we have

$$\alpha(4\alpha-\beta) = 3794\alpha - 2450\beta + 1001\gamma$$

$$= 1001(\gamma - 2\beta + 2\alpha) + 448(4\alpha - \beta).$$

Therefore

$$\alpha . 8^4 . 11 = 1001 . 8^5 . 4 + 448 . 8^4 . 11,$$

$$\alpha = 91 . 32 + 14 . 32,$$

$$\alpha = 32 . 105 \quad \text{.....................................(7)},$$

and from (1), (5), (6), (7) we get

$$\delta = -2^7 . 91.$$

Hence $\alpha = 32 . 105,\ \beta = -8^2 . 494,\ \gamma = 8^2 . 955,\ \delta = -2^7 . 91.$

Substituting these values for $\alpha, \beta, \gamma, \delta$, and noticing that the modular equation is a symmetric function of the quantities $x, y, -1$, it may easily be reduced to the form

$$2^{16}R^2P - 2^5R\,(105P^4 - 2^7 11 P^2Q + 2^{13}Q^2) + P^7 = 0,$$

where
$$P = x+y-1 = \kappa\lambda + \kappa'\lambda' - 1,$$
$$Q = xy - x - y = \kappa\lambda\kappa'\lambda' - \kappa\lambda - \kappa'\lambda',$$
$$R = -xy = -\kappa\lambda\kappa'\lambda',$$
and
$$P+Q+R+1 \equiv 0.$$

The modular equations in the cases where $\frac{n+1}{8} = \frac{p}{4}$ are, of course, much the most difficult to obtain by the "q-formulæ," but we can always determine very easily a few of the constants by that method, and then apply the principle explained in the last section to determine the remainder. In applying the principle we shall find it convenient to reject in the rationalisation those terms which do not bear on the final result, as we never require more than the coefficients of x^{2p-4} and x^2.

13. The theory developed in the last section applies equally to the case where $\frac{n+1}{8} = \frac{p}{2}$, as in the case $\frac{n+1}{8} = \frac{p}{4}$.

The modular equation in the former case is a relation between $\sqrt{\kappa\lambda}$ and $\sqrt{\kappa'\lambda'}$; hence for x^2 and y^2 we substitute

$$\frac{4y}{1+y^2+\sqrt{R}} \text{ and } \frac{1+y^2-\sqrt{R}}{1+y^2+\sqrt{R}} \text{ respectively,}$$

where
$$R = (y^2+1)^2 - x^4.$$

Put
$$1+y^2-\sqrt{R} \equiv N,\quad 1+y^2+\sqrt{R} \equiv D,$$

then
$$ND = x^4;$$

and, when we make the substitution, the result takes the form

$$y^{\frac{1}{2}}F = (\sqrt{N} - \sqrt{D})\,\Phi,$$

where F and Φ are rational functions of x^2 and y. On squaring, we obtain an equation of degree $2p$, but which contains no odd powers

of x. The roots of this equation, considering x^2 as the unknown quantity, are the squares of the roots of the assumed modular equation.

Hence we have a means, as in the case of $\frac{n+1}{8} = \frac{p}{4}$, of establishing relations among the constants.

This will be made more clear by reconsidering the case of $n = 19$, the method being perfectly general. The equation is

$$x^5+5x^4(y-1)+x^3(10y^2+ay+21)+x^2(10y^3+by^2-by-10)$$
$$+x(5y^4+ay^3-by^2+ay+5)+(y-1)^5 = 0;$$

and, making the above substitutions, we have

$$y^4\left[2^5y^2+2^3y\,(10\,\overline{N+D}+ax^2)+2\,(5\,\overline{N^2+D^2}+ax^2\,\overline{N+D}-bx^4)\right]$$
$$+(\sqrt{N}-\sqrt{D})\left[5\,.\,2^4y^2+2^3y\left\{10\,\overline{N+D}+(b+6)\,x^2\right\}\right.$$
$$\left.+(N^2+D^2-4x^2\,\overline{N+D}+6x^4)\right] = 0.$$

On transposing and squaring we have an equation of 10th degree containing only even powers of x. We proceed therefore exactly as in the case of $n = 13$.

14. We have seen that, when the modular equation is of an odd degree in x, y (whether x represents $\sqrt[4]{\kappa\lambda}$, $\sqrt{\kappa\lambda}$, or $\kappa\lambda$), then it is a symmetric function of the quantities $x, y, -1$, and its form can therefore be written down in terms of P, Q, R, where

$$P = x+y-1, \quad Q = xy-x-y, \quad R = -xy.$$

For example, the form of the modular equation for $n = 17$ is

$$aR^3+R^2(\beta P^3+\gamma PQ)+R\,(\delta P^6+\epsilon P^4Q+\theta P^2Q^2+\phi Q^3)+P^9 = 0,$$

x and y being $\kappa\lambda$ and $\kappa'\lambda'$ respectively.

Now, in determining the constants by the method of substitution already explained, we always found that, when we made the coefficient of a certain power of q to vanish in order to determine a constant, then at least the next highest vanished with it. There are, therefore, unnecessary terms somewhere. How are we to get rid of them?

The expressions $\frac{xy-x-y}{(x+y-1)^2}$ *and* $\frac{-xy}{(x+y-1)^3}$ *contain only even powers of* q.

Employ the notation

$$\sqrt[4]{\kappa} = \phi\,(\omega), \qquad \sqrt[4]{\lambda} = \phi\,(\Omega),$$
$$\sqrt[4]{\kappa'} = \psi\,(\omega), \qquad \sqrt[4]{\lambda'} = \psi\,(\Omega),$$

where we have always taken $\Omega = n\omega$, and therefore $\Omega+n = n\,(\omega+1)$

Now, changing the sign of q is equivalent to replacing ω by $\omega+1$, and therefore Ω by $\Omega+n$, and therefore $\sqrt[4]{\kappa\lambda}$ and $\sqrt[4]{\kappa'\lambda'}$ by $e^{\frac{1}{4}(n+1)\pi}\sqrt[4]{\frac{\kappa\lambda}{\kappa'\lambda'}}$ and $\frac{1}{\sqrt[4]{\kappa'\lambda'}}$, and therefore x and y by $-\frac{x}{y}$ and $\frac{1}{y}$. But this change makes no change whatever in the expressions

$$\frac{xy-x-y}{(x+y-1)^2} \text{ and } \frac{-xy}{(x+y-1)^2};$$

hence these can contain only even powers of q.

This is most important as it reduces very much the labour of expansion.

15. It is not difficult to obtain expressions for $\frac{Q}{P^2}$ and $\frac{R}{P^4}$, in terms of q^2. We have

$$\sqrt{\kappa}=\frac{\mathrm{H}\,(\mathrm{K})}{\Theta\,(\mathrm{K})},\quad \sqrt{\kappa'}=\frac{\Theta\,(0)}{\Theta\,(\mathrm{K})},\quad \sqrt{\lambda}=\frac{\mathrm{H}\,(\Lambda)}{\Theta\,(\Lambda)},\quad \sqrt{\lambda'}=\Theta\,(0,\lambda).$$

Hence, if x and y denote $\kappa\lambda$ and $\kappa'\lambda'$, as in $n=17$, we have

$$x=\frac{\mathrm{H}^2\,(\mathrm{K})\,\mathrm{H}^2\Lambda}{\Theta^2(\mathrm{K})\,\Theta^2\,(\Lambda)},\quad y=\frac{\Theta^2\,(0,\kappa)\,\Theta^2\,(0,\lambda)}{\Theta^2\,(\mathrm{K})\,\Theta^2\,(\Lambda)},$$

therefore

$$\frac{Q}{P^2}=\frac{\left\{\begin{array}{c}\mathrm{H}^2(\mathrm{K})\,\mathrm{H}^2(\Lambda)\,\Theta^2(0,\kappa)\,\Theta^2(0,\lambda)-\Theta^2\,(\mathrm{K})\,\Theta^2\,(\Lambda)\,\mathrm{H}^2\,(\mathrm{K})\,\mathrm{H}^2\,(\Lambda)\\ -\Theta^2(\mathrm{K})\,\Theta^2(\Lambda)\,\Theta^2(0,\kappa)\,\Theta^2(0,\lambda)\end{array}\right\}}{\left[\mathrm{H}^2(\mathrm{K})\,\mathrm{H}^2(\Lambda)+\Theta^2(0,\kappa)\,\Theta^2(0,\lambda)-\Theta^2\,(\mathrm{K})\,\Theta^2(\Lambda)\right]^2}.$$

Now $\mathrm{H}^2(\mathrm{K})\,\mathrm{H}^2(\Lambda)=16q^9\,(1+q^2+q^6+q^{12}+q^{20}+\dots)^2(1+q^{34}+\dots)^2$,

a function of odd powers; again

$$\Theta\,(0,\kappa)=1+2q^4+2q^{16}+\dots-(2q+2q^9+2q^{25}+\dots),$$

and $$\Theta\,(\mathrm{K})=1+2q^4+2q^{16}+\dots+(2q+2q^9+2q^{25}+\dots).$$

Hence it is obvious that $\Theta^2(0,\kappa)\,\Theta^2(0,\lambda)-\Theta^2(\mathrm{K})\,\Theta^2(\Lambda)$ is a function of odd powers only; we see therefore that

$$\frac{Q}{P^2}=\frac{f(q^2)}{q^2},\ \text{ and }\ \frac{R}{P^4}=q^6F\,(q^2).$$

Now the modular equation for $n=17$ may be written

$$\alpha\frac{R^2}{P^6}+\frac{R}{P^4}\left(\beta+\gamma\frac{Q}{P^2}\right)+\left(\delta+\epsilon\frac{Q}{P^2}+\theta\frac{Q^2}{P^4}+\phi\frac{Q^3}{P^6}\right)+\frac{P^4}{R}=0;$$

therefore $$\alpha q^{18}F^2(q^2)+q^{10}F(q^2)\left\{\beta q^2+\gamma f(q^2)\right\}$$
$$+\left\{\delta q^6+\epsilon q^4 f(q^2)+\theta q^2 f^2(q^2)+\phi f^3(q^2)\right\}+\frac{1}{F(q^2)}=0;$$

we therefore require the values of $f(q^2)$ and $\dfrac{1}{F(q^2)}$ as far as q^{18}, and $F(q^2)$ as far as q^8. These expressions are not at all difficult to obtain by the above method, and when found there remains no difficulty in determining the constants.

I shall merely state that in a similar manner it can be shown that $\dfrac{Q}{P^2}$ and $\dfrac{R}{P^3}$ are always functions of even powers of q, where x represents $\kappa\lambda$, $\sqrt{\kappa\lambda}$, or $\sqrt[4]{\kappa\lambda}$; only that we must remember that, when the degree of the equation is even, P, Q, and R denote the quantities $x+y+1$, $xy+x+y$, and xy.

The principles stated in the last section reduce the labour of calculation, unless in very difficult cases, within very narrow limits, inasmuch as we have got rid of unnecessary terms.

There will be no difficulty found in calculating the constants for $n=13$ by this method. I had not, however, noticed these facts until after the calculation had been performed. I shall therefore leave it in its present state. I also leave the completion of the discussion for $n=17$, until I have more time at my disposal. There is absolutely no difficulty about the subject, except occasionally a long numerical calculation.

16. The modular equation, in the case of $n=17$, I have found to be
$$2^{21}.3^3R^3+2^{10}R^2(7309P^3-2^8.117PQ)$$
$$+2^6.R(-287P^6+2^5.261P^4Q-2^{12}.15P^2Q^2+2^{17}Q^3)+P^9=0.$$
In endeavouring to discover some short method of determining the constants, I came across the following plan of solving these equations when we put $\kappa=\lambda'$ and $\kappa'=\lambda$.

It has been already explained, in connexion with the case of $n=13$, that, if in the modular equation $\left(\text{where } \dfrac{n+1}{8}=\dfrac{p}{4}\right)$ we suppose that equation arranged in powers of x, and then replace x and y by
$$\frac{4y^{\frac{1}{2}}}{1+y+\sqrt{R}} \text{ and } \frac{1+y-\sqrt{R}}{1+y+\sqrt{R}} \text{ respectively,}$$
we obtain, on rationalising, an equation which, when arranged in powers of x^2, has for roots the squares of the roots of the original.

This gives us an identity, and on comparing coefficients enables us to establish conditions amongst the constants.

Let us consider what form this identity takes when in it we assume $y = -1$. In this case

$$\frac{4y^3}{1+y+\sqrt{R}} \equiv \frac{4}{x} \quad\text{and}\quad \frac{1+y-\sqrt{R}}{1+y+\sqrt{R}} \equiv -1.$$

Hence, if in the assumed modular equation we put $y = -1$, and then change x into $\frac{4}{x}$, we obtain an equation whose roots can only differ in sign from the original equation. Thus, $f(x, y) = 0$ being the modular equation, we see that

$$f(x, -1) = 0 \quad\text{and}\quad f\left(\frac{4}{x}, -1\right) = 0$$

have roots which can only differ in sign. Of course it is not necessary that they should all differ in sign. Assuming $x = 2z$, this amounts to saying that

$$f(z, -1) = 0, \quad\text{and}\quad f\left(\frac{1}{z}, -1\right) = 0$$

have roots that can only differ in sign.

Now the modular equation for $n = 17$ is, in general,

$$aR^3 + R^2(\beta P^3 + \gamma PQ) + R(\delta P^6 + \epsilon P^4 Q + \theta P^2 Q^2 + \phi Q^3) + P^9 = 0.$$

Putting $y = -1$ and $x = 2z$, this becomes

$$a.2^3.z^3 + 2^2 z^2 (2^3\beta\,\overline{z-1}^3 + 2\gamma\,\overline{z-1}\;\overline{1-4z})$$

$$+ 2z(\delta.2^6\,\overline{z-1}^6 + 2^4\epsilon\,\overline{z-1}^4\;\overline{1-4z} + 2^2\theta\,\overline{z-1}^2\;\overline{1-4z}^2 + \phi\,\overline{1-4z}^3)$$

$$+ 2^9\,\overline{z-1}^9 = 0,$$

or

$$f(z, -1) = 0.$$

Changing z into $\frac{1}{z}$, we have

$$z^9 f\left(\frac{1}{z}, -1\right) \equiv a.2^3.z^6 + 2^2.z^4(2^3\beta\,\overline{1-z}^3 + 2\gamma z\,\overline{1-z}\;\overline{z-4})$$

$$+ 2z^2(\delta.2^6\,\overline{1-z}^6 + 2^4\epsilon z\,\overline{1-z}^4\,\overline{z-4} + 2^2\theta z^2\,\overline{1-z}^2\,\overline{z-4}^2 + \phi z^3\,\overline{z-4}^3) + 2^9(1-z)^9.$$

But this is exactly the expression we obtain if in the modular equation we wrote $\kappa = \lambda'$ and $\kappa' = \lambda$, and then denote $2\kappa\kappa'$ by z. Hence, then, to solve the modular equation under these last circumstances,

we have only to solve

$$f\left(\frac{1}{z}, -1\right) = 0,$$

$$f(z, -1) \equiv z^9-9z^8-4556z^7-106164z^6-16082z^5+30338z^4$$
$$+103428z^3-7108z^2+153z-1,$$

$$z^9 f\left(\frac{1}{z}, -1\right) \equiv z^9-153z^8+7108z^7-103428z^6-30338z^5+16082z^4$$
$$+106164z^3+4556z^2+9z-1.$$

It will be found at once that these expressions have as a common factor $(z-1)(z^4-80z^3-98z^2-80z+1)$, and that

$$f\left(\frac{1}{z}, -1\right) \equiv (z-1)(z^4-80z^3-98z^2-80z+1)(z^2-36z-1)^2,$$

and therefore

$$f(z, -1) \equiv (z-1)(z^4-80z^3-98z^2-80z+1)(z^2+36z-1)^2.$$

If therefore we put $\kappa=\lambda'$ and $\kappa'=\lambda$ in the modular equation for $n=17$ we obtain, for determining $2\kappa\kappa' \equiv z$, the equation

$$(z-1)(z^2-36z-1)^2(z^4-80z^3-98z^2-80z+1) = 0,$$

which can at once be solved, and thence the value of $2\kappa\kappa'$ when $\frac{K'}{K} = \sqrt{17}$. This method will apply to every case where $\frac{n+1}{8} = \frac{p}{4}$, and enable us to solve the equation for $2\kappa\kappa'$ when $\frac{K'}{K} = \sqrt{n}$.

In the above equation the factor $z-1 = 0$ corresponds to $K'/K = 1$, the factor $z^2-36z-1 = 0$ to $K'/K = \sqrt{13}$, and the reciprocal quartic $z^4-80z^3-98z^2-80z+1 = 0$ to $K'/K = \sqrt{17}$.

It is scarcely necessary to add any explanations to the following Table. It is given purely for ease of reference. Many of the results are of course old.

I did not take the trouble to calculate the equations for determining $2\kappa\kappa'\left(\text{when } \frac{K'}{K} = \sqrt{n}\right)$ for any difficult cases, except $n=13$ and $n=17$.

It is most remarkable that the factor $z^2-36z-1$, which corresponds to $\frac{K'}{K} = \sqrt{13}$, should occur in the case for $n = 17$.

Again, I believe that, $z^2+14z+1$, which occurs in the case of $\frac{K'}{K} = \sqrt{13}$, determines $2\kappa\kappa'$ when $\frac{K'}{K} = \sqrt{9} = 3$.

n	x	y	P	Q	R	Modular Equation.	$\frac{K'}{K}$	s	Equation for determining s.
1	$\kappa\lambda$	$\kappa'\lambda'$	$x+y-1$	$xy-x-y$	$-xy$	$P=0$	1	$2\kappa\kappa'$	$s-1=0$
3	$\sqrt{\kappa\lambda}$	$\sqrt{\kappa'\lambda'}$	$x+y-1$	$xy-x-y$	$-xy$	$P=0$			
5	$\kappa\lambda$	$\kappa'\lambda'$	$x+y-1$	$xy-x-y$	$-xy$	$P^3-32R=0$	$\sqrt{5}$	$2\kappa\kappa'$	$(s+1)(s^2+4s-1)=0$
5	...	...	....	....	......	$\kappa\lambda+\kappa'\lambda'+2\sqrt[3]{4\kappa\lambda\kappa'\lambda'}=1$			
7	$\sqrt[4]{\kappa\lambda}$	$\sqrt[4]{\kappa'\lambda'}$	$x+y-1$	$xy-x-y$	$-xy$	$P=0$			
11	$\sqrt{\kappa\lambda}$	$\sqrt{\kappa'\lambda'}$	$x+y-1$	$xy-x-y$	$-xy$	$P^3-16R=0$			
11	...	...	....	....	......	$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{4\kappa\lambda\kappa'\lambda'}=1$			
13	$\kappa\lambda$	$\kappa'\lambda'$	$x+y-1$	$xy-x-y$	$-xy$	$P^7-2^5R(105P^4-2^7.11P^2Q+2^{12}Q^2)+2^{16}R^2P=0$ ...	$\sqrt{13}$	$2\kappa\kappa'$	$(s+1)(s^2+14s+1)^2(s^2+36s-1)=0$
17	$\kappa\lambda$	$\kappa'\lambda'$	$x+y-1$	$xy-x-y$	$-xy$	$P^9+2^6R(-287P^6+2^5.261P^4Q-2^{12}.15P^2Q^2+2^{17}.Q^3)$ $+2^{10}R^2(7309P^3-2^8117PQ)+2^{21}.3^3R^3=0$	$\sqrt{17}$	$2\kappa\kappa'$	$(s-1)(s^2-36s-1)^2$ $\times(s^4-80s^3-98s^2-80s+1)=0$
19	$\sqrt{\kappa\lambda}$	$\sqrt{\kappa'\lambda'}$	$x+y-1$	$xy-x-y$	$-xy$	$P^5-112P^2R+256QR=0$			
23	$\sqrt[4]{\kappa\lambda}$	$\sqrt[4]{\kappa'\lambda'}$	$x+y-1$	$xy-x-y$	$-xy$	$P^3-4R=0$			
23	...	...	....	....	......	$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'}=1$			
31	$\sqrt[4]{\kappa\lambda}$	$\sqrt[4]{\kappa'\lambda'}$	$x+y+1$	$xy+x+y$	$+xy$	$(P^2-4Q)^2-4PR=0$			
47	$\sqrt[4]{\kappa\lambda}$	$\sqrt[4]{\kappa'\lambda'}$	$x+y+1$	$xy+x+y$	$+xy$	$(P^2-4Q)^3-R(28P^3+96PQ)-128R^2=0$			
47	..	...	....	....	......	$(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1-2\sqrt[4]{\kappa\lambda\kappa'\lambda'}-2\sqrt[4]{\kappa\lambda}-2\sqrt[4]{\kappa'\lambda'})$ $=\sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'}(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1)+4\sqrt[3]{2}\sqrt[6]{\kappa\lambda\kappa'\lambda'}$			

The Algebra of Multi-linear Partial Differential Operators.
By Captain P. A. MacMahon, R.A.

[*Read Dec. 8th*, 1887.]

§ 1.

In Vol. xviii., p. 61, *Proc. Lond. Math. Soc.*, I discussed a linear partial differential operator which was defined by

$$(\mu, \nu;\ m, n) \equiv \sum_{s=0}^{s=\infty} (\mu + s\nu) A_{s,m}\, \partial_{a_{n+s}};$$

where

$$A_{s,m} = \Sigma \frac{(m-1)!}{\kappa_0!\, \kappa_1!\, \kappa_2!\, \kappa_3!\, \dots}\, a^{\kappa_0} b^{\kappa_1} c^{\kappa_2} d^{\kappa_3} \dots \begin{pmatrix} \Sigma\kappa &= m \\ \Sigma t\kappa_t &= s \end{pmatrix}.$$

These operators were shown to form an alternating group, in that the alternant of any two of them resulted in another operator of the same class.

It will be convenient to call the successive operation of two operators P and Q their outer multiplication, and to write it

$$(P)\,(Q),$$

also their symbolic algebraic multiplication may be called their inner multiplication, and may be written

$$(PQ),$$

and the explicit operation of P upon Q, the latter being considered as a function of symbols of quantity only, may, for reasons which will subsequently appear, be termed the symbolic addition of P and Q, and may be written $(P\dagger Q)$.

Of these three operations the second only is in general commutative. We have then

$$(P)(Q) = (PQ) + (P\dagger Q),$$

P and Q being any linear operators whatever, and the main theorem (*loc. cit.*) was that, P and Q being any members of the multi-linear class above defined,

$$(P)(Q) - (Q)(P) = (P\dagger Q) - (Q\dagger P),$$
$$= \text{an operator of the same class.}$$

This result expresses that the alternant of P and Q, viz.

$$(P)(Q)-(Q)(P),$$

is another operator of the same general class, and hence the characteristic property of these operators, that they form an alternating group.

The multiplication theorem which was auxiliary to this result was

$$(\mu', \nu'; m', n')(\mu, \nu; m, n) = \{(\mu', \nu'; m', n').(\mu, \nu; m, n)\}$$
$$+\sum_{\kappa=0}^{\kappa=\infty}\left\{(m'+m-1)\frac{\mu'}{m'}+\kappa\nu'\right\}\{\mu+(n'+\kappa)\nu\}A_{\kappa, m'+m-1}\partial_{a_{n'+n+\kappa}},$$

which by analogy may be written

$$(\mu', \nu'; m', n')(\mu, \nu; m, n) = \{(\mu', \nu'; m', n').(\mu, \nu; m, n)\}$$
$$+\left\{(m'+m-1)\frac{\mu'}{m'}, \nu'; \mu+n'\nu, \nu; m'+m-1, n'+n\right\};$$

the operator last written is a multi-linear operator of six elements which arises from the theorem

$$(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)$$
$$=\left\{(m'+m-1)\frac{\mu'}{m'}, \nu'; \mu+n'\nu, \nu; m'+m-1, n'+n\right\}.$$

In the further development of the algebra, operators of 8, 10, 12, ... elements will arise; in fact

$$(\mu'', \nu''; m'', n'') \dagger \{(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)\}$$
$$=(\mu'', \nu''; m'', n'') \dagger \Big[\ldots+\sum_{s=0}^{s=\infty}\left\{(m'+m-1)\frac{\mu'}{m'}+(n''+s)\nu'\right\}$$
$$\times\{\mu+(n''+n'+s)\nu\}A_{n''+s, m'+m-1}\partial_{a_{n''+n'+n+s}}\Big]$$
$$=\sum_{s=0}^{s=\infty}\left\{(m''+m'+m-2)\frac{\mu''}{m''}+s\nu''\right\}\left\{(m'+m-1)\frac{\mu'}{m'}+n''\nu'+s\nu'\right\}$$
$$\times\{\mu+(n''+n')\nu+s\nu\}A_{s, m''+m'+m-2}\partial_{a_{n''+n'+n+s}};$$

or, finally, we have a formula introducing a multi-linear operator of 8 elements, viz.:

$$(\mu'', \nu''; m'', n'') \dagger \{(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)\}$$
$$=\Big[(m''+m'+m-2)\frac{\mu''}{m''}, \nu''; (m'+m-1)\frac{\mu'}{m'}+n''\nu', \nu';$$
$$\mu+(n''+n')\nu, \nu; m''+m'+m-2, n''+n'+n\Big\}.$$

The next formula involving an operator of 10 elements is, without difficulty, found to be

$$(\mu''', \nu'''; m''', n''') \dagger \left[(\mu'', \nu''; m'', n'') \dagger \left\{(\mu', \nu'; m', n') \dagger (\mu, \nu; m, n)\right\}\right]$$

$$= \left\{\begin{array}{c} (m'''+m''+m'+m-3)\dfrac{\mu'''}{m'''}, \ \nu''' \\ (m''+m'+m-2)\dfrac{\mu''}{m''}+m'''\nu'', \ \nu'' \\ (m'+m-1)\dfrac{\mu'}{m'}+(n'''+n'')\nu', \ \nu' \\ \mu+(n'''+n''+n')\nu, \ \nu \\ m'''+m''+m'+m-3, \ n'''+n''+n'+n \end{array}\right\},$$

where for conciseness the pairs of elements on the right-hand side have been written underneath one another.

By induction, the law of formation of the successive pairs of elements is easily established.

Section 2.

Explicit operation of a six-element upon a four-element operator.

Denoting by P, Q, R, any three four-element operators, we have

$$\begin{aligned}\{(P \dagger Q) \dagger R\} &= \{(P)(Q) \dagger (R)\} - \{(PQ) \dagger (R)\} \\ &= \{P \dagger (Q \dagger R)\} - \{(PQ) \dagger R\},\end{aligned}$$

showing that P, Q, and R are non-associative as regards symbolic addition, and this may be regarded as a theorem either for the expression of the explicit operation of the six-element operator $(P \dagger Q)$ upon the four-element operator R, as a result of explicit operations performed only upon R; or as the expression of the explicit operation of the symbolic product (PQ) upon R by means of explict operations without the prior performance of symbolic multiplication.

We may write the formula as follows,

$$(PQ \dagger R) = \{P \dagger (Q \dagger R)\} - \{(P \dagger Q) \dagger R\},$$

also $(R)(P \dagger Q) - (P \dagger Q)(R) = \{R \dagger (P \dagger Q)\} - \{(P \dagger Q) \dagger R\}$

$$= \{R \dagger (P \dagger Q)\} - \{P \dagger (Q \dagger R)\} + (PQ \dagger R);$$

so that, if R be lineo-linear, its alternant with $(P \dagger Q)$ is expressible by means of two operators of eight elements each.

Section 3.

The general multilinear operator may be expressed in terms of the lineo-linear operators

$$d_\lambda = (1, 0;\ 1, \lambda) = a_0\partial_{a_\lambda} + a_1\partial_{a_{\lambda+1}} + a_2\partial_{a_{\lambda+2}} + \dots,$$

for
$$\partial_{a_\lambda} = \frac{d_\lambda}{a_0} - \frac{h_1 d_{\lambda+1}}{a_0^2} + \frac{h_2 d_{\lambda+2}}{a_0^3} - \frac{h_3 d_{\lambda+3}}{a_0^4} + \dots,$$

where h_s is the product of a_0^s, and the total symmetric function of weight s of the roots of the equation

$$u \equiv a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \dots = 0 \quad (n = \infty);$$

and hence

$$\mu A_{0,m}\partial_{a_n} + (\mu+\nu)A_{1,m}\partial_{a_{n+1}} + (\mu+2\nu)A_{2,m}\partial_{a_{n+2}} + \cdots$$

$$= \mu A_{0,m}\left\{\frac{d_n}{a_0} - \frac{h_1}{a_0^2}d_{n+1} + \frac{h_2}{a_0^3}d_{n+2} - \dots\right\}$$

$$+ (\mu+\nu)A_{1,m}\left\{\frac{1}{a_0}d_{n+1} - \frac{h_1}{a_0^2}d_{n+2} + \frac{h_2}{a_0^3}d_{n+3} - \dots\right\}$$

$$+ (\mu+2\nu)A_{2,m}\left\{\frac{1}{a_0}d_{n+2} - \frac{h_1}{a_0^2}d_{n+3} + \frac{h_2}{a_0^3}d_{n+4} - \dots\right\}$$

$$+ \dots\dots$$

$$= \mu\frac{A_{0,m}}{a_0}d_n + \left\{(\mu+\nu)A_{1,m}\frac{1}{a_0} - \mu\frac{A_{0,m}}{a_0^2}h_1\right\}d_{n+1}$$

$$+ \left\{(\mu+2\nu)\frac{A_{2,m}}{a_0} - (\mu+\nu)\frac{A_{1,m}}{a_0^2}h_1 + \mu\frac{A_{0,m}}{a_0^3}h_2\right\}d_{n+2}$$

$$+ \dots\dots$$

$$+ \left\{(\mu+s\nu)\frac{A_{s,m}}{a_0} - (\mu+s\nu-\nu)\frac{A_{s-1,m}}{a_0^2}h_1 + \dots + (-)^s\mu\frac{A_{0,m}}{a_0^{s+1}}h_s\right\}d_{n+s}$$

$$+ \dots\dots.$$

Now, since
$$\frac{1}{m}u^m = A_{0,m} - A_{1,m}x + A_{2,m}x^2 - \dots,$$

$$u^{-1} = \frac{1}{a_0} + \frac{h_1}{a_0^2}x + \frac{h_2}{a_0^3}x^2 + \dots$$

We find, by multiplication and comparison of the coefficients of x^s,

$$\frac{m-1}{m}A_{s,m-1} = \frac{1}{a_0}A_{s,m} - \frac{h_1}{a_0^2}A_{s-1,m} + \dots (-)^s\frac{h_s}{a_0^{s+1}}A_{0,m},$$

further $$u^{m-1}u' = -A_{1,m}+2A_{2,m}x-3A_{3,m}x^2+\dots,$$
$$u^{m-2}u' = -A_{1,m-1}+2A_{2,m-1}x-3A_{3,m-1}x^2+\dots,$$

therefore

$$(-A_{1,m}+2A_{2,m}x-3A_{3,m}x^2+\dots)\left(\frac{1}{a_0}+\frac{h_1}{a_0^2}x+\dots\right)$$
$$= -A_{1,m-1}+2A_{2,m-1}x-3A_{3,m-1}x^2+\dots,$$

therefore

$$sA_{s,m-1} = s\frac{1}{a_0}A_{s,m}-(s-1)\frac{h_1}{a_0^2}A_{s-1,m}+\dots+(-)^{s-1}\frac{h_{s-1}}{a_0^s}A_{1,m};$$

whence the coefficient of d_{n+s} is

$$\left(\frac{m-1}{m}\mu+s\nu\right)A_{s,m-1}$$

and the operator becomes

$$\sum_{s=0}^{s=\infty}\left(\frac{m-1}{m}\mu+s\nu\right)A_{s,m-1}d_{n+s}.$$

The operator in the theory of pure reciprocants now takes the simpler and, in some respects, more convenient form

$$2ad_1+3bd_2+4cd_3+\dots.$$

Section 4.

The sub-group of Operators of Two Elements.

A special case of the general operator arises when the second element ν is zero; then we may without loss of generality put μ equal to unity, since it is common to every term, and we may represent the operator $(1, 0;\ m, n)$ by the shorter notation (m, n).

The multiplication theorem is

$$(m', n')(m, n) = \{(m', n')(m, n)\} + \frac{m'+m-1}{m'}(m'+m-1, n'+n),$$

an identity in which only operators of two elements occur.

This class thus constitutes an algebraic group in the sense that algebraic operations produce operators which may be always expressed by operators of the same class.

For the alternant of $(\mu', \nu';\ m', n')$ and (m, n), we find

$$(\mu', \nu';\ m', n')(m, n)-(m, n)(\mu', \nu';\ m', n')$$
$$= (\mu_1, \nu_1;\ m'+m-1, n'+n),$$

where $$\mu_1 = (m'+m-1)\left\{\frac{\mu'}{m'} - \frac{1}{m}(\mu'+n\nu')\right\},$$
$$\nu_1 = -\frac{m'-1}{m}\nu'.$$

This alternant will vanish, if $\mu_1 = \nu_1 = 0$.

CASE I.—If $\nu' = 0$, then
$$m' = m, \text{ or } m' = 1-m,$$
leading to the results
$$(m, n')(m, n) - (m, n)(m, n') = 0,$$
$$(1-m, n')(m, n) - (m, n)(1-m, n') = 0.$$

CASE II.—If $m' = 1$, then
$$\frac{\mu'}{\nu'} = \frac{n}{m-1},$$
leading to
$$(n, m-1;\ 1, n')(m, n) - (m, n)(n, m-1;\ 1, n') = 0.$$
Thus, in general, (m, n) has the three commutors
$$(m, n'),$$
$$(1-m, n'),$$
$$(n, m-1;\ 1, n');$$
the alternant is given by
$$\begin{vmatrix} (m', n'), & (m, n) \\ (m', n'), & (m, n) \end{vmatrix} = \frac{(m'+m-1)(m-m')}{m'm}(m'+m-1, n'+n),$$
to which may be added the result
$$\begin{vmatrix} (m'', n''), & (m', n'), & (m, n) \\ (m'', n''), & (m', n'), & (m, n) \\ (m'', n''), & (m', n'), & (m, n) \end{vmatrix}$$
$$= \frac{(m'+m-1)(m-m')}{m'm}\left\{(m'', n'')(m'+m-1, n'+n)\right\}$$
$$+ \frac{(m''+m'-1)(m'-m'')}{m''m'}\left\{(m, n)(m''+m'-1, n''+n')\right\}$$
$$+ \frac{(m+m''-1)(m''-m)}{mm''}\left\{(m', n')(m+m''-1, n+n'')\right\}.$$

SECTION 5.

Let π, ρ_0 be any two members of the alternating group, and further

$$(\pi)(\rho_0)-(\rho_0)(\pi)=\rho_1,$$
$$(\pi)(\rho_1)-(\rho_1)(\pi)=\rho_2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(\pi)(\rho_m)-(\rho\ \ )(\pi)=\rho_{m+1};$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

we have then the known theorem

$$(\pi)^n(\rho_0)=(\rho_0)(\pi)^n+n(\rho_1)(\pi)^{n-1}+\frac{n(n-1)}{2!}(\rho_2)(\pi)^{n-2}+\ldots.$$

If the subject of operation be previously operated upon by ρ_0, we have

$$(\pi)^n(\rho_0)^2=(\rho_0)^2(\pi)^n+n\left\{(\rho_0)(\rho_1)+(\rho_1)(\rho_0)\right\}(\pi)^{n-1}$$
$$+\frac{n(n-1)}{2!}\left\{(\rho_0)(\rho_2)+2(\rho_1)^2+(\rho_2)(\rho_0)\right\}(\pi)^{n-2}+\ldots,$$

which may be symbolically written

$$(\pi)^n(\rho_0)^2=(\rho+\rho)^0(\pi)^n+n(\rho+\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\rho+\rho)^2(\pi)^{n-2}+\ldots,$$

wherein $(\rho+\rho)^t$ denotes

$$(\rho_0)(\rho_t)+t(\rho_1)(\rho_{t-1})+\frac{t(t-1)}{2!}(\rho_2)(\rho_{t-2})+\ldots.$$

Let us assume

$$(\pi)^n(\rho_0)^s=(\overset{s}{\Sigma}\rho)^0(\pi)^n+n(\overset{s}{\Sigma}\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho)^2(\pi)^{n-2}+\ldots,$$

and then $(\pi)^n(\rho_0)^{s+1}=(\overset{s}{\Sigma}\rho)^0(\pi)^n(\rho_0)+n(\overset{s}{\Sigma}\rho)^1(\pi)^{n-1}(\rho_0)$

$$+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho)^2(\pi)^{n-2}(\rho_0)+\ldots,$$

or, by a previous theorem,

$$(\pi)^n(\rho_0)^{s+1}=(\overset{s}{\Sigma}\rho+\rho)^0(\pi)^n+n(\overset{s}{\Sigma}\rho+\rho)^1(\pi)^{n-1}$$
$$+\frac{n(n-1)}{2!}(\overset{s}{\Sigma}\rho+\rho)^2(\pi)^{n-2}+\ldots$$
$$=(\overset{s+1}{\Sigma}\rho)^0(\pi)^n+n(\overset{s+1}{\Sigma}\rho)^1(\pi)^{n-1}+\frac{n(n-1)}{2!}(\overset{s+1}{\Sigma}\rho)^2(\pi)^{n-2}+\ldots;$$

or, the law assumed true for the expansion of $(\pi)^n(\rho_0)^s$ is equally true for the expansion of $(\pi)^n(\rho_0)^{s+1}$.

Hence, by induction, the general law is established.

In particular, if $(\pi) = 0$,

$$(\pi)^n (\rho_0)^s \equiv (\overset{s}{\Sigma}\rho)^n.$$

We may write the result in the form

$$(\pi)^n (\rho_0)^s = (\overset{s}{\Sigma}\rho+\pi)^n;$$

and we easily reach the companion theorem

$$(\rho_0)^s (\pi)^n = (\pi-\overset{s}{\Sigma}\rho)^n.$$

More generally

$$f(\pi)(\rho_0) = (\rho_0)f(\pi)+\frac{\rho_1}{1!}f'(\pi)+\frac{\rho_2}{2!}f''(\pi)+\dots,$$

wherein $f(\pi)$ denotes any rational integral function of π; whence, proceeding as before, we find

$$f(\pi)(\rho_0)^s = f(\overset{s}{\Sigma}\rho+\pi);$$

and also $$(\rho_0)^s f(\pi) = f(\pi-\overset{s}{\Sigma}\rho).$$

Let now $$\phi(\rho_0) = \Sigma A_s(\rho_0)^s,$$

and then $$f(\pi)\,\phi(\rho_0) = \Sigma A_s f(\overset{s}{\Sigma}\rho+\pi);$$

or, if $$f_s = f(\overset{s}{\Sigma}\rho+\pi),$$

and $$\phi(f) = \Sigma A_s f_s,$$

then $$f(\pi)\,\phi(\rho_0) = \phi(f),$$

and $$\phi(\rho_0)\,f(\pi) = \phi(f'),$$

where $$f'_s = f(\pi-\overset{s}{\Sigma}\rho),$$

and $$\phi(f') = \Sigma A_s f'_s.$$

Section 6.

(P), (Q) denoting any two linear operators whatever, we have

$$(P)(Q) = (PQ)+(P\dagger Q),$$

and comparing this with the symmetric function relation

$$\Sigma\alpha^l\,\Sigma\alpha^m = \Sigma\alpha^l\beta^m + \Sigma\alpha^{l+m},$$

or, in the notation of partitions,

$$(l)(m) = (lm)+(l+m),$$

we see that, regarding the symbol $\dagger$ as expressing a symbolic addition, the linear operators (P), (Q) combine according to precisely the same law as single partition symmetric functions; the algebra of the operators is not, however, commutative, and we may in the first instance regard it as the algebra of symmetric functions freed from the restriction of being commutative in the two respects of outer multiplication and addition.

As regards three linear operators

$$(u_1),\ (u_2),\ (u_3),$$

we have the theorems

$$\begin{aligned}
(u_1)(u_2)(u_3) &= (u_1u_2u_3) + (u_1 \dagger u_2,\ u_3) + (u_1 \dagger u_2 \dagger u_3)^* \\
&\quad + (u_1 \dagger u_3,\ u_2) \\
&\quad + (u_2 \dagger u_3,\ u_1), \\
(u_1)\,(u_2u_3) &= (u_1u_2u_3) + (u_1 \dagger u_2,\ u_3) \\
&\quad + (u_1 \dagger u_3,\ u_2), \\
(u_1u_2)\,(u_3) &= (u_1u_2u_3) + (u_1 \dagger u_3,\ u_2) + (u_1u_2 \dagger u_3) \\
&\quad + (u_2 \dagger u_3,\ u_1),
\end{aligned}$$

wherein, in the expansion of $(u_1u_2)(u_3)$, the operator $(u_1u_2 \dagger u_3)$ is formed by multiplying (u_1) and (u_2) symbolically, and adding the result symbolically to (u_3).

It will be observed that, *quâ* the symbol $\dagger$, the suffixes are in numerical order.

Comparing these with the corresponding relations in symmetric functions, we observe perfect coincidence of theory, except in the case of the term $(u_1u_2 \dagger u_3)$.

But, if (u_3) be lineo-linear, this operator vanishes, and there is no longer any exception.

In general, an exception occurs whenever an operator is formed by the explicit operation of a symbolic product of linear operators upon a linear operator.

Any outer multiplication of operators, each of which is either a single linear operator or a symbolic product (inner multiplication) of linear operators, may in general be expanded in a series of symbolic products, each component of which is a linear operator.

---

* Observe that $(u_1 \dagger u_2 \dagger u_3)$ means $u_1 \dagger (u_2 \dagger u_3)$ and not $(u_1 \dagger u_2) \dagger u_3$, and that u_1, u_2, u_3 are associative as regards outer multiplication.

Restricting ourselves, in the first place, to outer multiplications of two operators, we may calculate the set of relations

$$(u_1)(u_2) = (u_1 u_2) + (u_1 \dagger u_2),$$

$$\begin{aligned}(u_1)(u_2 u_3) = (u_1 u_2 u_3) &+ (u_1 \dagger u_2,\ u_3) \\ &+ (u_1 \dagger u_3,\ u_2),\end{aligned}$$

$$\begin{aligned}(u_1 u_2)(u_3) = (u_1 u_2 u_3) &+ (u_1 \dagger u_3,\ u_2) + (u_1 u_2 \dagger u_3) \\ &+ (u_2 \dagger u_3,\ u_1),\end{aligned}$$

$$\begin{aligned}(u_1)(u_2 u_3 u_4) = (u_1 u_2 u_3 u_4) &+ (u_1 \dagger u_2,\ u_3 u_4) \\ &+ (u_1 \dagger u_3,\ u_2 u_4) \\ &+ (u_1 \dagger u_4,\ u_2 u_3),\end{aligned}$$

$$\begin{aligned}&(u_1 u_2)\,(u_3 u_4) \\ = (u_1 u_2 u_3 u_4) &+ (u_1 \dagger u_3,\ u_2 u_4) + (u_1 \dagger u_3,\ u_2 \dagger u_4) + (u_1 u_2 \dagger u_3,\ u_4) \\ &+ (u_1 \dagger u_4,\ u_2 u_3) + (u_1 \dagger u_4,\ u_2 \dagger u_3) + (u_1 u_2 \dagger u_4,\ u_3) \\ &+ (u_2 \dagger u_3,\ u_1 u_4) \\ &+ (u_2 \dagger u_4,\ u_1 u_3),\end{aligned}$$

$$\begin{aligned}(u_1 u_2 u_3)(u_4) = (u_1 u_2 u_3 u_4) &+ (u_1 \dagger u_4,\ u_2 u_3) + (u_1 u_2 \dagger u_4,\ u_3) + (u_1 u_2 u_3 \dagger u_4) \\ &+ (u_2 \dagger u_4,\ u_1 u_3) + (u_1 u_3 \dagger u_4,\ u_2) \\ &+ (u_3 \dagger u_4,\ u_1 u_2) + (u_2 u_3 \dagger u_4,\ u_1),\end{aligned}$$

$$\begin{aligned}(u_1)(u_2 u_3 u_4 u_5) = (u_1 u_2 u_3 u_4 u_5) &+ (u_1 \dagger u_2,\ u_3 u_4 u_5) \\ &+ (u_1 \dagger u_3,\ u_2 u_4 u_5) \\ &+ (u_1 \dagger u_4,\ u_2 u_3 u_5) \\ &+ (u_1 \dagger u_5,\ u_2 u_3 u_4),\end{aligned}$$

$$\begin{aligned}&(u_1 u_2)\,(u_3 u_4 u_5) \\ = (u_1 u_2 u_3 u_4 u_5) &+ (u_1 \dagger u_3,\ u_2 u_4 u_5) + (u_1 \dagger u_3,\ u_2 \dagger u_4,\ u_5) + (u_1 u_2 \dagger u_3,\ u_4 u_5) \\ &+ (u_1 \dagger u_4,\ u_2 u_3 u_5) + (u_1 \dagger u_4.\ u_2 \dagger u_5,\ u_3) + (u_1 u_2 \dagger u_4,\ u_3 u_5) \\ &+ (u_1 \dagger u_5,\ u_2 u_3 u_4) + (u_1 \dagger u_5,\ u_2 \dagger u_3,\ u_4) + (u_1 u_2 \dagger u_5,\ u_3 u_4) \\ &+ (u_2 \dagger u_3,\ u_1 u_4 u_5) + (u_1 \dagger u_4,\ u_2 \dagger u_3,\ u_5) \\ &+ (u_2 \dagger u_4,\ u_1 u_3 u_5) + (u_1 \dagger u_5,\ u_2 \dagger u_4,\ u_3) \\ &+ (u_2 \dagger u_5,\ u_1 u_3 u_4) + (u_1 \dagger u_3,\ u_2 \dagger u_5,\ u_4),\end{aligned}$$

$$
\begin{aligned}
(u_1u_2u_3)(u_4u_5) = (u_1u_2u_3u_4u_5) &+ (u_1 \dagger u_4,\ u_2u_3u_5) + (u_1u_2 \dagger u_4,\ u_3u_5) \\
&+ (u_1 \dagger u_5,\ u_2u_3u_4) + (u_1u_2 \dagger u_5,\ u_3u_4) \\
&+ (u_2 \dagger u_4,\ u_1u_3u_5) + (u_1u_3 \dagger u_4,\ u_2u_5) \\
&+ (u_2 \dagger u_5,\ u_1u_3u_4) + (u_1u_3 \dagger u_5,\ u_2u_4) \\
&+ (u_3 \dagger u_4,\ u_1u_2u_5) + (u_2u_3 \dagger u_4,\ u_1u_5) \\
&+ (u_3 \dagger u_5,\ u_1u_2u_4) + (u_2u_3 \dagger u_5,\ u_1u_4),
\end{aligned}
$$

$$
\begin{aligned}
&+ (u_1 \dagger u_4,\ u_2 \dagger u_5,\ u_3) + (u_1u_2 \dagger u_4,\ u_3 \dagger u_5) + (u_1u_2u_3 \dagger u_4,\ u_5) \\
&+ (u_1 \dagger u_5,\ u_2 \dagger u_4,\ u_3) + (u_1u_2 \dagger u_5,\ u_3 \dagger u_4) + (u_1u_2u_3 \dagger u_5,\ u_4) \\
&+ (u_1 \dagger u_4,\ u_3 \dagger u_5,\ u_2) + (u_1u_3 \dagger u_4,\ u_2 \dagger u_5) \\
&+ (u_1 \dagger u_5,\ u_3 \dagger u_4,\ u_2) + (u_1u_3 \dagger u_5,\ u_2 \dagger u_4) \\
&+ (u_2 \dagger u_4,\ u_3 \dagger u_5,\ u_1) + (u_2u_3 \dagger u_4,\ u_1 \dagger u_5) \\
&+ (u_2 \dagger u_5,\ u_3 \dagger u_4,\ u_1) + (u_2u_3 \dagger u_5,\ u_1 \dagger u_4),
\end{aligned}
$$

$$(u_1u_2u_3u_4)(u_5) = (u_1u_2u_3u_4u_5)$$

$$
\begin{aligned}
&+ (u_1 \dagger u_5,\ u_2u_3u_4) + (u_1u_2 \dagger u_5,\ u_3u_4) + (u_1u_2u_3 \dagger u_5,\ u_4) + (u_1u_2u_3u_4 \dagger u_5) \\
&+ (u_2 \dagger u_5,\ u_1u_3u_4) + (u_1u_3 \dagger u_5,\ u_2u_4) + (u_1u_2u_4 \dagger u_5,\ u_3) \\
&+ (u_3 \dagger u_5,\ u_1u_2u_4) + (u_1u_4 \dagger u_5,\ u_2u_3) + (u_1u_3u_4 \dagger u_5,\ u_2) \\
&+ (u_4 \dagger u_5,\ u_1u_2u_3) + (u_2u_3 \dagger u_5,\ u_1u_4) + (u_2u_3u_4 \dagger u_5,\ u_1) \\
&\qquad + (u_2u_4 \dagger u_5,\ u_1u_3) \\
&\qquad + (u_3u_4 \dagger u_5,\ u_1u_2).
\end{aligned}
$$

In these expansions it will be observed that explicit operation only takes place upon a single linear operator.

It is easy to see that the outer multiplication of two operators, each of which is a symbolic product of linear operators, may be always so expanded.

Considering, in general, the product

$$(u_1u_2u_3 \ldots u_m)\,(v_1v_2v_3 \ldots v_n),$$

there will arise a batch of operators corresponding to every partition of m, and every lower number, into n or fewer parts.

If, for instance, we fix the attention upon the batches corresponding to the partitions of p ($p \overline{\overline{<}}\, m$) into s ($s \overline{\overline{<}}\, n$) parts, we see that the total number of operators which occur in these batches depends, firstly, upon the number of ways in which it is possible to pack up p things in exactly s parcels; and secondly, upon the number of ways in

which s out of n things can be distributed amongst these parcels, one in each parcel.

The number of ways of choosing p out of m things is

$$\frac{m!}{p!\,(m-p)!},$$

and p things can be distributed into s parcels in a number of ways denoted by

$$\frac{1}{s!}\,\Delta^s(0^p),$$

in the notation of the calculus of finite differences.

Further, we can distribute s out of n things amongst these s parcels, one in each parcel in

$$\frac{n!}{(n-s)!} \text{ ways.}$$

Consequently, in the batches corresponding to the two numbers p and s, there will be a number of operators equal to

$$\frac{m!}{p!\,(m-p)!}\,\frac{1}{s!}\,\Delta^s(0^p)\,\frac{n!}{(n-s)!},$$

and in the aggregate of batches corresponding to the number p there will be

$$\frac{m!}{p!\,(m-p)!}\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p)$$

operators.

Giving p all values from 0 to m, we shall obtain the complete number of operators which appear in the expansion; this number thus is

$$\sum_{p=0}^{p=m}\frac{m!}{p!\,(m-p)!}\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p).$$

It will now be shown that this expression has the value

$$(n+1)^m.$$

First consider the summation

$$\sum_{s=1}^{s=n}\frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p),$$

and write

$$\frac{n!}{(n-s)!} = A_n^s$$

$$\frac{1}{s!}\,\Delta\,(0^p) = K_p$$

(see M. Maurice d'Ocagne "Sur une Classe de Nombres remarquables," *American Journal of Mathematics*, Vol. IX., No. 4, p. 366);

and then $$\sum_{s=1}^{s=n} \frac{n!}{s!\,(n-s)!}\,\Delta^s(0^p) = \Sigma A_n^s K_p^s$$

$$= A_n^1 K_p^1 + A_n^2 K_p^2 + A_n^3 K_p^3 + \dots A_n^n K_p^n$$

$$= (1+A_{n-1}^1)\,K_p^1 + (2A_{n-1}^1 + A_{n-1}^2)\,K_p^2 + (3A_{n-1}^2 + A_{n-1}^3)\,K_p^3 + \dots + (nA_{n-1}^{n-1})\,K_p^n$$

$$= K_p^1 + A_{n-1}^1\,(2K_p^2 + K_p^1) + A_{n-1}^2\,(3K_p^3 + K_p^2) + \dots + A_{n-1}^{n-1}\,(nK_p^n + K_p^{n-1})$$

$$= K_{p+1}^1 + A_{n-1}^1\,K_{p+1}^2 + A_{n-1}^2\,K_{p+1}^3 + \dots + A_{n-1}^{n-1}\,K_{p+1}^n$$

$$= n^p \text{ (loc. cit.)}.$$

Consequently the number we are in search of is

$$\sum_{p=0}^{p=m} \frac{m!}{p!\,(m-p)!}\,n^p$$

$$= (n+1)^m.$$

The theorem may be stated as follows :—

"The outer multiplication of two operators, the sinister and dexter being symbolic products of m and n linear operators respectively, may be expressed as a sum of $(n+1)^m$ operators, each of which is a symbolic product of linear operators."

In the case of the dexter being formed wholly of lineo-linear operators the theory is identical with that of the algebraic theory of symmetric functions.

Section 7.

Symbolic Addition of Operators.

Denoting by

$$u_1 u_2 u_3 \dots u_m, \quad v_1 v_2 v_3 \dots v_m,$$

operators of the m^{th} and n^{th} orders obtained by the symbolic multiplication of the linear operators $u_1, u_2 \dots, v_1, v_2 \dots$, we require the expansion of the operator $(u_1 u_2 \dots u_m \dagger v_1 v_2 \dots v_n)$ as a linear function of operators, each of which is a symbolic product of linear operators.

It will be shown that the number of operators occurring in the development is precisely n^m.

Consider a simple case of Leibnitz's theorem, viz., the continued performance of a single linear partial differential operation upon a product of two functions ϕ_1, ϕ_2.

If $(u)^m$ designates m successive operations of u, we have

$$\frac{(u)^m}{m!}\phi_1\phi_2 = \Sigma\frac{(u)^s\phi_1(u)^{m-s}\phi_2}{s!\,(m-s)!}.$$

It is to be proved that a perfectly valid theorem is obtained if herein we write (u^s) in place of $(u)^s$, where, as usual in this paper, (u^s) denotes the operator of the s^{th} order reached by raising u symbolically to the power s.

In fact, the theorem to be proved is

$$\frac{(u^m)}{m!}\phi_1\phi_2 = \Sigma\frac{(u^s)\,\phi_1\,(u^{m-s})\,\phi_2}{s!\,(m-s)!}.$$

In general, the most extended form of Leibnitz's theorem is capable of a similar dual interpretation, which may be established in the following manner:—

Suppose

$u_1, u_2, u_3, \dots u_s$ to be any linear operators whatever,

and put

$$\Theta = \Theta_0\partial_a + \Theta_1\partial_b + \Theta_2\partial_c + \dots = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_s u_s,$$

further let $\phi_1, \phi_2, \dots \phi_m$ be any m functions of $a, b, c, d, \dots,$

and put $\phi = \phi_1\phi_2\phi_3 \dots \phi_m$;

then

$$\phi_t(a+\Theta_0, b+\Theta_1, c+\Theta_2, \dots) = \phi_t + \Theta\phi_t + \frac{(\Theta^2)}{2!}\phi_t + \frac{(\Theta^3)}{3!}\phi_t + \dots,$$

and

$$\phi + \Theta\phi + \frac{(\Theta^2)}{2!}\phi + \frac{(\Theta^3)}{3!}\phi + \dots = \prod_{t=1}^{t=m}\left(\phi_t + \Theta\phi_t + \frac{(\Theta^2)}{2!}\phi_t + \frac{(\Theta^3)}{3!}\phi_t + \dots\right),$$

that is,

$$\begin{aligned}\phi + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_s u_s)\,\phi \\ + \frac{1}{2!}(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_s u_s)^2\,\phi + \dots \\ = \prod_{t=1}^{t=m}\Big\{\phi_t + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_s u_s)\,\phi_t \\ + \frac{1}{2!}(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_s u_s)^2\,\phi_t + \dots\Big\}.\end{aligned}$$

We now compare the coefficients of

$$\lambda_1^{x_1}\,\lambda_2^{x_2}\dots\lambda_s^{x_s}$$

on the two sides of this identity, and obtain a result which may be written in the form :—

$$\frac{(u_1^{\chi_1} u_2^{\chi_2} \dots u_s^{\chi_s})\,\phi}{\chi_1!\,\chi_2!\dots\chi_s!\,\phi} = \Sigma\Sigma\,\frac{(u_1^{a_1} u_2^{a_2} \dots u_s^{a_s})\,\phi_1}{a_1!\,a_2!\dots a_s!\,\phi_1}\;\frac{(u_1^{\beta_1} u_2^{\beta_2} \dots u_s^{\beta_s})\,\phi_2}{\beta_1!\,\beta_2!\dots\beta_s!\,\phi_2}$$

$$\dots\frac{(u_1^{\mu_1} u_2^{\mu_2} \dots u_s^{\mu_s})\,\phi_m}{\mu_1!\,\mu_2!\dots\mu_s!\,\phi_m},$$

where $$a_t+\beta_t+\gamma_t+\dots+\mu_t = \chi_t, \quad (t = 1, 2, \dots s),$$

and the double summation is in regard to every positive integral solution of these s equations and to every permutation of the ϕ's.

When we compare this result with that of Leibnitz, viz. :—

$$\frac{(u_1)^{\chi_1}(u_2)^{\chi_2}\dots(u_s)^{\chi_s}\phi}{\chi_1!\,\chi_2!\dots\chi_s!\,\phi} = \Sigma\Sigma\,\frac{(u_1)^{a_1}(u_2)^{a_2}\dots(u_s)^{a_s}\phi_1}{a_1!\,a_2!\dots a_s!\,\phi_1}\cdot\frac{(u_1)^{\beta_1}(u_2)^{\beta_2}\dots(u_s)^{\beta_s}\phi_2}{\beta_1!\,\beta_2!\dots\beta_s!\,\phi_2}$$

$$\dots\frac{(u_1)^{\mu_1}(u_2)^{\mu_2}\dots(u_s)^{\mu_s}\phi_m}{\mu_1!\,\mu_2!\dots\mu_s!\,\phi_m},$$

we establish its dual character.

Comparing either of these formulæ with the ordinary multinomial theorem, we see at once that the number of terms in the development

is $$m^{\Sigma\chi}.$$

Applying the theorem to the case of symbolic addition, we find in particular

$$(u_1u_2 \dagger v_1v_2) = (v_1,\, u_1u_2 \dagger v_2) + (u_1 \dagger v_1,\, u_2 \dagger v_2)$$
$$+ (v_2,\, u_1u_2 \dagger v_1) + (u_1 \dagger v_2,\, u_2 \dagger v_1)\,;$$

wherein, be it remembered, the operator

$$(v_1,\, u_1u_2 \dagger v_2)$$

is formed—

(i.) By multiplying u_1 and u_2 together symbolically.

(ii.) By then operating with this symbolic product upon v_2, considered as a function of the symbols of quantity only and not of the differential inverses.

(iii.) By finally multiplying the last result by v_1 symbolically.

Also

$$\begin{aligned}(u_1u_2u_3 \dagger v_1v_2) = \;& (v_1,\ u_1u_2u_3 \dagger v_2) + (u_1 \dagger v_1,\ u_2u_3 \dagger v_2)\\ &+ (v_2,\ u_1u_2u_3 \dagger v_1) + (u_1 \dagger v_2,\ u_2u_3 \dagger v_1)\\ &+ (u_2 \dagger v_1,\ u_1u_3 \dagger v_2)\\ &+ (u_2 \dagger v_2,\ u_1u_3 \dagger v_1)\\ &+ (u_3 \dagger v_1,\ u_1u_2 \dagger v_2)\\ &+ (u_3 \dagger v_2,\ u_1u_2 \dagger v_1).\end{aligned}$$

In general, in writing down the expansion of

$$(u_1u_2 \dots u_m \dagger v_1v_2 \dots v_n),$$

we shall obtain a batch of operators corresponding to every partition of m into n or fewer parts, and, as before remarked, the total number of operators is n^m.

SECTION 8.

A very important example of the symbolic interpretation of Leibnitz's formula occurs in the theory of symmetric functions.

In the result

$$\frac{(u^m)}{m!}\,\phi_1\phi_2 \dots \phi_n = \Sigma\,\frac{(u^{s_1})\,\phi_1\,(u^{s_2})\,\phi_2 \dots (u^{s_{n-1}})\,\phi_{n-1}\,(u^{m-\Sigma s})\,\phi_n}{s_1!\ s_2!\ \dots\ s_{n-1}!\ (m-\Sigma s)!},$$

put

$$D_t = \frac{(u^t)}{t!} = \frac{1}{t!}\,(a\partial_b + b\partial_c + c\partial_d + \dots)^t,$$

thus obtaining

$$D_m\phi_1\phi_2 \dots \phi_n = \Sigma D_{s_1}\phi_1 D_{s_2}\phi_2 \dots D_{s_{n-1}}\phi_{n-1} D_{m-\Sigma s}\phi_n.$$

Supposing $\phi_1, \phi_2, \dots \phi_n$ to be symmetric functions of the roots of the equation

$$ax^n - bx^{n-1} + cx^{n-2} - \dots = 0,$$

expressed by means of partitions, the effect of operating with D_t upon any partition containing a symbolic number t, is to take away one such number t; further

$$D_t\,(t) = 1,$$

and any partition, not containing a number t, is obliterated.

Hence the operation of D_m upon a compound symmetric function is performed by picking out the different partitions of m in all possible ways from the partitions of $\phi_1, \phi_2, \phi_3, \dots \phi_n$, one part only at a time from each partition.

Thus
$$
\begin{aligned}
D_6\,(541)(321)(21) = \;& (*\,41)(32\,*)(21)\\
&+(*\,41)(321)(2\,*)\\
&+(5*1)(3*1)(21)\\
&+(5*1)(321)(*\,1)\\
&+(5*1)(34\,*)(2\,*)\\
&+(54\,*)(*\,21)(*\,1),
\end{aligned}
$$

where the asterisks denote the partitions of 6, successively picked out.

The result then is

$$
\begin{aligned}
D_6\,(541)(321)(21) = \;& (41)(32)(21)+(41)(321)(2)\\
&+(51)(31)(21)+(51)(321)(1)\\
&+(51)(32)(2)+(54)(21)(1).
\end{aligned}
$$

From any symmetric function identity, we can, by repeating operations similar to the above, derive a number of other identities.

In particular, in the theory of invariants, we can from any syzygy between covariants derive a number of lower syzygies.

The operation of "decapitation," whether of a single or compound symmetric function of a degree θ, is seen to be merely the performance as above of the operation D_θ.

Also
$$
\begin{aligned}
& D_\theta\, b^\beta c^\gamma d^\delta \ldots, \qquad (\beta+\gamma+\delta+\ldots=\theta),\\
&= D_\theta\,(1)^\beta\,(1^2)^\gamma\,(1^3)^\delta\ldots,\\
&= a^\theta\,(1)^\gamma\,(1^2)^\delta\ldots,\\
&= a^\theta b^\gamma c^\delta\ldots,
\end{aligned}
$$

showing that a symmetric function of degree θ belonging to the equation
$$bx^n-cx^{n-1}+dx^{n-2}-\ldots=0$$

is transformed by the operation of D_θ into one appertaining to the equation
$$ax^n-bx^{n-1}+cx^{n-2}-\ldots=0.$$

Confocal Paraboloids. *By* A. G. Greenhill.

[*Read December 8th,* 1887.]

The geometrical and analytical theory of confocal central quadrics has received considerable attention from its important applications to problems in Hydrodynamics, Electricity, Magnetism, and Attractions; but except for § 154, Chapter x., Vol. I., of Maxwell's *Electricity*, the corresponding theorems and applications of confocal paraboloids have not received special treatment;* and it is the object of this article to develop this mathematical treatment from an independent standpoint.

It will be found analytically interesting and instructive to carry this out, as the elliptic functions required in the general case of confocal central quadrics degenerate in the special case of confocal paraboloids into the ordinary circular and hyperbolic functions; and consequently the problems discussed do not require a knowledge of anything more than the properties of the functions employed in elementary mathematics.

1. Taking the ordinary system of three rectangular axes Ox, Oy, Oz in space, and two points S, S' on the axis of x, each at a distance a from the origin O, then the two foci S and S', and the two coordinate planes zOx, xOy are sufficient to define a system of confocal paraboloids.

Any point A being taken in the axis of x as the vertex of a paraboloid, the two principal sections of the surface made by the coordinate planes zOx and xOy will be the parabolas in these planes, having a common vertex at A and foci S and S' respectively; these parabolas may conveniently be called the *principal* or *directing parabolas* of the paraboloid.

If A is taken anywhere between the foci S and S', the paraboloid will be *hyperbolic*; but if A is taken anywhere beyond S or S' on either side, the paraboloid will be *elliptic*.

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* [I have just received *Parabolische Koordinaten*, von Dr. Karl Baer, Frankfurt a/O, 1888. A. G. G., 19th April, 1888.]

2. Suppose a vertex A_1 taken beyond S on the positive side of the axis of x at a distance from O, which we shall denote by $a \cosh \alpha$; then

$$SA_1 = a(\cosh \alpha - 1) = 2a \sinh^2 \tfrac{1}{2}\alpha,$$

$$S'A_1 = a(\cosh \alpha + 1) = 2a \cosh^2 \tfrac{1}{2}\alpha;$$

and the equations of the directing parabolas in the coordinate planes of zOx and xOy, with common vertex at A_1 and foci S and S', are

$$y = 0, \quad z^2 = 8a \sinh^2 \tfrac{1}{2}\alpha\,(a \cosh \alpha - x),$$

$$z = 0, \quad y^2 = 8a \cosh^2 \tfrac{1}{2}\alpha\,(a \cosh \alpha - x);$$

and therefore the equation of the corresponding elliptic paraboloid is

$$\frac{y^2}{\cosh^2 \frac{1}{2}\alpha} + \frac{z^2}{\sinh^2 \frac{1}{2}\alpha} = 8a(a \cosh \alpha - x) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

For, putting $y = 0$ and $z = 0$ alternately in equation (1), the equations of the corresponding directing parabolas are obtained.

3. Next suppose a vertex A_2 taken between S and S', at a distance from the origin O, which we shall denote by $a \cos \beta$; then

$$A_2S = a(1 - \cos \beta) = 2a \sin^2 \tfrac{1}{2}\beta,$$

$$S'A_2 = a(1 + \cos \beta) = 2a \cos^2 \tfrac{1}{2}\beta.$$

The equations of the directing parabolas being now

$$y = 0, \quad z^2 = 8a \sin^2 \tfrac{1}{2}\beta\,(x - a \cos \beta),$$

$$z = 0, \quad y^2 = 8a \cos^2 \tfrac{1}{2}\beta\,(a \cos \beta - x);$$

the equation of the corresponding hyperbolic paraboloid will be

$$\frac{y^2}{\cos^2 \frac{1}{2}\beta} - \frac{z^2}{\sin^2 \frac{1}{2}\beta} = 8a(a \cos \beta - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These hyperbolic paraboloids will have generating lines, parallel to the asymptotic planes

$$\frac{y^2}{\cos^2 \frac{1}{2}\beta} - \frac{z^2}{\sin^2 \frac{1}{2}\beta} = 0.$$

4. Lastly, suppose a vertex A_3 taken beyond S', at a distance $a \cosh \gamma$ from the origin O; then

$$A_3S = a(\cosh \gamma + 1) = 2a \cosh^2 \tfrac{1}{2}\gamma,$$

$$A_3S' = a(\cosh \gamma - 1) = 2a \sinh^2 \tfrac{1}{2}\gamma;$$

and the equations of the directing parabolas being

$$y = 0, \quad z^2 = 8a \cosh^2 \tfrac{1}{2}\gamma\,(a \cosh \gamma + x),$$

$$z = 0, \quad y^2 = 8a \sinh^2 \tfrac{1}{2}\gamma\,(a \cosh \gamma + x),$$

the equation of the corresponding elliptic paraboloid will be

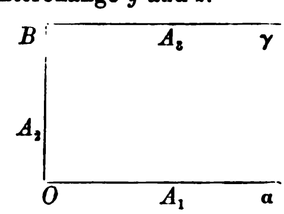
$$\frac{y^2}{\sinh^2 \frac{1}{2}\gamma}+\frac{z^2}{\cosh^2 \frac{1}{2}\gamma}=8a\,(a\cosh\gamma+x)\;\ldots\ldots\ldots\ldots\ldots(3).$$

These equations (1), (2), and (3) represent a system of orthogonal confocal paraboloids in their simplest canonical form; and the parameters α, β, γ are the equivalents of Lamé's thermometric parameters for confocal ellipsoids and hyperboloids.

5. Solving these equations (1), (2), (3) for x, y, z in terms of α, β, γ, we find

$$\left.\begin{aligned} x &= a\,(\cosh\alpha+\cos\beta-\cosh\gamma)\\ y &= 4a\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma\\ z &= 4a\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma \end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

so that to agree with the corresponding expressions given by Maxwell, *Electricity and Magnetism*, Vol. I., p. 190, we must invert the positive direction of the axis of x, and interchange y and z.



The whole series of surfaces and of values of x, y, z is obtained by making α range from ∞ to 0, β from 0 to π, and γ from 0 to ∞.

Since $\qquad \cos\beta=\cosh i\beta,$

$$\cosh\gamma=-\cosh\,(i\pi+\gamma),$$

we may take a period parallelogram of infinite length, open at one end and bounded by the lines

$$y=0,\quad x=0,\quad \text{and}\quad y=\pi;$$

and then the vector of a point moving round the perimeter of the period parallelogram will give the series of confocal paraboloids; the vector being α anywhere on A_1O, $i\beta$ on OA_2B, and $i\pi+\gamma$ on BA_3; so that now, writing α', β', γ' for α, $i\beta$ and $i\pi+\gamma$, we shall obtain the symmetrical expressions for x, y, z in terms of α', β', γ',

$$\left.\begin{aligned} x &= a\,(\cosh\alpha'+\cosh\beta'+\cosh\gamma')\\ y &= -4ia\cosh\tfrac{1}{2}\alpha'\cosh\tfrac{1}{2}\beta'\cosh\tfrac{1}{2}\gamma'\\ z &= -4a\sinh\tfrac{1}{2}\alpha'\sinh\tfrac{1}{2}\beta'\sinh\tfrac{1}{2}\gamma' \end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots(5),$$

so that

$$y+iz=-4ia\left\{\exp\tfrac{1}{2}(\alpha'+\beta'+\gamma')+\exp\tfrac{1}{2}(\alpha'-\beta'-\gamma')\right.$$
$$\left.+\exp\tfrac{1}{2}(-\alpha'+\beta'-\gamma')+\exp\tfrac{1}{2}(-\alpha'-\beta'+\gamma')\right\}.$$

6. The generating lines of the paraboloids are real only on the hyperbolic paraboloids given by equation (2); and their equations are of the form

$$\left.\begin{aligned} \frac{y}{\cos\frac{1}{2}\beta} \pm \frac{z}{\sin\frac{1}{2}\beta} &= 4a\lambda \\ \frac{y}{\cos\frac{1}{2}\beta} \mp \frac{z}{\sin\frac{1}{2}\beta} &= \frac{2(a\cos\beta - x)}{\lambda} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

so that the projections of the generating lines on the plane yOz are two sets of parallel lines inclined at an angle β.

This is well seen in the cardboard model of this surface made by Brill, of Darmstadt, which exhibits the series of different forms of confocal hyperbolic paraboloids made by the deformation of the model and its generating lines, when the angle β between the two sets of parallel planes of cardboard is altered; the focal parabolas being obtained in the two positions in which the model is flattened out.

With the values of x, y, z given in (4),

$$\lambda = \cosh\tfrac{1}{2}\alpha \sinh\tfrac{1}{2}\gamma \pm \sinh\tfrac{1}{2}\alpha \cosh\tfrac{1}{2}\gamma = \sinh\tfrac{1}{2}(\gamma \pm \alpha);$$

so that, keeping β constant, then, along a generating line of the corresponding hyperbolic paraboloid, $\gamma \pm \alpha$ is constant.

7. Employing Maxwell's notation, in Chapter x., *Electricity and Magnetism*, let us denote by ds_1, ds_2, ds_3 the elements of the normal to the surfaces α, β, γ; then

$$\left(\frac{ds_1}{d\alpha}\right)^2 = \left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2,$$

$$\begin{aligned} \frac{1}{a^2}\left(\frac{ds_1}{d\alpha}\right)^2 &= \sinh^2\alpha + 4\sinh^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\sinh^2\tfrac{1}{2}\gamma + 4\cosh^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta\cosh^2\tfrac{1}{2}\gamma \\ &= \sinh^2\alpha + \tfrac{1}{2}(\cosh\alpha - 1)(1+\cos\beta)(\cosh\gamma - 1) \\ &\qquad + \tfrac{1}{2}(\cosh\alpha + 1)(1-\cos\beta)(\cosh\gamma + 1) \\ &= \sinh^2\alpha + 1 - \cos\beta\cosh\gamma + \cosh\gamma\cosh\alpha - \cosh\alpha\cos\beta \\ &= (\cosh\alpha - \cos\beta)(\cosh\alpha + \cosh\gamma); \end{aligned}$$

and, similarly,

$$\frac{1}{a^2}\left(\frac{ds_2}{d\beta}\right)^2 = (\cosh\alpha - \cos\beta)(\cos\beta + \cosh\gamma),$$

$$\frac{1}{a^2}\left(\frac{ds_3}{d\gamma}\right)^2 = (\cosh\alpha + \cosh\gamma)(\cos\beta + \cosh\gamma).$$

Denoting by l_1, m_1, n_1 the direction-cosines of the normal to the surface α, then

$$l_1 = \frac{dx}{ds_1} = \frac{dx}{d\alpha}\bigg/\frac{ds_1}{d\alpha} = \frac{\sinh\alpha}{\sqrt{\{(\cosh\alpha-\cos\beta)(\cosh\alpha+\cosh\gamma)\}}},$$

$$m_1 = \frac{2\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma}{\sqrt{\{(\cosh\alpha-\cos\beta)(\cosh\alpha+\cosh\gamma)\}}},$$

$$n_1 = \frac{2\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma}{\sqrt{\{(\cosh\alpha-\cos\beta)(\cosh\alpha+\cosh\gamma)\}}}.$$

Writing D_1^2 for $\cos\beta+\cosh\gamma$, D_2^2 for $\cosh\alpha+\cosh\gamma$, D_3^2 for $\cosh\alpha-\cos\beta$, by analogy with Maxwell's notation, then l_2, m_2, n_2 and l_3, m_3, n_3 denoting the direction-cosines of the normals to the surfaces β and γ, a similar investigation proves that

$$l_2 = -\frac{\sin\beta}{D_3D_1}, \qquad m_2 = -\frac{2\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma}{D_3D_1},$$

$$n_2 = \frac{2\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma}{D_3D_1};$$

$$l_3 = -\frac{\sinh\gamma}{D_1D_2}, \qquad m_3 = \frac{2\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma}{D_1D_2},$$

$$n_3 = \frac{2\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma}{D_1D_2}.$$

Thus

$$l_2l_3+m_2m_3+n_2n_3 = 0$$

$$l_3l_1+m_3m_1+n_3n_1 = 0,$$

$$l_1l_2+m_1m_2+n_1n_2 = 0;$$

verifying that the surfaces α, β, γ form a triply orthogonal system.

8. Laplace's equation with α, β, γ for variables now becomes (Maxwell, § 148)

$$\nabla^2 V = D_1^2\frac{d^2V}{d\alpha^2}+D_2^2\frac{d^2V}{d\beta^2}+D\,\frac{d^2V}{d\gamma^2} = 0,$$

or

$$\nabla^2 V = (\cos\beta+\cosh\gamma)\frac{d^2V}{d\alpha^2}+(\cosh\gamma+\cosh\alpha)\frac{d^2V}{d\beta^2}$$

$$+(\cosh\alpha-\cos\beta)\frac{d^2V}{d\gamma^2} = 0,$$

Supposing $V = AB\Gamma$, where A is a function of α only, B of β, and Γ of γ, then Laplace's equation is equivalent to

$$\frac{1}{A}\frac{d^2A}{d\alpha^2} = g\cosh\alpha + h, \quad \frac{1}{B}\frac{d^2B}{d\beta^2} = -g\cos\beta - h,$$

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{d\gamma^2} = -g\cosh\gamma + h,$$

three degenerate forms of Lamé's equation.

Laplace's equation is also satisfied if V is a linear function of α, β, γ.

As an Electrostatic example, suppose two elliptic paraboloids, denoted by α_1 and α_2, are electrified to potentials V_1 and V_2 respectively; then the electric potential in the interspace will be

$$V = \frac{\alpha - \alpha_2}{\alpha_1 - \alpha_2} V_1 + \frac{\alpha_1 - \alpha}{\alpha_1 - \alpha_2} V_2;$$

and the electrification σ_1, at any point of the surface α_1, will be given by

$$4\pi\sigma_1 = -\frac{dV}{ds_1} = -\frac{V_1 - V_2}{\alpha_1 - \alpha_2}\frac{d\alpha}{ds_1} = -\frac{V_1 - V_2}{\alpha_1 - \alpha_2}\frac{D_2 D_3}{a};$$

with a similar expression for the electrification σ_2 at any point of the surface α_2.

When two surfaces β_1 and β_2 or γ_1 and γ_2 are electrified to potentials V_1 and V_2, similar expressions to the above hold for the potential in the interspace, and for the electrification on either surface.

With regard to the geometrical interpretation of D_1, D_2, and D_3, we notice that, if A_1, A_2, A_3 denote the vertices of the paraboloids α, β, γ,

$$D_1^2 = a \,.\, A_2A_3, \quad D_2^2 = a \,.\, A_3A_1, \quad D_3^2 = a \,.\, A_1A_2.$$

With confocal central quadrics, D_2 and D_3 are the semi-axes of the central section of α which is conjugate to the diameter through the point of intersection of the surfaces α, β, γ, and these are parallel to ds_2 and ds_3.

But with paraboloids the centre has gone off to an infinite distance and a different geometrical interpretation must be devised.

9. As a Hydrodynamical application, consider the disturbance in the motion of infinite liquid flowing in the direction xO with uniform velocity U, due to the presence of a fixed obstacle in the shape of the elliptic paraboloid α_1; then we must seek to determine a velocity function ϕ satisfying Laplace's equation and also the conditions that

$$\frac{d\phi}{dx} = -U, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0, \quad \text{when } a = \infty,$$

and
$$\frac{d\phi}{ds_1} = 0, \quad \text{when } \alpha = \alpha_1.$$

This can be effected by supposing the velocity function of the form

$$\phi = U(A\alpha - x),$$

obviously satisfying Laplace's equation as the equation of continuity, and then determining A so as to satisfy the boundary conditions.

Now
$$\frac{d\phi}{dx} = U\left(A\frac{d\alpha}{dx} - 1\right) = U\left(Al_1\frac{d\alpha}{ds_1} - 1\right)$$
$$= U\left(\frac{A}{a}\frac{\sinh\alpha}{D_2^2 D_3^2} - 1\right),$$

which when $\alpha = \infty$ becomes $-U$, since $\sinh\alpha/D_2^2D_3^2$ is then ultimately zero; and similarly

$$\frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0, \quad \text{when} \quad \alpha = \infty.$$

Next, when $\alpha = \alpha_1$,

$$\frac{d\phi}{ds_1} = U\left(A\frac{d\alpha}{ds_1} - l_1\right) = 0,$$

so that
$$A = l_1\frac{ds_1}{d\alpha} = a\sinh\alpha_1;$$

and therefore the expression for the velocity function is

$$\phi = U(a\alpha\sinh\alpha_1 - x).$$

Similar investigations for a fixed obstacle in the shape of the hyperbolic paraboloid β_1, will show that the velocity function

$$\phi = U(a\beta\sin\beta_1 - x);$$

while the velocity function when the fixed obstacle is the elliptic paraboloid γ_1 is

$$\phi = U(x - a\gamma\sinh\gamma_1),$$

the liquid being now supposed originally flowing uniformly with velocity U in the direction Ox, in order to flow over the outer convex surface of the paraboloid γ_1.

A paraboloid receiving the stream of liquid on the concave side would stop the stream, instead of merely deflecting it, as the convex side does, and the preceding investigations are no longer applicable.

10. Next suppose the infinite stream of liquid originally flowing parallel to the axis of y with uniform velocity V, and disturbed by the presence of a fixed obstacle in the shape of the elliptic paraboloid α_1; to investigate the form of the velocity function ϕ.

We shall find that the required conditions can be satisfied by supposing ϕ to be composed of two terms, one term being

$$Vy = 4aV\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma,$$

and the other term being of the form

$$AV\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma.$$

For the first term Vy satisfies Laplace's equation, and so also does the second; so that now, putting

$$\phi = V(A\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma - y),$$

we must seek to determine A from the boundary conditions.

As before, $\dfrac{d\phi}{dx} = 0,\ \dfrac{d\phi}{dy} = -V,\ \dfrac{d\phi}{dz} = 0,$ when $\alpha = \infty$;

also $$\frac{d\phi}{ds_1} = V\left(\tfrac{1}{2}A\cosh\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma\frac{d\alpha}{ds_1} - m_1\right) = 0,$$

when $\alpha = \alpha_1$, so that $A = 4a\tanh\tfrac{1}{2}\alpha_1$; and therefore

$$\begin{aligned}\phi &= V(4a\tanh\tfrac{1}{2}\alpha_1\sinh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma - y)\\ &= 4aV\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma(\tanh\tfrac{1}{2}\alpha_1\sinh\tfrac{1}{2}\alpha - \cosh\tfrac{1}{2}\alpha)\\ &= 4aV\operatorname{sech}\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}(\alpha_1-\alpha)\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma.\end{aligned}$$

If the stream was originally parallel to the axis of z, and the same fixed obstacle α_1 was introduced, then we should find, as before,

$$\begin{aligned}\phi &= W(4a\coth\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma - z)\\ &= 4aW\operatorname{cosech}\tfrac{1}{2}\alpha_1\cosh\tfrac{1}{2}(\alpha-\alpha_1)\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma.\end{aligned}$$

The corresponding expressions for the velocity function ϕ when the fixed obstacle is the hyperbolic paraboloid β_1, or the elliptic paraboloid γ_1, are now easily written down; for instance, we shall find, for the surface β_1,

$$\phi = -4aV\sec\tfrac{1}{2}\beta_1\cos\tfrac{1}{2}(\beta-\beta_1)\cosh\tfrac{1}{2}\alpha\sinh\tfrac{1}{2}\gamma,$$

or $$\phi = 4aW\operatorname{cosec}\tfrac{1}{2}\beta_1\cos\tfrac{1}{2}(\beta-\beta_1)\sinh\tfrac{1}{2}\alpha\cosh\tfrac{1}{2}\gamma,$$

according as the liquid was originally streaming parallel to the axis of y or z; while the corresponding expressions for the surface γ_1 are

$$\phi = 4aV\operatorname{cosech}\tfrac{1}{2}\gamma_1\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cosh\tfrac{1}{2}(\gamma-\gamma_1),$$

and $$\phi = -4aW\operatorname{sech}\tfrac{1}{2}\gamma_1\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}(\gamma-\gamma_1).$$

11. Suppose the interspace between α_1 and α_2 filled with liquid, and now suppose the surface α_1 to be moved with velocity U_1, and α_2 with velocity U_2, parallel to Ox; to determine the velocity function ϕ of the initial motion of the liquid filling the interspace.

This is obtained by putting

$$\phi = A\alpha + Bx,$$

thus satisfying Laplace's equation of continuity; and then A and B are determined from the conditions that

$$\frac{d\phi}{ds_1} = U_1 l_1, \quad \text{when} \quad \alpha = \alpha_1;$$

$$\frac{d\phi}{ds_2} = U_2 l_1, \quad \text{when} \quad \alpha = \alpha_2.$$

Now
$$\frac{d\alpha}{ds_1} = l_1 \bigg/ \frac{dx}{d\alpha} = l_1 / a \sinh \alpha,$$

and
$$\frac{dx}{ds_1} = l_1,$$

so that dividing out l_1,

$$U_1 = \frac{A}{a} \operatorname{cosech} \alpha_1 + B,$$

$$U_2 = \frac{A}{a} \operatorname{cosech} \alpha_2 + B,$$

whence A and B are determined. Then

$$\phi = a \frac{\left\{ \begin{array}{c} (U_1 - U_2)\, a \sinh \alpha_1 \sinh \alpha_2 \\ -(U_1 \sinh \alpha_1 - U_2 \sinh \alpha_2)(\cosh \alpha + \cos \beta - \cosh \gamma) \end{array} \right\}}{\sinh \alpha_1 - \sinh \alpha_2}.$$

Supposing $\alpha_1 > \alpha_2$, then, for the infinite liquid outside the surface α_1, the motion due to the velocity U_1 of α_1 is given by the velocity function
$$\phi = aU_1\, \alpha \sinh \alpha_1;$$

while, for the liquid filling the interior of the surface α_2, the velocity function is simply
$$\phi = U_2 x.$$

If the surfaces α_1 and α_2 had been started with velocities V_1 and V_2 parallel to Oy, then we should have to put

$$\phi = (A \sinh \tfrac{1}{2}\alpha + B \cosh \tfrac{1}{2}\alpha) \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma,$$

satisfying the equation of continuity; and then the boundary

conditions
$$\frac{d\phi}{ds_1} = V_1 m_1, \text{ when } \alpha = \alpha_1,$$

$$\frac{d\phi}{ds_1} = V_2 m_1, \text{ when } \alpha = \alpha_2,$$

lead to the equations

$$A \cosh \tfrac{1}{2}\alpha_1 + B \sinh \tfrac{1}{2}\alpha_1 = 4a\, V_1 \sinh \tfrac{1}{2}\alpha_1,$$

$$A \cosh \tfrac{1}{2}\alpha_2 + B \sinh \tfrac{1}{2}\alpha_2 = 4a\, V_2 \sinh \tfrac{1}{2}\alpha_2,$$

for the determination of A and B; so that

$$\phi = 4a \operatorname{cosech} \tfrac{1}{2}(\alpha_1 - \alpha_2)$$
$$\times \left\{ V_1 \sinh \tfrac{1}{2}\alpha_1 \sinh \tfrac{1}{2}(\alpha - \alpha_2) + V_2 \sinh \tfrac{1}{2}\alpha_2 \sinh \tfrac{1}{2}(\alpha_1 - \alpha) \right\} \cos \tfrac{1}{2}\beta \sinh \tfrac{1}{2}\gamma.$$

When the surfaces α_1 and α_2 have initial velocities W_1 and W_2 respectively parallel to Oz, then we must put the velocity function of the liquid in the interspace

$$\phi = (A \sinh \tfrac{1}{2}\alpha + B \cosh \tfrac{1}{2}\alpha) \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma;$$

and determine A and B from the boundary conditions

$$\frac{d\phi}{ds_1} = W_1 n_1, \quad \text{when } \alpha = \alpha_1,$$

$$\frac{d\phi}{ds_1} = W_2 n_1, \quad \text{when } \alpha = \alpha_2;$$

thus leading to the equations

$$A \cosh \tfrac{1}{2}\alpha_1 + B \sinh \tfrac{1}{2}\alpha_1 = 4a W_1 \cosh \tfrac{1}{2}\alpha_1,$$

$$A \cosh \tfrac{1}{2}\alpha_2 + B \sinh \tfrac{1}{2}\alpha_2 = 4a W_2 \cosh \tfrac{1}{2}\alpha_2,$$

for A and B; and finally giving

$$\phi = 4a \operatorname{cosech} \tfrac{1}{2}(\alpha_1 - \alpha_2)$$
$$\times \left\{ W_1 \cosh \tfrac{1}{2}\alpha_1 \cosh \tfrac{1}{2}(\alpha - \alpha_2) - W_2 \cosh \tfrac{1}{2}\alpha_2 \cosh \tfrac{1}{2}(\alpha_1 - \alpha) \right\} \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma.$$

Similar expressions can easily be written down for the motion of the liquid in the interspace between the two surfaces β_1 and β_2 or γ_1 and γ_2, due to arbitrary velocities V_1 and V_2 parallel to Oy, or W_1 and W_2 parallel to Oz, imparted to the surfaces.

12. As another example, suppose liquid filling the interspace of the surfaces β_1 and β_2 to be set in motion by communicating an angular

velocity p_1 to the surface β_1, and an angular velocity p_2 to the surface β_2, each about the axis of x; to determine ϕ, the velocity function of the initial motion of the liquid.

We must make ϕ satisfy the conditions

$$\nabla^2\phi = 0,$$

and $$\frac{d\phi}{ds_2} = \left\{\begin{array}{l}\text{normal component of the velocity of the}\\ \text{surface } \beta_1 \text{ due to the angular velocity } p_1\end{array}\right\}$$

$$= -p_1 zm_2 + p_1 yn_2$$

$$= p_1(yn_2 - zm_2), \quad \text{when } \beta = \beta_1$$

$$\frac{d\phi}{ds_2} = p_2(yn_2 - zm_2), \quad \text{when } \beta = \beta_2,$$

$$= 2ap_2 \frac{\sinh\alpha \sinh\gamma}{D_3 D_1}.$$

The proper form to assume for the velocity function is

$$\phi = (A\cos\beta + B\sin\beta)\sinh\alpha\sinh\gamma,$$

and then, when $\beta = \beta_1$,

$$\frac{d\phi}{ds_2} = \frac{-A\sin\beta_1 + B\cos\beta_1}{aD_3D_1}\sinh\alpha\sinh\gamma$$

$$= \frac{2ap_1}{D_3D_1}\sinh\alpha\sinh\gamma,$$

so that the variable factors $\sinh\alpha\sinh\gamma$ and D_3D_1 cancel, and then

$$-A\sin\beta_1 + B\cos\beta_1 = 2a^2p_1,$$

and similarly $$-A\sin\beta_2 + B\cos\beta_2 = 2a^2p_2,$$

whence A and B can be determined; and then

$$\phi = 2a^2\operatorname{cosec}(\beta_1-\beta_2)\left\{p_2\cos(\beta_1-\beta) - p_1\cos(\beta-\beta_2)\right\}\sinh\alpha\sinh\gamma.$$

If the interspace had been bounded by the surfaces α_1 and α_2, then we should have had

$$\phi = (A\cosh\alpha + B\sinh\alpha)\sin\beta\sinh\gamma,$$

and A and B determined by the equations

$$A\sinh\alpha_1 + B\cosh\alpha_1 = 2a^2p_1,$$

$$A\sinh\alpha_2 + B\cosh\alpha_2 = 2a^2p_2;$$

and then

$$\phi = 2a^2 \operatorname{cosech}(\alpha_1-\alpha_2)\,\{p_1 \cosh(\alpha-\alpha_2) - p_2 \cosh(\alpha_1-\alpha)\} \sin\beta \sinh\gamma.$$

If $p_1 = 0$, and $\alpha_1 = \infty$, then

$$A+B=0,$$

and $$A = -B = -2a^2 p_2 e^{\alpha_2},$$

so that $$\phi = -2a^2 p_2 e^{-\alpha+\alpha_2} \sin\beta \sinh\gamma,$$

the velocity function due to the rotation of the surface α_2 about the axis of x with angular velocity p_1, in infinite liquid surrounding this elliptic paraboloid on the outside.

But, if $\alpha_2 = 0$, then

$$\phi = 2a^2 \operatorname{cosech}\alpha_1 \,\{ p_1 \cosh\alpha - p_2 \cosh(\alpha_1-\alpha)\} \sin\beta \sinh\gamma\,;$$

and, if $p_2 = 0$ also, then

$$\phi = 2a^2 p_1 \operatorname{cosech}\alpha_1 \cosh\alpha \sin\beta \sinh\gamma,$$

the velocity function of liquid inside the elliptic paraboloid α_1; but as pointed out by one of the referees of this paper, this state of motion implies that the focal parabola for which $\alpha = 0$, and therefore $z = 0$,

$$y^2 = 8a\,(a-x),$$

must be looked upon as a fixed boundary.

When this boundary is removed, the motion of the liquid inside the elliptic paraboloid α_1, due to a rotation p_1 about Ox, will be given by a velocity function of the form

$$\phi = Ayz$$
$$= 2a^2 A \sinh\alpha \sin\beta \sinh\gamma,$$

and then we shall find, as before,

$$A = p_1 \operatorname{sech}\alpha_1\,;$$

so that $$\phi = p_1 yz \operatorname{sech}\alpha_1.$$

When the hyperbolic paraboloid β_1 is rotated about Ox with angular velocity p_1, then the motion of infinite liquid, on either side of the surface, is given by the velocity function

$$\phi = p_1 yz \sec\beta_1.$$

13. When the surfaces a_1 and a_2 are made to rotate with angular velocities q_1 and q_2 about the axis Oy, the velocity function of the initial motion in the interspace is more complicated, the boundary conditions being now

$$\frac{d\phi}{ds_1} = q_1\,(zl_1 - xn_1), \text{ when } a = a_1,$$

or

$$\frac{d\phi}{da} = aq_1\,(zl_1 - xn_1)\,D_2D_3$$

$$= 2a^2 q_1 \left\{2\sinh\tfrac{1}{2}a\sinh a - (\cosh a + \cos\beta - \cosh\gamma)\cosh\tfrac{1}{2}a\right\}\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma$$

$$= 2a^2 q_1\,(\cosh a - \cos\beta + \cosh\gamma - 2)\cosh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

when $a = a_1$; and

$$\frac{d\phi}{ds_1} = q_2\,(zl_1 - xn_1), \text{ when } a = a_2,$$

for all values of β and γ.

The form of the velocity function must be inferred by analogy from the corresponding expressions for confocal central quadrics.

Inside the surface a_2, the velocity function of the liquid motion would be of the form

$$\phi = Cxz$$

$$= 4Ca^2\,(\cosh a + \cos\beta - \cosh\gamma)\sinh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma;$$

and, noticing that the terms a and $\cosh\frac{1}{2}a\sin\frac{1}{2}\beta\cosh\frac{1}{2}\gamma$ give the motion of the liquid due to translations parallel to Ox and Oy, we are led to infer that the required velocity function must be built up partly of terms of the form

$$(\cosh a + \cos\beta - \cosh\gamma)\cosh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma$$

and

$$a\sinh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma.$$

But, substituted in Laplace's equation of continuity of § 8, we find

$$\nabla^2(\cosh a + \cos\beta - \cosh\gamma)\cosh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma$$
$$= -2\,(\cos\beta + \cosh\gamma)\cosh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

$$\nabla^2 a\sinh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma = (\cos\beta + \cosh\gamma)\cosh\tfrac{1}{2}a\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma;$$

so that these two terms must be combined in the form

$$\left\{(\cosh a + \cos\beta - \cosh\gamma)\cosh\tfrac{1}{2}a + 2a\sinh\tfrac{1}{2}a\right\}\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

in order that the equation of continuity may be satisfied.

To these terms may be added the terms

$$(\cosh \alpha + \cos \beta - \cosh \gamma) \sinh \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma$$

and

$$(P \cosh \tfrac{1}{2}\alpha + Q \sinh \tfrac{1}{2}\alpha) \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma,$$

obviously satisfying the equation of continuity; so that now in the general case we may put

$$\phi = \{(\cosh \alpha + \cos \beta - \cosh \gamma)(A \cosh \tfrac{1}{2}\alpha + B \sinh \tfrac{1}{2}\alpha)$$
$$+ 2A\alpha \sinh \tfrac{1}{2}\alpha + P \cosh \tfrac{1}{2}\alpha + Q \sinh \tfrac{1}{2}\alpha\} \sin \tfrac{1}{2}\beta \cosh \tfrac{1}{2}\gamma,$$

and now we have sufficient disposable constants A, B, P, Q to satisfy the boundary conditions when $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

Similar expressions can be constructed for the surfaces β_1 and β_2, or γ_1 and γ_2.

14. The velocity function

$$\phi = xyz$$
$$= 2a^3 (\cosh \alpha + \cos \beta - \cosh \gamma) \sinh \alpha \sin \beta \sinh \gamma$$

satisfies the equation of continuity, and gives the motion of the liquid inside a surface due to a torsional strain imparted to the surface about a principal axis.

The velocity function of the motion of the liquid in the interspace between two surfaces, due to arbitrary torsional strains of the surfaces, may then be constructed by analogy with the solution of the corresponding problem for confocal central quadrics, being built up of terms of the form

$$\alpha \sinh \alpha \sin \beta \sinh \gamma,$$

$$(\cosh \alpha + \cos \beta - \cosh \gamma) \begin{matrix}\cosh \\ \sinh\end{matrix}\, \alpha \, \begin{matrix}\cos \\ \sin\end{matrix}\, \beta \, \begin{matrix}\cosh \\ \sinh\end{matrix}\, \gamma,$$

in addition to the terms employed in the previous solutions.

Similar investigations will enable us to determine the induced magnetism in sheets of soft iron, bounded by two confocal paraboloids of the same kind, due to a magnetic field of potential

$$Ax + By + Cz + Pyz + Qzx + Rxy + Sxyz.$$

January 12*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. J. M. Dodds, M.A., Fellow of St. Peter's College, Cambridge; and Mr. G. G. Morrice, M.A., M.B., Trinity College, Cambridge, were elected members; and Mr. E. W. Hobson was admitted into the Society.

The following communications were made:—

The Theory of Distributions: Captain P. A. MacMahon, R.A.

On the Analogues of the Nine-Points Circle in Space of Three Dimensions: S. Roberts, F.R.S.

On a Theorem analogous to Gauss's in Continued Fractions, with applications to Elliptic Functions: L. J. Rogers, M.A.

A Theorem connecting the Divisors of a certain Series of Numbers: Dr. Glaisher, F.R.S.

On Reciprocal Theorems in Dynamics: Prof. H. Lamb, F.R.S.

The following presents were received:—

"Proceedings of the Royal Society," Nos. 259 and 260.

"Educational Times," for January, 1888.

"Nautical Almanac," for 1891.

"Annals of Mathematics," Vol. III., Number 5.

"Bulletin des Sciences Mathématiques," December 1887, and January 1888.

"Annales de l'École Polytechnique de Delft," Tome III., 3me Livr.

"Acta Mathematica," XI., 1.

"Annali di Matematica," Tome XV., Fasc. 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band XI., Stück 11.

"Jahrbuch über die Fortschritte der Mathematik," Band XVII., H. 1.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo I., No. 5.

"Bollettino delle Pubblicazioni Italiane," Nos. 47 and 48.

Pamphlets by Maurice d'Ocagne: "Sur une Classe de Nombres remarquables"; "Sur la Relation entre les Rayons de Courbure de deux Courbes Polaires Réciproques"; "Les Coordonnées parallèles de Points"; "Sur les Courbes Algébriques de degré quelconque"; Quelques Propriétés du Triangle"; "Les Coordonnées Cycliques."

"American Journal of Mathematics," Vol. X., No. 2; Baltimore, January, 1888.

On Reciprocal Theorems in Dynamics.

By HORACE LAMB, M.A., F.R.S.

[*Read Jan.* 12*th*, 1888.]

In a recent paper on Least Action*, von Helmholtz has given a reciprocal theorem of great generality, which may be stated as follows :—

Consider any natural motion of a conservative system between two configurations A and A' through which it passes at times t and t' respectively, and let $t'-t=\tau$. Let $q_1, q_2, \ldots$ be the coordinates of the system, and $p_1, p_2, \ldots$ the component momenta, at time t, and let the values of the same quantities at time t' be distinguished by accents. As the system is passing through the configuration A, let a small impulse δp_r of any type be given to it; and let the consequent alteration in any coordinate q_s after the time τ be denoted by $\delta q'_s$. Next consider the *reversed* motion of the system, in virtue of which it would, if undisturbed, pass from the configuration A' to the configuration A in the time τ. Let a small impulse $\delta p'_s$ be applied as it is passing through the configuration A', and let the consequent change in the coordinate q_r, after a time τ, be δq_r. The theorem in question asserts that

$$\delta q_r : \delta p'_s = \delta q'_s : \delta p_r \quad \text{.............................(1).}$$

If the coordinates q_r, q_s be of the same kind (*e.g.*, both lines or both angles), the statement of the theorem may be simplified by supposing $\delta p'_s = \delta p_r$, in which case

$$\delta q_r = \delta q'_s.$$

In words, the change produced in the time τ by a small initial impulse of any type in the coordinate of any other (or of the same) type, in the *direct* motion, is equal to the change produced in the same time by a small initial impulse of the second type in the coordinate of the first type, in the *reversed* motion.

The proof given by von Helmholtz is based on the properties of Hamilton's "characteristic function"

$$S = \int_t^{t'} (T-V)\,dt \quad \text{..........................(2)}$$

where T and V are the kinetic and potential energies of the system

* Crelle, t. 100, pp. 137 and 213.

and S is supposed expressed in terms of the initial and final coordinates $q_1, q_2, \ldots$ and $q'_1, q'_2, \ldots$, and the time τ. Under these circumstances we have the relations

$$p'_r = \frac{dS}{dq'_r}, \quad p_r = -\frac{dS}{dq_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

from which the theorem is deduced without much difficulty.

Von Helmholtz has also given a second reciprocal theorem, to which reference will be made further on.

In searching for a more general result which should include these theorems (and possibly others) as particular cases, I was led to recognise that the desired generalisation is already contained in a remarkable formula established by Lagrange in the "Mécanique Analytique,"* by way of prelude to his theory of the variation of arbitrary constants. Starting from his equations of motion in generalised coordinates, he proves that

$$\frac{d}{dt}\Sigma\left\{\delta p_r . \Delta q_r - \Delta p_r . \delta q_r\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where the variation-symbols δ and Δ refer to any two slightly disturbed natural motions of the system. To call attention to this somewhat neglected theorem of Lagrange, and to some of the consequences which flow from it, is the main object of this paper.

In the first place, integrating from t to t', we have

$$\Sigma\left(\delta p_r . \Delta q_r - \Delta p_r . \delta q_r\right) = \Sigma\left(\delta p'_r . \Delta q'_r - \Delta p'_r . \delta q'_r\right) \ldots\ldots\ldots\ldots(5).$$

In this form, it may be noted, Lagrange's result follows very readily from the Hamiltonian relations (3). Writing for shortness

$$\frac{d^2 S}{dq_r dq_s} = (r, s), \quad \frac{d^2 S}{dq_r dq'_s} = (r, s'),$$

we have $\qquad \delta p_r = -\Sigma_s (r, s)\, \delta q_s - \Sigma_s (r, s')\, \delta q'_s,$

with a similar expression for Δp_r. Hence

$$\begin{aligned}\Sigma\left(\delta p_r . \Delta q_r - \Delta p_r . \delta q_r\right) &= -\Sigma_r\left\{\Sigma_s (r, s)\, \delta q_s + \Sigma_s (r, s')\, \delta q'_s\right\} \Delta q_r \\ &\quad + \Sigma_r\left\{\Sigma_s (r, s)\, \Delta q_s + \Sigma_s (r, s')\, \Delta q'_s\right\} \delta q_r \\ &= \Sigma_r \Sigma_s (r, s')\left\{\delta q_r . \Delta q'_s - \Delta q_r . \delta q'_s\right\} \ldots\ldots\ldots\ldots(6).\end{aligned}$$

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* Bertrand's edition, t. i., pp. 300 et seq. The theorem appears to have been first published in a memoir read to the Institute, March 13th, 1809 (Cayley, *Report on Theoretical Dynamics*).

The same value is obtained in like manner for the expression on the right-hand side of (5).

The reciprocal theorem above stated is an immediate consequence of Lagrange's formula. For, suppose all the δq to vanish, and likewise all the δp with the exception of δp_r. Again, suppose all the $\Delta q'$ to vanish, and likewise all the $\Delta p'$ except $\Delta p'_s$. The formula (5) then reduces to

$$\delta p_r \,.\, \Delta q_r = -\Delta p'_s \,.\, \delta q'_s,$$

or

$$\delta q'_s : \delta p_r = \Delta q_r : -\Delta p'_s,$$

which is equivalent to Von Helmholtz's result, since we may suppose the symbol Δ to refer to the reversed motion, provided we change the signs of the Δp.

A slight extension may be given to the statement of the reciprocal theorem from the consideration that the expression

$$\Sigma\,(\delta p_r \,.\, \Delta q_r - \Delta p_r \,.\, \delta q_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

is a *covariant*, and that the coordinate systems employed on the two sides of (5) may therefore (if we please) be different. To see this we may recall Hamilton's variational equation

$$\delta S = -E\delta\tau + \Sigma p'_r \,.\, \delta q'_r - \Sigma p_r \,.\, \delta q_r \quad \ldots\ldots\ldots\ldots\ldots (8),$$

in which E denotes the total energy. It is known that $\Sigma p \,.\, \delta q$ is covariant,* and we may therefore suppose the initial and final coordinates in terms of which (with τ) S is expressed to belong to different systems. The relations (3) will still hold, and from these (as we have seen) the proof of Lagrange's formula immediately follows, through equation (6).†

The freedom we have thus gained of using different coordinates in the two configurations contemplated in the reciprocal theorem is of importance in the optical applications to be referred to further on.

Some good illustrations of the theorem are afforded by the case of a single particle. For instance, in elliptic motion about the centre, if a small velocity δv in the direction of the normal be communicated to the particle as it is passing through either extremity of the major axis, the tangential deviation produced after a quarter-period is easily found to be $\delta v/\mu^{\frac{1}{2}}$, where μ is the "absolute force." And it is readily

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* It is in fact $= \Sigma m\,(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z)$, where x, y, z are the rectangular coordinates of any particle m of the system. The proof of this forms part of the usual investigation of Lagrange's equations of motion in generalised coordinates.

† The covariant property of (7) was suggested to me as probable by Mr. J. Larmor. It may also be established by direct transformation of coordinates, but the work is rather long.

verified that a tangential velocity δv, communicated at the extremity of the minor axis, produces after a quarter-period an equal normal deviation $\delta v/\mu^{\frac{1}{2}}$.

Again, in the corpuscular theory of light, take the case of a medium symmetrical about an axis (*e.g.*, an optical system), and consider two points P, P' on the axis, and let V, V' be the corresponding velocities of light. At P let a small impulse be applied at right angles to the axis so as to produce an angular deflection $\delta\theta$, and let β' be the consequent lateral deviation at P'. In like manner, in the reversed motion, let a small deflection $\delta\theta'$ at P' produce a lateral deviation β at P. The reciprocal theorem asserts that

$$\frac{\beta}{V'\delta\theta'} = \frac{\beta'}{V\delta\theta};$$

that is, the "apparent distance"* of P from P' is to that of P' from P in the ratio of the refractive indices at P' and P respectively.

When the restriction as to symmetry about an axis is abandoned, it is convenient to adopt independent systems of coordinates at P and P'. Taking these points as origins of rectangular systems x, y, z and x', y', z' respectively, the axes of z, z' being tangential to the ray, the lateral deviations at P' due to impulses $\delta\dot{x}$, $\delta\dot{y}$ at P will be given by equations of the form

$$\delta x' = A\delta\dot{x} + B\delta\dot{y},$$

$$\delta y' = C\delta\dot{x} + D\delta y.$$

The reciprocal theorem then shows that the deviations at P due to impulses $\delta\dot{x}'$, $\delta\dot{y}'$ at P' will be given by

$$\delta x = A\delta\dot{x}' + C\delta\dot{y}',$$

$$\delta y = B\delta\dot{x}' + D\delta\dot{y}'.$$

Hence, if σ' be the section at P' of a small pencil of rays proceeding from P and forming there a solid angle ω, and if σ, ω' have similar meanings with regard to a pencil from P', we shall have

$$\frac{\sigma'}{V^2\omega} = \frac{d\,(\delta x', \delta y')}{d\,(\delta\dot{x}, \delta\dot{y})} = AD - BC.$$

The same value is obtained for $\sigma/V'^2\omega'$, so that, if μ, μ' be the refractive indices,

$$\frac{\sigma'}{\omega} : \frac{\sigma}{\omega'} = V^2 : V'^2 = \mu'^2 : \mu^2,$$

which is the theorem of "apparent distance" in its extended form.

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* *Cf.* Lord Rayleigh, *Phil. Mag.*, June, 1886, p. 472.

Most, if not all, of the reciprocal relations already known in Dynamics appear to range themselves as particular cases under our present theorem.

Thus, if the system be originally at rest, and if the time τ be infinitely short, we may put

$$\delta q'_s = \dot{q}'_s \cdot \tau,$$

and, on account of the linearity of the relations between the momenta and the velocities, the restriction to infinitely small impulses δp_r may be dropped. Hence the velocity of type s produced by an impulse of type r is equal to the velocity of type r produced by an equal impulse of type s.

Again, applying the theorem to the case of small periodic disturbances from a configuration of equilibrium, we are led to the reciprocal relation which is discussed at length by Lord Rayleigh in his "Theory of Sound."* This includes as particular cases the important principles of acoustical and optical reciprocity formulated long ago by Von Helmholtz, although in the case of a continuous medium some care is necessary to recognise the displacements and impulses of corresponding types. The acoustical principle is to the effect that in a uniform mass of air (or other gas), bounded in whole or in part by rigid or perfectly elastic walls, the variations in density at a point P' due to a simple source of sound at P are identical in amplitude and phase with those produced at P by an equal source at P'. "In this theorem equal sources of sound are those produced by the periodic introduction and abstraction of equal quantities of fluid, or something whose effect is the same." The statement needs modification when the nature of the medium is different at P and at P'; but this is of no great interest in Acoustics.

The optical principle is as follows, in (as nearly as possible) Von Helmholtz's own words :†—"Let a ray of light from the point P arrive after any number of reflections, refractions, &c., at the point P'. At P draw two planes a_1, a_2 through the direction of the ray, and at right angles to one another. Let two similar planes a'_1, a'_2 be drawn through the direction of its ray at P'. The following result may then be proved: If a quantity I of light polarised in the plane a_1 proceeds from P in the direction of the ray in question, and if of this the quantity I' arrives at P' polarised in the plane a'_1; then reciprocally, when a quantity I of light polarised in the plane a'_1 proceeds from P', the same quantity I' of light polarised in the plane a_1 arrives at P."

* T. i., § 107, &c.

† *Physiologische Optik*, p. 169, or *Gesammelte Werke*, t. ii., p. 136.

Von Helmholtz goes on to say that the light may on its path be subject to any amount of single or double refraction, reflection, dispersion, and diffraction.*

The statement may perhaps be made a little more precise, if we define I to mean the *intensity* of a source of light at P, polarised in the plane a_1, as measured by the energy emitted per second, and J the intensity of the component polarised in the plane a_1' of the light which arrives at P', as measured by the energy which falls per second on unit area placed perpendicular to the ray. If I' and J' have similar meanings for light proceeding from P' to P, we have

$$J' : I = J : I'.$$

This supposes the medium to be the same at P' as at P. When this is not the case, we are, I think, obliged to enter into some consideration of the mechanism by which light is propagated, although the final result is independent of the particular theory adopted on this point. To do this here would lead us too far; the result, whether we adopt the elastic solid or the electro-magnetic theory of light, is (I find)

$$\frac{J'}{I} : \frac{J}{I'} = \mu'^2 : \mu^2,$$

where μ, μ' are the refractive indices at P and P', respectively. This law, which is thus proved by purely dynamical reasoning, is identical with that established in other ways, and under somewhat narrower restrictions, by Von Helmholtz and Clausius.

In some further applications of the reciprocal theorem care must be taken that in the "reversed" motion the reversal is complete, and extends to every part of the system. For example, if the system contain gyrostats in rotation, the rotation of each one of these must be reversed.

Again, the propagation of sound in a moving atmosphere has been recognised† as a case to which Von Helmholtz's principle of acoustic reciprocity, as above stated, does not apply. In fact, if P' be to the leeward of P, the intensity at P' due to a source at P will, in consequence of refraction, be greater than that at P due to a source at P' The reciprocity is, however, restored if when we transfer the source to P' we also reverse the wind.

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* He also includes absorption, which does not come within the scope of the general reciprocal theorem which is the subject of this paper. It *is* covered, of course, by Lord Rayleigh's principle, which does not exclude the action of dissipative forces.

† *Theory of Sound*, § 111. *ad fin.*

Another interesting example is furnished by the magnetic rotation of the plane of polarised light. Von Helmholtz himself called attention to the fact that his optical principle of reciprocity does not hold when the ray in its course traverses a medium possessing the rotatory property in virtue of magnetic influence. Now that we have obtained a dynamical basis for the principle, we can assert that, if the phenomenon is susceptible of a dynamical explanation at all, there must be some motion in the medium independently of the luminiferous disturbance. Also the reciprocal relation must necessarily hold if we are able to reverse this latent motion when we transfer the source of light from P to P'. As a matter of fact, it does hold if we reverse the magnetic field. This indicates that the motion in question is of a dipolar character; and the only* motion of this kind which we can associate with the magnetic field is one of rotation about the lines of force. We are thus led by the general theorem of dynamical reciprocity to Sir W. Thomson's well-known argument, in a very compact form.

At the end of the paper on "Least Action," von Helmholtz has given a second reciprocal theorem. In this the motion through the configuration A is supposed to be varied by a slight change in the value of one of the coordinates (say q_r), the momenta being all unaltered, and the subject of the theorem is the consequent variation $\delta p'_s$ in any one of the momenta after the time τ. In proving this theorem, Hamilton's function

$$Q = \int_t^{t'} (T - V + \Sigma \dot{p}_r \, . \, q_r)\, dt,$$

which is supposed expressed in terms of the initial and final *momenta*, and the time τ, is employed. This function possesses the properties

$$\frac{dQ}{dp_r} = q_r, \quad \frac{dQ}{dp'_r} = -q'_r.$$

This second reciprocal theorem, like the former one, is an immediate deduction from the Lagrangian formula (5). Let all the δp vanish, and all the δq save δq_r. Again, let all the $\Delta p'$ vanish, as also the $\Delta q'$ with the exception of $\Delta q'_s$. The formula reduces to

$$-\Delta p_r \, . \, \delta q_r = \delta p'_s \, . \, \Delta q'_s,$$

or

$$\delta p'_s : \delta q_r = -\Delta p_r : \Delta q'_s,$$

* [*May*, 1888. This is perhaps stated too absolutely. For example, an arrangement of vortex rings whose axes are tangential to the lines of force would possess the requisite dipolar quality, however improbable it may be on other grounds.]

which is the theorem in question, if we make the symbol Δ refer to the reversed motion by changing the sign of Δp_r.

As an example, consider an optical system symmetrical about an axis, and let F and F' be the principal foci. The above theorem, applied to the corpuscular hypothesis, shows that the convergence at F' of a parallel beam from F is to the convergence at F of a parallel beam of equal breadth from F' in the inverse ratio of the refractive indices at F' and F respectively. This is equivalent to Gauss's result that the two focal lengths are to one another directly as the corresponding indices.

A third reciprocal theorem may be obtained by making all the δq and all the $\Delta p'$ to vanish, whilst all the δp vanish save δp_r, and all the $\Delta q'$ save $\Delta q'_s$. Under these circumstances, we have

$$\delta p_r \, . \, \Delta q_r = \delta p'_s \, . \, \Delta q'_s,$$

or

$$\delta p'_s : \delta p_r = \Delta q_r : \Delta q'_s.$$

The optical interpretation of this is that the angular divergence at P' of a pencil of rays from P is to its divergence at P as the breadth at P of a parallel beam from P' is to its breadth at P.*

The first reciprocal theorem, however, is perhaps the one most worth preserving in a separate form. For cases which do not immediately come under it, it is best to have recourse directly to the Lagrangian formula. As a final application of this, let P, P' be conjugate foci of an optical system. We may then suppose the δq and the $\delta q'$ all to vanish, and likewise all the δp and $\delta p'$ save those with the suffix r. This leads to

$$\delta p_r \, . \, \Delta q_r = \delta p'_r \, . \, \Delta q'_r.$$

Writing

$$\delta p_r = V . \, \delta\theta, \quad \delta p'_r = V' . \, \delta\theta',$$

$$\Delta q_r = \beta, \quad \Delta q'_r = \beta',$$

where $\delta\theta$ and $\delta\theta'$ are the divergences at P and P' of a ray between these points, and β and β' are the breadths of conjugate images, then

$$V . \, \delta\theta \, . \, \beta = V' . \, \delta\theta' \, . \, \beta',$$

or

$$\mu\beta \, . \, \delta\theta = \mu'\beta' \, . \, \delta\theta',$$

which is the well-known optical law of Lagrange.

* [Provided the refractive indices at P and P' are equal.]

On the Analogues of the Nine-Points Circle in Space of Three Dimensions, and connected Theorems. By Samuel Roberts.

[*Read January* 12*th*, 1888.]

1. When one analogue to a plane theorem exists in space of three dimensions, we are apt to find others which have rival claims. This is so with respect to the nine-points circle in many of its relations to the associated triangle. For two spheres present themselves immediately as entitled *primâ facie* to the rank of analogues. The middle points of the sides of a triangle may be regarded either as the centres of circles described on the sides as diameters, or as the centroids of the sides. We may therefore consider, as corresponding to the nine-points circle, the sphere passing through the centres of the circles circumscribed about the faces of a tetrahedron (or, say, through the centres of the spheres circumscribed about the faces and having their planes respectively for diametral planes), or else the sphere passing through the centroids of the faces.

The difficulty, however, at once occurs, that the altitudes of a tetrahedron form four generators of the same system of a hyperboloid, and do not co-intersect, except when it degenerates to a cone. If, therefore, this common intersection be insisted on, we must forego obtaining any general analogue, or else subject the tetrahedron to conditions.

M. Prouhet, adopting the latter alternative, obtained an analogue in the case of the "orthogonal" tetrahedron, in which the altitudes pass through the same point. This sphere passes through the centroids and orthocentres of the faces. I shall hereafter refer particularly to a paper by Signor Carmelo Intrigila, in which he extends the analogy to the general tetrahedron. His sphere of twelve points passes through the centroids of the faces.

2. There appear to be some reasons for giving the first rank as an analogue to the sphere which passes through the centres of the circles circumscribed about the faces.

The orthocentre of a triangle possesses, in addition to the property from which it derives its name, a further characteristic which equally defines it, viz., it is the isogonal conjugate of the centre of the circumscribed circle with respect to the triangle. Moreover, in regard to the triangle the following theorems exist:—

(*a*) If a pair of points are isogonal conjugates, the orthogonal projections of the points on the sides lie on a circle.

This circle has for a diameter the major axis of the ellipse inscribed in the triangle, and having the points in question for its foci.

(*b*) If two circles are given, the centres of all circles which are bisected by either of the circles and orthogonally cut by the other, lie on a circle coaxal with the given circles, and whose centre is the middle point between their centres.

(*c*) If any circle be described about a given point as centre, and circles be described on the intercepts made by it on the sides of a triangle as diameters, the radical centre of the circles is the isogonal conjugate of the centre of the first-named circle with respect to the triangle.

(*d*) Combining these results, we see that, if a pair of points are isogonal conjugates with respect to a triangle, and two circles be described about them as centres, such that each cuts orthogonally the circles described on the intercepts made on the sides by the other as diameters, the six orthogonal projections of the points on the sides lie on a circle whose centre is the middle point between the centres of the first-named circles, and which is coaxal with them.

If one of the pairs of circles is circumscribed about the triangle, the other is the "polar" circle, and the derived circle is the nine-points circle.

The latter circle may therefore be described as the locus of the centres of circles which are bisected by one or other of the circumscribed and polar circles, and cut orthogonally by the other.

3. Now, the theorems (*a*), (*b*), (*c*), and (*d*) have close analogues in solid space. In fact, we only have to substitute "sphere" for "circle," "tetrahedron" for "triangle," "face" for "side," and adapt in minor ways the phraseology.

Thus the following theorem may be ennunciated (*Educational Times*, August, 1887) :—(*D*) If a pair of points are isogonal conjugates with regard to a tetrahedron, and two spheres be constructed about them as centres such that either of the spheres cuts orthogonally the spheres constructed on the intercepts made on the faces by the other as diametral sections, the eight orthogonal projections of the pair of points on the faces lie on a sphere whose centre is the middle point between the centres of the first-named spheres, and which has a common section with them.

The derived sphere has for its diameter the major axis of the ellipsoid of revolution inscribed in the tetrahedron and having the pair of points for its foci (Neuberg, "Sur le Tétraèdre," t. XXXVII., *Mémoires publiés par l'Académie Royale de Belgique*, 1884).

Prof. Neuberg has further pointed out that the isogonal conjugate of the centre of the circumscribed sphere is the centre of the sphere inscribed in the tetrahedron formed by connecting its four orthogonal projections on the faces of the original tetrahedron (*Educational Times*, December, 1887).

Let A, B, C, D be the vertices of a tetrahedron, and let O_2 be the isogonal conjugate of O, the centre of the circumscribed sphere, and O_1 the middle point between O and O_2. Then, producing AO_1, BO_1, CO_1, DO_1, respectively, to A', B', C', D', so that $AO_1 = O_1A'$, $BO_1 = O_1B'$, &c., we get the vertices of another tetrahedron symmetric with, equal, and inversely homothetic to the original one, and which has O_2 for the centre of the circumscribed sphere, O being its isogonal conjugate with respect to the new tetrahedron.

Hence, by the symmetry of the figure, the sphere about O_1 as centre, according to the before mentioned conditions, is common to the two tetrahedrons, and the sixteen orthogonal projections of O, O_2 on the faces lie on the sphere about O_1 as centre. The sixteen points are the extremities of eight diameters of this sphere, and the line joining O or O_2 to a vertex of the tetrahedron is perpendicular to the plane through the orthogonal projections of O_2 or O, as the case may be, on the faces meeting at that vertex.

The sixteen points correspond to the twelve points determined on the nine-points circle of a triangle, by a precisely analogous construction, and comprising amongst them the nine-points from which the circle derives its name.

4. To bring out more clearly the analogies of these constructions, I have recourse to certain equations which retain marked geometrical characters. It will be convenient to collect in the first instance some results relating to a triangle.

Let the sides opposite the vertices A, B, C of a triangle be denoted by $\overline{bc}$, $\overline{ca}$, $\overline{ab}$ respectively; then in triangular coordinates the equation of a circle whose radius is ρ and the coordinates of whose centre are α_1, β_1, γ_1 is

$$\Sigma\, \overline{ab}^2\, \alpha\beta - \Sigma \left\{\alpha_1 \left(\overline{ab}^2\beta + \overline{ac}^2\gamma\right)\right\} \Sigma\alpha + \left(\Sigma\overline{ab}^2\, \alpha_1\beta_1 + \rho^2\right)(\Sigma\alpha)^2 = 0,$$

where the symbol Σ denotes the summation of similar combinations of the quantities involved.

When the coordinates of any point are substituted for α, β, γ in the above left-hand expressions, the result is minus the square of the tangent from that point to the circle, or minus the power of the point *with respect to* the circle. If, then, T, T_1, T_2 are the tangents from

the vertices A, B, C, we may write the equation in the form

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma T^2 \alpha \,.\, \Sigma\alpha = 0.$$

Let t^2, t_1^2, t_2^2 be the powers of the vertices with respect to circles drawn on the opposite sides respectively as diameters. The equations of these circles will be

$$\Sigma \overline{ab}^2 \alpha\beta - t^2 \alpha\, \Sigma\alpha = 0,\ \ \Sigma \overline{ab}^2 \alpha\beta - t_1^2 \beta\, \Sigma\alpha = 0,\ \ \Sigma \overline{ab}^2 \alpha\beta - t_2^2 \gamma\, \Sigma\alpha = 0 \ldots (1).$$

The equation of the circle cutting them orthogonally is

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma \left(\frac{1}{t^2 \Sigma \frac{1}{t^2}} (\overline{ab}^2 \beta + \overline{ac}^2 \gamma) \right) \Sigma\alpha + \frac{1}{\Sigma \frac{1}{t^2}} (\Sigma\alpha)^2 = 0.$$

This is the polar circle. The equation may be obtained by taking the Jacobian of the left-hand expressions of the equations (1), or perhaps more readily by first determining the coordinates of the centre of the circumscribed circle in the form $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, $\frac{t_2^2}{2p_2^2}$, where p, p_1, p_2 are the altitudes of the triangle. The coordinates of the isogonal conjugate are therefore $1/t^2 \Sigma \frac{1}{t^2}$, $1/t_1^2 \Sigma \frac{1}{t^2}$, $1/t_2^2 \Sigma \frac{1}{t^2}$.

The equation of the circumscribed circle, $\Sigma \overline{ab}^2 \alpha\beta = 0$, may be written in the form

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma \left(\frac{t^2}{2p^2} (\overline{ab}^2 \beta + \overline{ac}^2 \gamma) \right) \Sigma\alpha + 2R^2 (\Sigma\alpha)^2 = 0 \ \ldots\ldots (2).$$

The equation of the circle passing through the centres of the circles described on the sides as diameters (*i.e.*, the nine-points circle) is therefore

$$2 \Sigma \overline{ab}^2 \alpha\beta - \Sigma \left(\frac{1}{t^2 \Sigma \frac{1}{t^2}} (\overline{ab}^2 \beta + \overline{ac}^2 \gamma) \right) \Sigma\alpha + \frac{1}{\Sigma \frac{1}{t^2}} (\Sigma\alpha)^2 = 0 \ \ldots (3).$$

For the nine-points circle is coaxal with the circumscribed and polar circles, and its centre has for triangular coordinates

$$\tfrac{1}{2}\left(1/t^2 \Sigma \frac{1}{t^2} + t^2/2p^2\right),\ \ \tfrac{1}{2}\left(1/t_1^2 \Sigma \frac{1}{t^2} + t_1^2/2p_1^2\right),\ \ \tfrac{1}{2}\left(1/t_2^2 \Sigma \frac{1}{t^2} + t_2^2/2p_2^2\right).$$

To obtain its equation, therefore, we have only to add the left-hand member of (2) to the left-hand member of the equation of the polar circle, and equate the result to zero. But in (2) the terms following $\Sigma \overline{ab}^2 \alpha\beta$ are *identically zero*, and we may omit them.

We have also the following equalities

$$\Sigma \frac{t^2}{2p^2} = 1,$$

$$\frac{t^2}{2p^2}\overline{ab}^2 + \frac{t_1^2}{2p_2^2}\overline{bc}^2 = \frac{t^2}{2p^2}\overline{ac}^2 + \frac{t_1^2}{2p_1^2}\overline{bc}^2 = \frac{t_1^2}{2p_1^2}\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\overline{bc}^2 = 2R^2,$$

which may be verified by means of

$$2t^2 = \overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2, \quad 2t_1^2 = \overline{ab}^2 + \overline{bc}^2 - \overline{ac}^2, \quad 2t_2^2 = \overline{bc}^2 + \overline{ca}^2 - \overline{ab}^2.$$

By substituting $t^2/2p^2$, $t_1^2/2p_1^2$, $t_2^2/2p_2^2$ for α, β, γ in (2) and (3), we get

$$R^2 + R_1^2 - d^2 = 1/\Sigma \frac{1}{t^2} = 2\left(R_2^2 - \frac{d^2}{4}\right) \quad \ldots\ldots\ldots\ldots\ldots(4),$$

where R_1, R_2 are the radii of the circles, and d is the distance between the centres of the circumscribed and polar circles.

The equations of the polar and nine-points circles may also be written

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma t^2\alpha \, \Sigma\alpha = 0, \quad 2\Sigma \overline{ab}^2 \alpha\beta - \Sigma t^2\alpha \, \Sigma\alpha = 0;$$

from which we get

$$R^2 - d^2 - 3/\Sigma \frac{1}{t^2} = R^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Eliminating $\Sigma \frac{1}{t^2}$ and d^2, we get

$$R^2 = 4R_2^2,$$

and eliminating R and $\Sigma \frac{1}{t^2}$,

$$R^2 + 2R_1^2 = d^2,$$

lastly,

$$\Sigma \frac{1}{t^2} + \frac{1}{R_1^2} = 0.$$

5. Now, referring to a tetrahedron and tetrahedral coordinates, we find precisely similar forms. The equation of a sphere whose radius is ρ, and the coordinates of whose centre are α_1, β_1, γ_1, δ_1, is

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma \left\{\alpha_1 (\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta)\right\} \Sigma\alpha + (\Sigma \overline{ab}^2 \alpha_1\beta_1 + \rho^2)(\Sigma\alpha)^2 = 0,$$

where $\overline{ab}$ denotes the edge connecting the vertices A, B of the tetrahedron $ABCD$, and so on.

If T, T_1, T_2, T_3 are the tangents from the vertices respectively to the sphere, its equation may be written

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma (T^2\alpha) \Sigma\alpha = 0.$$

Denoting the tangents from the vertices to the spheres having for diametral sections the circles circumscribed about the opposite faces by t, t_1, t_2, t_3, the equations of those spheres are

$$\Sigma\,\overline{ab}^2\,\alpha\beta - t^2\alpha\,\Sigma\alpha = 0,\quad \Sigma\,\overline{ab}^2\,\alpha\beta - t_1^2\beta\,\Sigma\alpha = 0,\quad \Sigma\,\overline{ab}^2\,\alpha\beta - t_2^2\gamma\,\Sigma\alpha = 0,$$
$$\Sigma\,\overline{ab}^2\,\alpha\beta - t_3^2\delta\,\Sigma\alpha = 0.$$

And the equation of the sphere cutting these orthogonally (the Jacobian of the left-hand expressions equated to zero) is

$$\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{1}{t^2\,\Sigma\frac{1}{t^2}}\left(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta\right)\right)\Sigma\alpha + \frac{1}{\Sigma\frac{1}{t^2}}\,(\Sigma\alpha)^2 = 0 \ldots (6).$$

The equation of the circumscribed sphere $\Sigma\,\overline{ab}^2\,\alpha\beta = 0$ is identical with

$$\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{t^2}{2p^2}\left(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta\right)\right)\Sigma\alpha + 2R^2\,(\Sigma\alpha)^2 = 0,$$

R being the radius, p, p_1, p_2, p_3 being the altitudes of the tetrahedron, and the tetrahedral coordinates of the centre being $t^2/2p^2$, $t_1^2/2p_1^2$, &c., as I shall presently show.

Consequently, the equation of the sphere passing through the centres of the spheres having the circles circumscribed about the faces for diametral sectors is, as in the case of the triangle,

$$2\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{1}{t^2\,\Sigma\frac{1}{t^2}}\left(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta\right)\right)\Sigma\alpha + \frac{1}{\Sigma\frac{1}{t^2}}\,(\Sigma\alpha)^2 = 0 \ldots (7).$$

Also the following equalities hold—

$$\Sigma\,\frac{t^2}{2p^2} = 1,$$

$$\frac{t_1^2}{2p_1^2}\,\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\,\overline{ac}^2 + \frac{t_3^2}{2p_3^2}\,\overline{ad}^2 = \frac{t^2}{2p^2}\,\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\,\overline{bc}^2 + \frac{t_3^2}{2p_3^2}\,\overline{bd}^2$$
$$= \frac{t^2}{2p^2}\,\overline{ac}^2 + \frac{t_1^2}{2p_1^2}\,\overline{bc}^2 + \frac{t_3^2}{2p_3^2}\,\overline{cd}^2 = \frac{t^2}{2p^2}\,\overline{ad}^2 + \frac{t_1^2}{2p_1^2}\,\overline{bd}^2 + \frac{t_2^2}{2p_2^2}\,\overline{cd}^2 = 2R^2.$$

By means of the general expression in tetrahedral coordinates for the distance between two points, we have

$$t_3^2 = \frac{\Sigma\,\overline{ad}^2\,\overline{bc}^2\,(\overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2) - 2\,\overline{ab}^2\,\overline{bc}^2\,\overline{ac}^2}{\Sigma\,\overline{bc}^2\,(\overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2)}$$
$$= \frac{\overline{ad}^2\sin 2A + \overline{bd}^2\sin 2B + \overline{cd}^2\sin 2C - 4\Delta_3}{\sin 2A + \sin 2B + \sin 2C},$$

if A, B, C are the angles of the face ABC and Δ_3 is the area.

And there are corresponding expressions for t_2^2, t_1^2, t_3^2.

Let the faces opposite the vertices A, B, C, D of the tetrahedron of reference be denoted by Δ, Δ_1, Δ_2, Δ_3 respectively, and let V be its volume. Then

$$\Sigma \frac{t^2}{2p^2} = \frac{1}{18V^2}\left(t^2\Delta^2 + t_1^2\Delta_1^2 + t_2^2\Delta_2^2 + t_3^2\Delta_3^2\right).$$

But $$t^2 = \frac{1}{16\Delta^2}\left\{\Sigma\,\overline{ba}^2\,\overline{cd}^2\left(\overline{bc}^2 + \overline{bd}^2 - \overline{cd}^2\right) - 2\,\overline{bc}^2\,\overline{bd}^2\,\overline{cd}^2\right\},$$

&c. &c.,

and $$16\,\Sigma t^2\Delta^2 = 2\,.\,144V^2.$$

Hence $$\Sigma \frac{t^2}{2p^2} = 1.$$

Since the *four-plane* coordinates of the isogonal conjugate to the centre of the circumscribed sphere are as $\frac{p}{t^2}$, $\frac{p_1}{t_1^2}$, $\frac{p_2}{t_2^2}$, $\frac{p_3}{t_3^2}$, those of the centre in question are as $\frac{t^2}{p}$, $\frac{t_1^2}{p_1}$, $\frac{t_2^2}{p_2}$, $\frac{t_3^2}{p_3}$; and the *tetrahedral* coordinates are therefore $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, $\frac{t_2^2}{2p_2^2}$, $\frac{t_3^2}{2p_3^2}$.

If, then, the radii of the spheres (6) and (7) are R_1, R_2, and d denotes the distance between the centres of the circumscribed sphere and (6), we have

$$R^2 + R_1^2 - d^2 = \frac{1}{\Sigma \frac{1}{t^2}} = 2\left(R_2^2 - \frac{d^2}{4}\right).$$

The equations of the spheres (6) and (7) may be written also

$$\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\,(t^2\alpha)\,\Sigma\alpha = 0, \quad 2\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\,(t^2\alpha)\,\Sigma\alpha = 0\,;$$

from which we derive the additional equation

$$R^2 - d^2 - \frac{4}{\Sigma \frac{1}{t^2}} = R_1^2.$$

These expressions differ in form from the expressions (4), (5), only by the substitution of the numerator 4 for 3 over $\Sigma \frac{1}{t^2}$ in the last equation. This slight difference, however, creates ultimately a fault in the general analogy of more importance. Eliminating $1/\Sigma \frac{1}{t^2}$ and

d^2, we get

$$12R_2^2 = 3R^2 + R_1^2.$$

Eliminating $1/\Sigma\frac{1}{t^2}$ and R_2^2, we get

$$3R^2 + 5R_1^2 = 3d^2;$$

and similarly
$$\frac{1}{\Sigma\frac{1}{t^2}} + \tfrac{2}{3}R_1^2 = 0, \quad R^2 = 5R_2^2 + \frac{d^2}{4}.$$

6. It thus becomes necessary, in order to maintain the analogies, to admit the claim of the sphere passing through the centroids of the faces. Analogues, however, have different ranks, and I think the sphere we have been discussing is entitled to precedence. I propose to state, as fairly as I can, the case of the other analogue in the light of Signor Intrigila's paper "Sul Tetraedro" (*Rendiconti della Societá Reale di Napoli*, Anno 22, 1883).

The four altitudes p, p_1, p_2, p_3 of a tetrahedron $ABCD$ are four generators of the same system of a hyperboloid. Also the normals to the faces at the several orthocentres are generators of the other system belonging to the same hyperboloid. This theorem is attributed to Joachimstal (*Grunert*, t. xxxii., p. 109). Signor Intrigila determines by an easy geometrical construction the centre I of the hyperboloid.

If O, O_1 are the centres of the circumscribed sphere and the sphere through the centroids of the faces, he shows that I, O_1 and the centre of gravity G of the tetrahedron lie in one straight line, G being the middle point.* Moreover, I and G are the centres of direct and inverse similitude of the spheres. The modulus is 3. These relations are analogous to those of the circumscribed and nine-points circle, and do not hold in the previous case.

In tetrahedral coordinates, since $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, &c. are the coordinates of the centre O, and those of G are $\frac{1}{4}$, $\frac{1}{4}$, &c., we find for the coordinates of O_1

$$\tfrac{1}{3}\left(1 - \frac{t^2}{2p^2}\right), \quad \tfrac{1}{3}\left(1 - \frac{t_1^2}{2p_1^2}\right), \text{ \&c.},$$

and for those of I,

$$\tfrac{1}{2}\left(1 - \frac{t^2}{p^2}\right), \quad \tfrac{1}{2}\left(1 - \frac{t_1^2}{p_1}\right), \text{ \&c.}$$

* The author quotes Joachimstal as giving this result also (*Nouvelles Annales*, t. xviii., 1859, p. 266).

The corresponding coordinates in the case of the triangle are the coordinates of the centre of the nine-points circle,

$$\tfrac{1}{2}\left(1-\frac{t^2}{2p^2}\right),\quad \tfrac{1}{2}\left(1-\frac{t_1^2}{2p_1^2}\right),\ \&c.,$$

and those of the orthocentre

$$\left(1-\frac{t^2}{p^2}\right),\quad \left(1-\frac{t_1^2}{p_1^2}\right),\ \&c.$$

Signor Intrigila shows that the sphere which passes through the centroids of the faces also divides the distances of the vertices of the tetrahedron from the centre I of the hyperboloid into two parts whose ratio is 2, and passes through a point on each face, which is the harmonic conjugate of the orthocentre of the face with respect to the orthogonal projections thereon of the opposite vertex, and the centre I of the hyperboloid. Thus twelve noteworthy points are determined.

The paper of Signor Intrigila contains, beyond these results, several theorems of a more general character, but lying outside the field of analogy to the plane case. He also makes application of his results to equifacial and orthogonal tetrahedra, demonstrating in the latter case that his theorems relative to the general tetrahedron become identical with those of M. Prouhet, as indeed they evidently must since the sphere in question is determined by passing through the centroids of the faces.

7. In the case of the "orthogonal" tetrahedron, we find a third analogue of the nine-points circle distinct from the two already mentioned.

The equation of the polar sphere is

$$\Sigma\, \overline{ab}^2\, \alpha\beta - \Sigma\, T^2\alpha \,.\, \Sigma\, \alpha = 0,$$

where $\Sigma\, T^2\alpha$ means, as before,

$$T^2\alpha + T_1^2\beta + T_2^2\gamma + T_3^2\delta,$$

and the coefficients are the powers of the vertices relative to the sphere, and where

$$\overline{ab}^2 = T^2 + T_1^2,\quad \overline{ac}^2 = T^2 + T_2^2,\quad \overline{ad}^2 = T^2 + T_3^2,\ \&c.,$$

which imply, of course, the usual relations

$$\overline{ab}^2 + \overline{cd}^2 = \overline{ac}^2 + \overline{bd}^2 = \overline{ad}^2 + \overline{bc}^2 = \Sigma\, T^2,$$

and further

$$2T^2 = \overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2 = \overline{ab}^2 + \overline{ad}^2 - \overline{bd}^2 = \overline{ac}^2 + \overline{ad}^2 - \overline{cd}^2,$$

$$\&c. \qquad\qquad \&c.$$

But these are the expressions determining the powers of the vertices of each face with respect to the circles drawn on the opposite sides as diameters. The sections of the polar sphere by the plane of the faces are, therefore, the polar circles of those faces. The equation

$$2\Sigma\overline{ab}^2\,\alpha\beta-\Sigma\,T^2\alpha\;\Sigma\alpha=0$$

represents the sphere passing through the intersection of the polar and circumscribed spheres, and having its centre midway between their centres, *i.e.*, at the centre of gravity of the tetrahedron. The sections of this sphere by the planes of the face are the nine-points circles of the faces. Professor Wolstenholme has noted this and other properties of this sphere ["Exercices sur le Tétraèdre," *Nouvelles Annales* (2), x., 451, 452]. It is evidently a close analogue of the nine-points circle, although the centres of the polar sphere and the circumscribed sphere are not isogonal conjugates.

It is instructive to consider the following figure. If through any point P we draw lines AP, BP, CP, DP from the vertices of a tetrahedron, and produce them respectively to the points A', B', C', D', so that $AP = PA'$, &c., the last-named points are the vertices of an equal and inversely homothetic tetrahedron. If further, we make the condition that $B'C'$ shall meet AD in A'', $B'D'$ shall meet AC in B'', and so forth, the point P must be the common centroid of the two tetrahedrons, and the six points, say, A'', B'', C'', E'', F'', H'' can be connected in four ways with two corresponding vertices, as (A, A'), (B, B'), &c., so as to form parallelepipeds. If the tetrahedron is orthogonal, we may have $C''F''$, $H''B''$ perpendicular to $F''H''$, and $A''F''$, $E''B'$ perpendicular to $E''F''$, &c. Consequently, A'', B'', C'', E'', F'', H'' lie on a sphere, as before stated. But we may also regard the figure as *in plano*, or as the orthographic projection, and those six points lie on a conic whose centre is G. If the conic becomes a circle, it is in the projection, the nine-points circle of each of two projected faces, and the projections of the two vertices not on those faces will be the centres of the circumscribed circles and the orthocentres.

The imposition of conditions on the tetrahedron manifestly increases the number of analogues. We might establish another, for instance, by taking the isogonal conjugate of the intersection of the altitudes, constructing a sphere which should cut orthogonally the spheres having for diametral sections the circular intercepts on the faces made by the polar sphere. The interest, however, attaching to these analogies is much diminished by the limitations involved.

Thursday, February 9th, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. A. E. Hough Love and G. G. Morrice were admitted into the Society.

The following communications were made:—

Further remarks on the Theory of Distributions: Captain P. A. MacMahon, R.A.

The Free and Forced Vibrations of an Elastic Spherical Shell containing a given Mass of Liquid: A. E. H. Love, B.A.

On the Volume generated by a Congruency of Lines: R. A. Roberts, M.A.

On Isoscelians: R. Tucker, M.A.

The following presents were received:—

"Educational Times," for February, 1888.

"Bulletin de la Société Mathématique de France," Tome xv., No. 7.

"Beiblätter zu den Annalen der Physik und Chemie," Band xi., Stück 12; Band xii., Stück 1.

"Journal für die reine und angewandte Mathematik," Band cii., Heft 3.

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Isoscelians. *By* R. Tucker, M.A.

[*Read Feb. 9th*, 1888.]

Lines PQ, $P'Q'$, drawn as in Fig. 1, so as to make $\angle AQP = A = \angle AQ'P'$, I propose to call *Isoscelians*, because the triangles AQP, $AQ'P'$ are isosceles triangles. PQ is a positive isoscelian, and $P'Q'$ a negative isoscelian; *i.e.*, an isoscelian is positive or negative according as the angle (here A) is made with AB or AC, *i.e.*, in the cyclical or counter-cyclical order of the letters A, B, C.

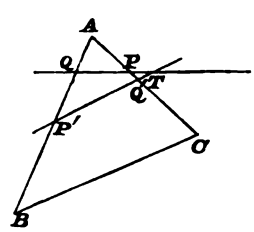

Fig. 1.

All isoscelians of the same affection (positive or negative) for any angle (A say) are, of course, parallels, and any two of opposite affection are anti-parallels; and the median lines, with respect to the common vertex for one such set of isoscelians, are evidently the symmedian lines of a set of opposite affection; hence the median lines of opposite sets are isogonal lines.*

In Fig. 2, DEF is the pedal triangle, and $\alpha\beta\gamma$ the medial triangle;

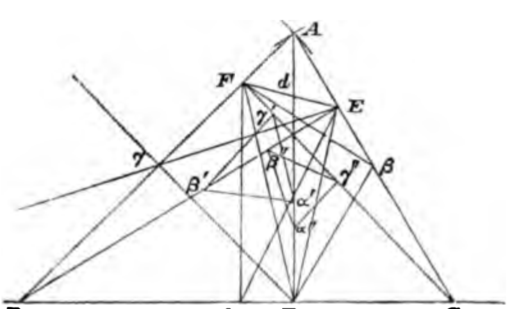

Fig. 2.

therefore, since the six points are on the nine-point circle, we have

$$\angle AF\beta = \angle \beta\alpha\gamma = A,$$

* The nomenclature throughout is that adopted by Rev. T. C. Simmons in his excellent article on "Recent Geometry" (pp. 99—184 of the Rev. J. J. Milne's *Companion to the Weekly Problem Papers*).

and $F\beta$ ($D\gamma$, $E\alpha$) are positive isoscelians, and $E\gamma$ ($F\alpha$, $D\beta$) are negative isoscelians for the angles A, B, C respectively.

For the moment, call AP (Fig. 1) λ; then equation to BP is

$$\frac{\alpha}{c(b-\lambda)} = \frac{\gamma}{a\lambda},$$

and to CQ is
$$\frac{\alpha}{b(c-2\lambda\cos A)} = \frac{\beta}{2a\lambda\cos A};$$

hence BP, CQ intersect on the hyperbola

$$a\alpha\beta + (c-2b\cos A)\,\beta\gamma - 2a\cos A\,\gamma\alpha = 0.$$

This curve evidently passes through A, B, C, A' ($A\alpha$ being joined and produced to A', so that $\alpha A' = \alpha A$) and has α for its centre; it also, of course, passes through the intersection of CF, $B\beta$.*

In like manner, BQ', CP' intersect on the hyperbola

$$a\gamma\alpha + (b-2c\cos A)\,\beta\gamma - 2a\cos A\,\alpha\beta = 0,$$

which passes also through A, B, C, A' and the intersection of BE, $C\gamma$, and has α for its centre.

Like results obtain for the other angles.

Now, take $AP' = \mu \,.\, AP$, with $AP = \lambda$; then equation to CQ is

$$\frac{\alpha}{b(c-2\lambda\cos A)} = \frac{\beta}{2a\lambda\cos A},$$

to BQ' is
$$\frac{\alpha}{c(b-2\lambda\mu\cos A)} = \frac{\gamma}{2\lambda\mu a\cos A},$$

and these intersect on the hyperbola

$$(\mu c - b)\,\beta\gamma - a\gamma\alpha + \mu a\,\alpha\beta = 0.$$

Similarly, CP', BP intersect on the curve

$$\mu a\,\gamma\alpha + (\mu b - c)\,\beta\gamma - a\alpha\beta = 0.$$

Both these curves have α for their centre and pass through A, B, C, A'; the former passes through the orthocentre if $\mu = c/b$, and the latter does so if $\mu = b/c$. There are, of course, four other hyperbolas for the other angles.

If we take AL, BM, CN; AL', BM', CN' (the points L, L' being

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* This point γ'' (in figure) is $c\cos B$, $c\cos A$, $a\cos B$; hence it and the corresponding points hereinafter mentioned are the six points of (xxvi.) of my paper on "'Cosine' Orthocentres of a Triangle" (*Mess. of Math.*, Vol. XVII., p. 100).

on BC, &c.) for the median lines of the positive and negative isoscelians respectively, it will be found that

$$\begin{array}{llll} BM,\ CN & \text{intersect in} & 2\cos C,\ 1,\ 4\cos B\cos C & (\pi_1), \\ CN,\ AL & \quad ,, & 4\cos C\cos A,\ 2\cos A,\ 1 & (\pi_2), \\ AL,\ BM & \quad ,, & 1,\ 4\cos A\cos B,\ 2\cos B & (\pi_3), \\ BM',\ CN' & \quad ,, & 2\cos B,\ 4\cos B\cos C,\ 1 & (\pi_1'), \\ CN',\ AL' & \quad ,, & 1,\ 2\cos C,\ 4\cos C\cos A & (\pi_2'), \\ AL',\ BM' & \quad ,, & 4\cos A\cos B,\ 1,\ 2\cos A & (\pi_3'). \end{array}$$

It is readily seen that π_1, π_1'; π_2, π_2'; π_3, π_3' are inverse points, and therefore foci of in-conics. Again,

$$\begin{array}{llll} BM,\ CN' & \text{intersect in} & 1,\ 2\cos C,\ 2\cos B & (\pi_1''), \\ CN,\ AL' & \quad ,, & 2\cos C,\ 1,\ 2\cos A & (\pi_2''), \\ AL,\ BM' & \quad ,, & 2\cos B,\ 2\cos A,\ 1 & (\pi_3''), \\ BM',\ CN & \quad ,, & 2\cos B\cos C,\ \cos B,\ \cos C & (\pi_1'''), \\ CN',\ AL & \quad ,, & \cos A,\ 2\cos C\cos A,\ \cos C & (\pi_2'''), \\ AL',\ BM & \quad ,, & \cos A,\ \cos B,\ 2\cos A\cos B & (\pi_3'''). \end{array}$$

As before, π_1'', π_1''', &c., are inverse points. Further, if O be the circumcentre, the first set lie respectively on AD, BE, CF, and the second set on AO, BO, CO.

If in Fig. 1 we take $AP = \lambda$, $AP' = \lambda'$, and denote $a\beta\gamma + b\gamma a + c a\beta$, $a a + b\beta + c\gamma$, by C and L respectively, then the equations to the circles APQ, $AP'Q'$ are

$$aC = L\left[(c-2\lambda\cos A)\,\beta + (b-\lambda)\,\gamma\right],$$

$$aC = L\left[(c-\lambda')\,\beta + (b-2\lambda'\cos A)\,\gamma\right],$$

and their radical axis (AT) is

$$\beta\,(\lambda' - 2\lambda\cos A) = \gamma\,(\lambda - 2\lambda'\cos A).$$

This will be a median through A if

$$\lambda' : \lambda = b + 2c\cos A : c + 2b\cos A\,;$$

a symmedian, if

$$\lambda' : \lambda = c + 2b\cos A : b + 2c\cos A\,;$$

coincide with AO, if

$$\lambda' : \lambda = \cos(A-B) : \cos(C-A)\,;$$

coincide with AD, if

$$\lambda' : \lambda = \cos(C-A) : \cos(A-B).$$

I proceed to find the point P through which, if positive isoscelians be drawn, their intersections with the sides are concyclic.

The isoscelians for A, B, C are $F'PE$, $D'PF$, $E'PD$, and P is α, β, γ.

Then $$^*BD\sin B = \gamma + \alpha\frac{\sin(A-C)}{\sin 2C}, \quad BD'/BF = 2\cos B,$$

$$BF'\sin B = \alpha + \gamma\frac{\sin(A-B)}{\sin A}, \quad CE'/CD = 2\cos C,$$

$$CD'\sin C = \beta + \frac{\alpha\sin(B-C)}{\sin B},$$

$$CE\sin C = \alpha - \beta\frac{\sin(A-B)}{\sin 2A}.$$

Hence, because (by hyp.) $BD \cdot BD' = BF \cdot BF'$, we get

$$2\cos B\left(\gamma + \alpha\frac{\sin(A-C)}{\sin 2C}\right) = \alpha + \gamma\frac{\sin(A-B)}{\sin A},$$

i.e., $$\alpha\sin A\left[\sin 2C - 2\cos B\sin(A-C)\right] = \gamma\sin 2C\left[2\sin A\cos B - \sin(A-B)\right],$$

or $$\alpha\sin A\sin 2A = \gamma\sin C\sin 2C = \beta\sin B\sin 2B.$$

We arrive at the same result if we work with negative isoscelians.

The point is analogous to the Lemoine-point, which for parallels gives the "T. R." circle, and for antiparallels gives the cosine circle.

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* Since (see Fig. A)

$BD\sin B = \gamma + PD\sin DPH$, and $\alpha = PD\sin PDC$, and $DPH = KPE' = A - C$.

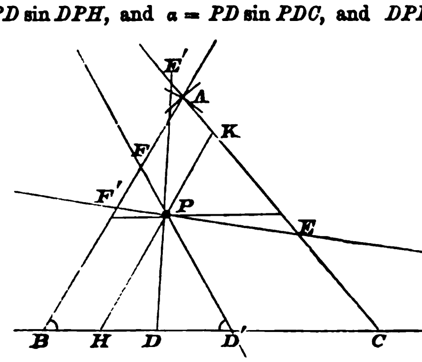

FIG. A.

The circles in the present case do not seem to possess any simple property. The equation to the positive circle may be obtained, with some trouble, by assuming it to be

$$C = L\,(\lambda'\alpha+\mu'\beta+\nu'\gamma),$$

and, using the property that this circle and APQ have for radical axis PQ, whose equation is

$$-2a\alpha\,\lambda\cos A+b\beta\,(c-2\lambda\cos A)+2c\gamma\,(b-\lambda)\cos A=0\,;$$

λ is here

$$= (c\sin A\sin 2C+b\sin 2A\sin 2B)$$
$$/\,(\sin 2A\sin 2B+\sin 2B\sin 2C+\sin 2C\sin 2A).$$

The point P is readily constructed thus:—in Fig. 2, through O draw a parallel to $F\beta$ cutting AD in K and AB in Q, through K draw KR parallel to AB to meet BC in R; then AR passes through P, for

$$BR : RC = QK : KO = Fd : dE = FD : DE = \sin 2B : \sin 2C.$$

Let DEF (Fig. 3) be the positive in-isoscelian triangle of the circle

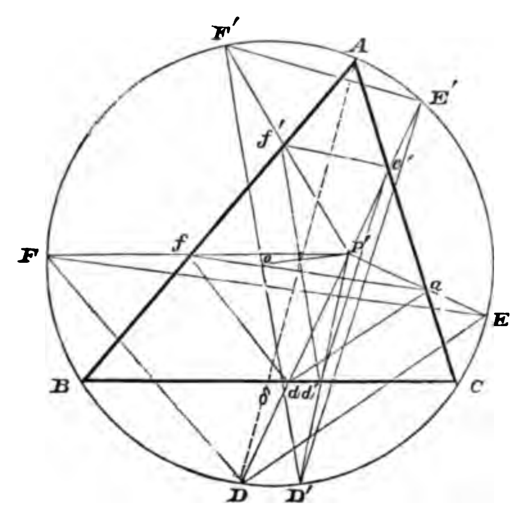

FIG. 3.

ABC, then its angles are

$$D=2C-B,\quad E=2A-C,\quad F=2B-A\,;*$$

* *Cf.* "The 'Sine-Triple-Angle' Circle" (*Mess. of Math.*, Vol. XVI., pp. 125, 126).

hence, if $\angle CAD = \theta$, we have, from the quadrilateral $BFEC$,

because $\angle CEF = \angle FED + \angle CAD = 2A - C + \theta$,

and $\angle ABF = \angle ADB = \angle C\delta D - B = C + \theta - B$,

$$2A - C + \theta + B + C + \theta - B = \pi,$$

therefore $2\theta = \pi - 2A$.

This result enables us easily to construct the isoscelian triangle.

Make $\angle CBD = \angle OBC = 90^\circ - A$,

then D is one vertex of the isoscelian triangle. Similarly, make

$$ABE = 90^\circ - B, \quad \text{and} \quad BCF = 90^\circ - C,$$

and DEF is the positive in-isoscelian triangle. In like manner, make

$$BAD' = 90^\circ - A, \quad CBE' = 90^\circ - B, \quad ACF' = 90^\circ - C,$$

then $D'E'F'$ is the negative in-isoscelian triangle of ABC.

If now we take the positive in-isoscelian triangle of the triangle ABC, viz., def, it is clear that its sides are parallel to those of DEF, and Dd, Ee, Ff will intersect in the centre of perspective P' of the two triangles. To find its coordinates, we have, if AD cut BC in δ,

$$AD = 2R\cos(A-B), \quad A\delta = b\sin C / \cos(A-C),$$

therefore $D\delta = 2R\left[\cos(A-B)\cos(A-C) - \sin B \sin C\right] / \cos(A-C)$

$$= -2R\cos A \cos 2A / \cos(A-C),$$

therefore α-coordinate of $D = 2R\cos A\cos 2A$;

hence P' is the point

$$\cos A\cos 2A : \cos B\cos 2B : \cos C\cos 2C.$$

From the symmetry it is evident that the centre of perspective of $D'E'F'$, $d'e'f'$ is the same point P'. [This result readily follows also from the geometrical fact that P' is the external centre of similitude of the pairs of circles DEF, def; $D'E'F'$, $d'e'f'$; *i.e.*, of the circum-circle (ABC) and the "sine-triple-angle" circle.]

When we have got P', we can readily construct the "sine-triple-angle" circle.

Let AP' cut BC in ρ, then

$$B\rho : C\rho = \sin 4C : \sin 4B.$$

If O' be the nine-point centre, in Fig. 2, and DG, DH perpendiculars on EO', FO', then

$$DG : DH = \sin 4C : \sin 4B.$$

Erect perpendiculars Bg, Ch, equal respectively to DG, DH, on opposite sides of BC; if gh cut BC in ρ, we have

$$B\rho : C\rho = Bg : Ch = \sin 4C : \sin 4B.$$

Hence P' is determined.*

The preceding results could be considerably extended, but they embody the most interesting results which I have, at present, arrived at.

[*Note.*—Professor Neuberg has noticed that "le point P est l'isotomique du centre du cercle ABC." This important remark follows at once from the fact that the trilinear coordinates of isotomic points are thus related—

$$a^2\alpha_1\alpha_2 = b^2\beta_1\beta_2 = c^2\gamma_1\gamma_2.$$

The determination of P is now simple, for, if AO cuts BC in t, then AP will cut BC in t', where $Bt = Ct'$.

Two circles similar to the S.T.A. can be obtained for the in-quad. of a circle, which have a common circumcircle. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. Drawing the isoscelians, we have

$$\left.\begin{aligned} a &= 2x\cos A + y \\ b &= 2y\cos B + z \\ c &= 2z\cos C + w \\ d &= 2w\cos D + x \end{aligned}\right\},$$

whence

$$x\begin{vmatrix} 2\cos A & 1 & 0 & 0 \\ 0 & 2\cos B & 1 & 0 \\ 0 & 0 & 2\cos C & 1 \\ 1 & 0 & 0 & 2\cos D \end{vmatrix} = \begin{vmatrix} a & 1 & 0 & 0 \\ b & 2\cos B & 1 & 0 \\ c & 0 & 2\cos C & 1 \\ d & 0 & 0 & 2\cos D \end{vmatrix},$$

or $$x(1-16\cos A\cos B\cos C\cos D) = \mu x$$

$$= d - 2c\cos D + 4b\cos C\cos D - 8a\cos B\cos C\cos D.$$

Similarly, from $$\left.\begin{aligned} a &= x' + 2y'\cos B \\ b &= y' + 2z'\cos C \\ c &= z' + 2w'\cos D \\ d &= w' + 2x'\cos A \end{aligned}\right\},$$

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* The point P', the inverse of P, and the point $\cos^3 A : \cos^3 B : \cos^3 C$, all lie on the line $a(b^2-c^2)\sec A + \ldots + \ldots = 0$, which is the line connecting the circumcentre and the "sine-triple-angle" centre (see *Mess. of Math.*, l. c., p. 126).

we have

$$\mu x' = a - 2b \cos B + 4c \cos B \cos C - 8d \cos B \cos C \cos D.$$

The angles of the two quadrilaterals $EFGH$, $E'F'G'H'$ are easily found to be

$$2B-A,\quad 2C-B,\quad 2D-C,\quad 2A-D;$$

$$2A-B,\quad 2B-C,\quad 2C-D,\quad 2D-A.$$

I have not succeeded in getting simple expressions for the intercepts on the sides, or a simple construction for the in-isoscelian quadrilaterals of A, B, C, D.

April, 1888.]

The Free and Forced Vibrations of an Elastic Spherical Shell containing a given Mass of Liquid. By A. E. H. Love, B.A.

[*Read Feb. 9th*, 1888.]

This paper contains an application of the methods of Professor Lamb's papers on the "Vibrations of Elastic Spheres and Spherical Shells" (*Proceedings*, Vols. XIII. and XIV.), to the discussion of the forced vibrations of an elastic spherical shell containing a given mass of liquid and rotating slowly about a diameter, the whole being subject to gravitation and to the action of external disturbing bodies supposed periodic in respect of time. I have prefixed an account of the free vibrations of the shell with and without the liquid nucleus. A summary of aims, methods, and results will be found at the end of the paper.

1. The equations of vibration of an elastic solid mass are

$$\left.\begin{aligned} m\frac{\partial\theta}{\partial x} + n\nabla^2 u &= \rho\frac{\partial^2 u}{\partial t^2} \\ m\frac{\partial\theta}{\partial y} + n\nabla^2 v &= \rho\frac{\partial^2 v}{\partial t^2} \\ m\frac{\partial\theta}{\partial z} + n\nabla^2 w &= \rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

u, v, w being the displacements parallel to the coordinate axes, and θ the cubical dilatation

$$\partial u/\partial x + \partial v/\partial y + \partial w/dz.$$

Differentiating these with respect to x, y, z and adding, we obtain in the usual way the equation

$$(m+n)\nabla^2\theta = \rho\frac{\partial^2\theta}{\partial t^2} \quad\text{.........................(2).}$$

Now, suppose the solid is performing vibrations in period $2\pi/p$, we have to suppose u, v, w proportional to e^{ipt}, and the equations become, writing $(m+n)h^2 = p^2\rho = nk^2$,

$$(\nabla^2+h^2)\,\theta = 0 \quad\text{..............................(3),}$$

$$\left.\begin{aligned}(\nabla^2+k^2)\,u = \left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial x},\quad (\nabla^2+k^2)\,v = \left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial y}\\ (\nabla^2+k^2)\,w = \left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial z}\end{aligned}\right\}\quad\text{.........(4).}$$

The last three equations (4) are satisfied by putting

$$u = -\frac{1}{h^2}\frac{\partial\theta}{\partial x},\quad v = -\frac{1}{h^2}\frac{\partial\theta}{\partial y},\quad w = -\frac{1}{h^2}\frac{\partial\theta}{\partial z} \quad\text{...............(5),}$$

where θ satisfies $(\nabla^2+h^2)\,\theta = 0$.

Hence the complete solution of the equations of vibration (1) will consist of, first, the general solution of (3), secondly, the general solution of the equations

$$\left.\begin{aligned}(\nabla^2+k^2)\,u = 0,\quad (\nabla^2+k^2)\,v = 0,\quad (\nabla^2+k^2)\,w = 0\\ \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0\end{aligned}\right\}\quad\text{.......(6),}$$

and thirdly, the particular solutions (5).

2. In what follows, we suppose the solid to be bounded by two concentric spheres. In the first place we have to find the solution of

$$(\nabla^2+h^2)\,\theta = 0$$

within such a space. Assume $\theta = R_i S_i$, where S_i is a spherical surface harmonic of order i, and R_i a function of r the distance of a point from the centre of the spheres; the equation for R_i is

$$\frac{d^2}{d(hr)^2}(rR_i) + (rR_i) - \frac{i\,.\,i+1}{(hr)^2}(rR_i) = 0 \quad\text{..............(7).}$$

This is the case of Riccati's equation, which is integrable in finite terms, and the solution is

$$rR_i = (hr)^{i+1}\left(\frac{1}{hr}\frac{d}{d(hr)}\right)^{i+1}(A'e^{\iota hr} + B'e^{-\iota hr}),$$

where A', B' are arbitrary complex constants.

This may be written

$$R_i = r^i \left(\frac{1}{r}\frac{d}{dr}\right)^i \left(A\,\frac{\sin hr}{hr} + B\,\frac{\cos hr}{hr}\right) \quad\text{.................(8).}$$

And hence, ω_i, Ω_i denoting spherical solid harmonics of positive degree i, we may take

$$\theta = \sum_{i=0}^{i=\infty} \{\omega_i \psi_i(hr) + \Omega_i \Psi_i(hr)\} \quad\text{.....................(9),}$$

where $r^i\Psi_i(hr)$ and $r^i\psi_i(hr)$ are the two particular integrals of the equation (7) for R_i, of which $\psi_i(hr)$ is the one which does not become infinite for $r = 0$.

The first term of (9) is the solution used by Prof. Lamb in his discussion of the vibrations of an elastic sphere.

3. *Properties of the functions ψ and Ψ.*

The differential equation satisfied by ψ_i and Ψ_i is found from (7) to be

$$\left(\frac{d^2}{dr^2} + \frac{2(i+1)}{r}\frac{d}{dr} + h^2\right) \begin{matrix}\psi_i \\ \Psi_i\end{matrix}(hr) = 0,$$

and, observing that $\left(\frac{1}{r}\frac{d}{dr}\right)^2 = \frac{1}{r^2}\frac{d^2}{dr^2} - \frac{1}{r^3}\frac{d}{dr}$, this is

$$\left[r^2\left(\frac{1}{r}\frac{d}{dr}\right)^2 + \frac{2i+3}{r}\frac{d}{dr} + h^2\right] \begin{matrix}\psi_i \\ \Psi_i\end{matrix}(hr) = 0 \quad\text{.........(10).}$$

But, since
$$\psi_i(hr) = \left(\frac{1}{r}\frac{d}{dr}\right)^i \left(\frac{\sin hr}{hr}\right),$$
$$\Psi_i(hr) = \left(\frac{1}{r}\frac{d}{dr}\right)^i \left(\frac{\cos hr}{hr}\right),$$

it follows that
$$\frac{d}{dr}\left\{\begin{matrix}\psi_i \\ \Psi_i\end{matrix}(hr)\right\} = r\left\{\begin{matrix}\psi_{i+1} \\ \Psi_{i+1}\end{matrix}(hr)\right\} \quad\text{.................(11),}$$

so that

$$r^2\left\{\begin{matrix}\psi_{i+2} \\ \Psi_{i+2}\end{matrix}(hr)\right\} + (2i+3)\left\{\begin{matrix}\psi_{i+1} \\ \Psi_{i+1}\end{matrix}(hr)\right\} + h^2\left\{\begin{matrix}\psi_i \\ \Psi_i\end{matrix}(hr)\right\} = 0;$$

hence the ψ and Ψ satisfy the same difference equation

$$r^2\psi_{i+1}(hr) + (2i+1)\,\psi_i(hr) + h^2\psi_{i-1}(hr) = 0.$$

Combining this with (11), we find, for ψ and Ψ,

$$\frac{d\psi_i}{dr} = -\frac{1}{r}\{h^2\psi_{i-1} + (2i+1)\,\psi_i\}.$$

It is convenient now to change the notation, so that

$$\left.\begin{aligned} \psi_i(hr) &= (-)^i (1.3 \ldots 2i+1) \left\{ \frac{1}{hr} \frac{d}{d(hr)} \right\}^i \left(\frac{\sin hr}{hr} \right) \\ \text{and } \Psi_i(hr) &= (-)^i (1.3 \ldots 2i+1) \left\{ \frac{1}{hr} \frac{d}{d(hr)} \right\}^i \left(\frac{\cos hr}{hr} \right) \end{aligned}\right\} \ldots\ldots(12).$$

Then the above relations are replaced by

$$\left.\begin{aligned} r \frac{d}{dr} \psi_{i-1}(hr) &= -\frac{h^2 r^2}{2i+1} \psi_i(hr) = (2i-1) \left\{ \psi_{i-2}(hr) - \psi_{i-1}(hr) \right\} \\ \text{and} \quad \psi_i(hr) - \psi_{i-1}(hr) &= \frac{h^2 r^2}{(2i+1)(2i+3)} \psi_{i+1}(hr) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(13),$$

with exactly similar relations between the functions Ψ.

This change of notation does not affect the form of θ, which is still given by (9).

4. We have now to consider the solution of equations (6). Professor Lamb has shown that within a sphere the general solution is expressed by

$$u = \Sigma \left[\psi_i(kr) \left(\frac{\partial \phi_{i+1}}{\partial x} + y \frac{\partial \chi_i}{\partial z} - z \frac{\partial \chi_i}{\partial y} \right) - \frac{i+1}{i+2} \frac{k^2 r^{2i+5}}{(2i+3)(2i+5)} \psi_{i+2}(kr) \frac{\partial}{\partial x} \left(\frac{\phi_{i+1}}{r^{2i+3}} \right) \right] \ldots(14),$$

with similar expressions for v and w, where ϕ_{i+1}, χ_i are spherical solid harmonics whose order is indicated by the suffixes.

The solutions of the equations

$$(\nabla^2 + k^2)\, u = 0, \quad (\nabla^2 + k^2)\, v = 0, \quad (\nabla^2 + k^2)\, w = 0$$

within a sphere are functions of the form

$$\Sigma \left[\psi_i(kr)\, V_i \right],$$

where V_i denotes a spherical solid harmonic, and the equation

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$$

reduces the number of independent harmonics from three to two. The reasoning employed by Professor Lamb proves that, if u, v, w are three functions of the above form, where ψ_i satisfies the differential equation

$$\frac{d^2}{d(kr)^2} (r^{i+1} \psi_i) + \left[1 - \frac{i(i+1)}{(kr)^2} \right] (r^{i+1} \psi_i) = 0,$$

and if $$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$$

then the most general form of solution for u, v, w is that given by (14). We conclude that the most general values of u, v, w which are of the forms

$$\Sigma\left[\psi_i(kr)\,\omega_i + \Psi_i(kr)\,\Omega_i\right],$$

where ψ_i, Ψ_i are the functions defined by equations (12) with k in place of h, and which also satisfy the relation

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$$

are three expressions of which the type is

$$\begin{aligned} u = \Sigma\Big[\psi_i(kr)\left(\frac{\partial\phi_{i+1}}{\partial x} + y\frac{\partial\chi_i}{\partial z} - z\frac{\partial\chi_i}{\partial y}\right) \\ - \frac{i+1}{i+2}\,\frac{k^2 r^{2i+5}}{(2i+3)(2i+5)}\,\psi_{i+2}(kr)\,\frac{\partial}{\partial x}\left(\frac{\phi_{i+1}}{r^{2i+3}}\right)\Big] \\ + \Sigma\Big[\Psi_i(kr)\left(\frac{\partial\Phi_{i+1}}{\partial x} + y\frac{\partial X_i}{\partial z} - z\frac{\partial X_i}{\partial y}\right) \\ - \frac{i+1}{i+2}\,\frac{k^2 r^{2i+5}}{(2i+3)(2i+5)}\,\Psi_{i+2}(kr)\,\frac{\partial}{\partial x}\left(\frac{\Phi_{i+1}}{r^{2i+3}}\right)\Big] \quad \ldots\ldots(15), \end{aligned}$$

where the ϕ, Φ, χ, X are spherical solid harmonics, and the values of v and w are to be obtained from that of u by cyclical interchange of the letters x, y, z.

Adding to (15) the particular solutions (5) and rewriting, we have

$$\begin{aligned} u = -\frac{1}{h^2}\frac{\partial}{\partial x}\left[\Sigma\left\{\omega_i\psi_i(hr)\right\}\right] \\ + \Sigma\Big[\psi_i(kr)\left(y\frac{\partial\chi_i}{\partial z} - z\frac{\partial\chi_i}{\partial y}\right) + \psi_{i-1}(kr)\frac{\partial\phi_i}{\partial x} \\ - \frac{i}{i+1}k^2 r^2\psi_{i+1}(kr)\,\frac{r^{2i+1}}{(2i+1)(2i+3)}\,\frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right)\Big] \\ - \frac{1}{h^2}\frac{\partial}{\partial x}\left[\Sigma\left\{\Omega_i\psi_i(hr)\right\}\right] \\ + \Sigma\Big[\Psi_i(kr)\left(y\frac{\partial X_i}{\partial z} - z\frac{\partial X_i}{\partial y}\right) + \Psi_{i-1}(kr)\frac{\partial\Phi_i}{\partial x} \\ - \frac{i}{i+1}k^2 r^2\Psi_{i+1}(kr)\,\frac{r^{2i+1}}{(2i+1)(2i+3)}\,\frac{\partial}{\partial x}\left(\frac{\Phi_i}{r^{2i+1}}\right)\Big] \ldots(16), \end{aligned}$$

with expressions for v and w to be obtained from this by cyclical interchange of the letters x, y, z.

Equation (16) is in a form suitable for the calculation of the surface-tractions.

5. If F, G, H be the components of the surface-traction at any sphere $r=$ const., we know that for a solid sphere Fr takes the form

$$\Sigma\left[p_i\left(y\frac{\partial\chi_i}{\partial z}-z\frac{\partial\chi_i}{\partial y}\right)+a_i\frac{\partial\omega_i}{\partial x}+b_i\frac{\partial}{\partial x}\left(\frac{\omega_i}{r^{2i+1}}\right)+c_i\frac{\partial\phi_i}{\partial x}+d_i\frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right)\right],$$

where p_i, a_i, b_i, c_i, d_i are functions of r, and Gr, Hr are to be obtained by cyclical interchange of the letters x, y, z. No reductions are made in obtaining this except such as arise from the χ, ϕ, ω being solid harmonics, and the ψ satisfying equations (10), (13), with, if necessary, k in the place of h. It follows that in the case of the shell

$$Fr=\Sigma\left[p_i\left(y\frac{\partial\chi_i}{\partial z}-z\frac{\partial\chi_i}{\partial y}\right)+a_i\frac{\partial\omega_i}{\partial x}+b_i\frac{\partial}{\partial x}\left(\frac{\omega_i}{r^{2i+1}}\right)+c_i\frac{\partial\phi_i}{\partial x}+d_i\frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right)\right]$$
$$+\Sigma\left[P_i\left(y\frac{\partial X_i}{\partial z}-z\frac{\partial X_i}{\partial y}\right)+A_i\frac{\partial\Omega_i}{\partial x}+B_i\frac{\partial}{\partial x}\left(\frac{\Omega_i}{r^{2i+1}}\right)+C_i\frac{\partial\Phi_i}{\partial x}+D_i\frac{\partial}{\partial x}\left(\frac{\Phi_i}{r^{2i+1}}\right)\right] \quad\text{............(17)},$$

where the P, A, B, C, D are to be obtained from the p, a, b, c, d by writing Ψs everywhere in place of ψs.

6. The values of p, a, b, c, d, as given by Professor Lamb, are equivalent to

$$\left.\begin{aligned}
p_i&=n\left[(i-1)\,\psi_i(kr)+kr\,\psi_i'(kr)\right]\\
a_i&=-\frac{n}{h^2}\left[2(i-1)\,\psi_{i-1}(hr)-\frac{k^2r^2}{2i+1}\,\psi_i(hr)\right]\\
b_i&=-\frac{nk^2}{h^2}\,\frac{r^{2i+3}}{2i+1}\left[\psi_i(hr)+\frac{2(i+2)}{k^2r^2}\,hr\,\psi_i'(hr)\right]\\
c_i&=n\left[2(i-1)\,\psi_{i-1}(kr)-\frac{k^2r^2}{2i+1}\,\psi_i(kr)\right]\\
d_i&=-\frac{i}{i+1}\,n\,\frac{k^2r^{2i+3}}{2i+1}\left[\psi_i(kr)+\frac{2(i+2)}{k^2r^2}\,kr\psi_i'(kr)\right]
\end{aligned}\right\}\quad\text{......(18)},$$

where $\quad\psi_i'(hr)=\dfrac{d}{d(hr)}\,\psi_i(hr),\quad \psi_i'(kr)=\dfrac{d}{d(kr)}\,\psi_i(kr).$

7. We consider two cases, (1) when the shell is perfectly free, and (2) when it contains a given mass of liquid.

Let the radii of the external and internal surfaces be a and b, and suppose both surfaces free from stress, then F, G, H vanish when

$r = a$ and when $r = b$. In future we shall denote by p_i, a_i ... the values taken by the quantities given in (18) when $r = a$, and by p'_i, a'_i ... their values when $r = b$, and similarly for P_i, A_i ... P'_i ... . The surface conditions are six, of the form

$$\Sigma\left\{\begin{array}{l} p_i\left(y\dfrac{\partial\chi_i}{\partial z}-z\dfrac{\partial\chi_i}{\partial y}\right)+a_i\dfrac{\partial\omega_i}{\partial x}+b_i\dfrac{\partial}{\partial x}\left(\dfrac{\omega_i}{r^{2i+1}}\right)+c_i\dfrac{\partial\phi_i}{\partial x}+d_i\dfrac{\partial}{\partial x}\left(\dfrac{\phi_i}{r^{2i+1}}\right) \\ +P_i\left(y\dfrac{\partial X_i}{\partial z}-z\dfrac{\partial X_i}{\partial y}\right)+A_i\dfrac{\partial\Omega_i}{\partial x}+B_i\dfrac{\partial}{\partial x}\left(\dfrac{\Omega_i}{r^{2i+1}}\right)+C_i\dfrac{\partial\Phi_i}{\partial x}+D_i\dfrac{\partial}{\partial x}\left(\dfrac{\Phi_i}{r^{2i+1}}\right)\end{array}\right\}$$

$$= 0 \quad \text{(19)},$$

viz., two others are to be got by writing y, z ... for x ... in cyclical order, and three by changing p_i, a_i ... P_i ... into p'_i, a'_i ... P'_i ... .

Now, we may show that these equations require that, for all values of i,

$$\left.\begin{array}{r} p_i\left(y\dfrac{\partial\chi_i}{\partial z}-z\dfrac{\partial\chi_i'}{\partial y}\right)+P_i\left(y\dfrac{\partial X_i}{\partial z}-z\dfrac{\partial X_i}{\partial y}\right)=0 \\ a_i\omega_i+c_i\phi_i+A_i\Omega_i+C_i\Phi_i=0 \\ b_i\omega_i+d_i\phi_i+B_i\Omega_i+D_i\Phi_i=0 \end{array}\right\} \quad \text{(20)},$$

and similar equations with accented letters.

For, take the three equations such as (19), which hold at the surface $r = a$, multiply them by x, y, z, and add; thus

$$i\,(a_i\omega_i+c_i\phi_i+A_i\Omega_i+C_i\Phi_i)-\frac{(i+1)}{a^{2i+1}}\,(b_i\omega_i+d_i\phi_i+B_i\Omega_i+D_i\Phi_i)=0 \quad \text{(21)}.$$

Again, consider the function

$$U = \Sigma\left\{\begin{array}{l} p_i\left(y\dfrac{\partial\chi_i}{\partial z}-z\dfrac{\partial\chi_i}{\partial y}\right)+P_i\left(y\dfrac{\partial X_i}{\partial z}-z\dfrac{\partial X_i}{\partial y}\right)+a_i\dfrac{\partial\omega_i}{\partial x}+A_i\dfrac{\partial\Omega_i}{\partial x}+C_i\dfrac{\partial\Phi_i}{\partial x}+c_i\dfrac{\partial\phi_i}{\partial x} \\ +\left(\dfrac{r}{a}\right)^{2i+1}\left\{b_i\dfrac{\partial}{\partial x}\left(\dfrac{\omega_i}{r^{2i+1}}\right)+B_i\dfrac{\partial}{\partial x}\left(\dfrac{\Omega_i}{r^{2i+1}}\right)+d_i\dfrac{\partial}{\partial x}\left(\dfrac{\phi_i}{r^{2i+1}}\right)+D_i\dfrac{\partial}{\partial x}\left(\dfrac{\Phi_i}{r^{2i+1}}\right)\right\}\end{array}\right\}$$

This is finite and continuous, and satisfies $\nabla^2 U = 0$ within the sphere $r = a$, and $= 0$ at its surface; hence it $= 0$ throughout the sphere. The same is true of the functions V, W got from this by cyclical interchange of the letters x, y, z. Hence, differentiating U, V, W with respect to x, y, z respectively, and adding, we find

$$(2i+3)(i+1)\left[b_i\omega_i+d_i\phi_i+B_i\Omega_i+D_i\Phi_i\right]=0 \quad \text{(22)}.$$

From (19), (21), and (22) we at once obtain (20).

Observe, now, that precisely the same reasoning applies to the equations derived from (19) by accenting the letters. And so, considering what happens within a sphere $r = b$, using functions formed from U, V, W by accenting the letters p, a, ..., and writing $\left(\frac{r}{b}\right)^{2i+3}$ in the b, d terms, we see that the equations (20) hold also at the surface $r = b$ if we accent the letters. Further, it is to be noticed that all the harmonics occurring in any one of the equations (20), and the similar set we have just considered, are of the same order i; and therefore, dividing all the equations (20) by a^i, and all the similar equations with accented letters by b^i, we shall be left with only surface harmonics which are the same in the two sets of equations, multiplied by constants which are different in the two sets of equations; *i.e.*, we have

$$\left.\begin{aligned} p_i \frac{1}{r^i}\left(y\frac{\partial\chi_i}{\partial z} - z\frac{\partial\chi_i}{\partial y}\right) + P_i \frac{1}{r^i}\left(y\frac{\partial X_i}{\partial z} - z\frac{\partial X_i}{\partial y}\right) &= 0 \\ p_i' \frac{1}{r^i}\left(y\frac{\partial\chi_i}{\partial z} - z\frac{\partial\chi_i}{\partial y}\right) + P_i' \frac{1}{r^i}\left(y\frac{\partial X_i}{\partial z} - z\frac{\partial X_i}{\partial y}\right) &= 0 \end{aligned}\right\} \quad \text{.........(23),}$$

and

$$\left.\begin{aligned} \frac{1}{r^i}\left(a_i\omega_i + c_i\phi_i + A_i\Omega_i + C_i\Phi_i\right) &= 0 \\ \frac{1}{r^i}\left(b_i\omega_i + d_i\phi_i + B_i\Omega_i + D_i\Phi_i\right) &= 0 \\ \frac{1}{r^i}\left(a_i'\omega_i + c_i'\phi_i + A_i'\Omega_i + C_i'\Phi_i\right) &= 0 \\ \frac{1}{r^i}\left(b_i'\omega_i + d_i'\phi_i + B_i'\Omega_i + D_i'\Phi_i\right) &= 0 \end{aligned}\right\} \quad \text{...............(24).}$$

In each of the equations in each of these sets the same harmonic functions occur, and, on eliminating them, we find the equations for the frequency, viz.,

$$p_i P_i' - p_i' P_i = 0 \text{.............................(25),}$$

and

$$\begin{vmatrix} a_i & c_i & A_i & C_i \\ b_i & d_i & B_i & D_i \\ a_i' & c_i' & A_i' & C_i' \\ b_i' & d_i' & B_i' & D_i' \end{vmatrix} = 0 \text{ ....(26).}$$

Just as in the case of the sphere, the modes of vibration are of two types, one purely transverse depending on the χ, X, and with a frequency equation (25), which involves only the modulus of

rigidity, and the other in which the displacement is partly longitudinal (radial), with a frequency equation (26), which involves both the rigidity and the resistance to compression.

8. When the shell contains liquid, the equations (21) will still hold for the outer surface, but at the inner we shall have

$$Fr = p_1 x, \quad Gr = p_1 y, \quad Hr = p_1 z,$$

where p_1 is the fluid pressure.

The condition of constancy of volume of the liquid is fulfilled, since the normal displacement at any sphere is proportional to a spherical surface harmonic.

The normal velocity at any sphere is

$$\frac{\partial}{\partial t}\left(\frac{ux+vy+wz}{r}\right),$$

and we find

$$ux+vy+wz = -\frac{1}{h^2}\, r\,\frac{\partial\theta}{\partial r} + \Sigma\left[i\psi_i(kr)\,\phi_i + i\Psi_i(kr)\,\Phi_i\right]$$

[to obtain this we have only to multiply (16) by x, y, z, add, and use the difference equation for the ψ, Ψ],

also
$$r\frac{\partial\theta}{\partial r} = \left\{r\frac{\partial\psi_i(hr)}{\partial r}+i\right\}\omega_i + \left\{r\frac{\partial\Psi_i(hr)}{\partial r}+i\right\}\Omega_i$$
$$= \left\{(2i+1)\,\psi_{i-1}(hr)-(i+1)\,\psi_i(hr)\right\}\omega_i$$
$$+\left\{(2i+1)\,\Psi_{i-1}(hr)-(i+1)\,\Psi_i(hr)\right\}\Omega_i.$$

Hence
$$-\frac{1}{h^2}\frac{\partial\theta}{\partial r} = \Sigma\left[e_i\,\omega_i + E_i\Omega_i\right]\frac{a^i}{r^i}, \text{ when } r=a,$$

and
$$= \Sigma\,(e_i'\,\omega_i + E_i'\Omega_i)\,\frac{b^i}{r_i}, \text{ when } r=b,$$

where
$$\left.\begin{aligned} e_i &= \frac{1}{h^2 a}\left\{(i+1)\,\psi_i(ha)-(2i+1)\,\psi_{i-1}(ha)\right\} \\ E_i &= \frac{1}{h^2 a}\left\{(i+1)\,\Psi_i(ha)-(2i+1)\,\Psi_{i-1}(ha)\right\} \end{aligned}\right\} \quad \text{..........(27)},$$

and e_i', E_i' are what these become when in them we put b for a.

Hence the normal velocity at the surface $r=b$ is

$$\iota p e^{\iota pt}\,\Sigma\left\{ib^{i-1}\left[\psi_i(kb)\frac{\phi_i}{r^i}+\Psi_i(kb)\frac{\Phi_i}{r^i}\right]+b^i\left[e_i'\frac{\omega_i}{r^i}+E_i'\frac{\Omega_i}{r^i}\right]\right\} \text{......(28)},$$

where the parts in [ ] are surface harmonics of order i.

The velocity-potential P for the motion of the liquid is

$$P = \iota p e^{\iota pt} \Sigma \left[\psi_i(kb)\,\phi_i + \Psi_i(kb)\,\Phi_i + \frac{b}{i}\,(e_i'\omega_i + E_i'\Omega_i) \right].$$

Hence, if σ be the density of the fluid, the pressure p_1 is

$$p_1 = \sigma p^2 \Sigma \left[\psi_i(kb)\,\phi_i + \Psi_i(kb)\,\Phi_i + \frac{b}{i}\,(e_i'\omega_i + E_i'\Omega_i) \right],$$

omitting the terms of the second order arising from the squares of the velocities, and constants which do not depend upon the strain.

Hence, at the surface $r = b$,

$$\begin{aligned} Fr = \sigma p^2 \Sigma \Bigg[\psi_i(kb)\,\frac{b^3}{2i+1} \left\{ \frac{\partial \phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right) \right\} \\ + \Psi_i(kb)\,\frac{b^3}{2i+1} \left\{ \frac{\partial \Phi_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x}\left(\frac{\Phi_i}{r^{2i+1}}\right) \right\} \Bigg] \\ + \sigma p^2 \Sigma \frac{b^3}{i.2i+1} \left[e_i' \left\{ \frac{\partial \omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x}\left(\frac{\omega_i}{r^{2i+1}}\right) \right\} + E_i' \left\{ \frac{\partial \Omega_i}{\partial x} - b^{2i+1} \frac{\partial}{\partial x}\left(\frac{\Omega_i}{r^{2i+1}}\right) \right\} \right]. \end{aligned}$$

The equations which hold at $r = b$ are thus of the same form as in the case of the free vibrations, and we deduce from them that equations (23) still hold, leading to the frequency equation (25); but, instead of the two equations of (24) which contain accented letters, we have the two following equations:—

$$\left. \begin{aligned} &\left\{ \begin{aligned} &\left(a_i' - \frac{\sigma p^2 b^3}{i.2i+1}\,e_i'\right) \frac{\omega_i}{r^i} + \left(c_i' - \frac{\sigma p^2 b^3}{2i+1}\,\psi_i(kb)\right) \frac{\phi_i}{r^i} \\ &+ \left(A_i' - \frac{\sigma p^2 b^3}{i.2i+1}\,E_i'\right) \frac{\Omega_i}{r^i} + \left(C_i' - \frac{\sigma p^2 b^3}{2i+1}\,\Psi_i(kb)\right) \frac{\Phi_i}{r^i} \end{aligned} \right\} = 0 \\ &\left\{ \begin{aligned} &\left(b_i' + \frac{\sigma p^2 b^{2i+4}}{i.2i+1}\,e_i'\right) \frac{\omega_i}{r^i} + \left(d_i' + \frac{\sigma p^2 b^{2i+3}}{2i+1}\,\psi_i(kb)\right) \frac{\phi_i}{r^i} \\ &+ \left(B_i' + \frac{\sigma p^2 b^{2i+4}}{i.2i+1}\,E_i'\right) \frac{\Omega_i}{r^i} + \left(D_i' + \frac{\sigma p^2 b^{2i+3}}{2i+1}\,\Psi_i(kb)\right) \frac{\Phi_i}{r^i} \end{aligned} \right\} = 0 \end{aligned} \right\} \quad \ldots\text{(29).}$$

We may write these for shortness

$$\left. \begin{aligned} &\frac{1}{r^i}(a_i''\omega_i + c_i''\phi_i + A_i''\Omega_i + C_i''\Phi_i) = 0 \\ \text{and} \qquad &\frac{1}{r^i}(b_i''\omega_i + d_i''\phi_i + B_i''\Omega_i + D_i''\Phi_i) = 0 \end{aligned} \right\},$$

and the frequency equation for vibrations of this type becomes

$$\begin{vmatrix} a_i'', & c_i'', & A_i'', & C_i'' \\ b_i'', & d_i'', & B_i'', & D_i'' \\ a_i, & c_i, & A_i, & C_i \\ b_i, & d_i, & B_i, & D_i \end{vmatrix} = 0 \quad \text{......................(30)},$$

which differs from the equation (26), which holds when there is no liquid, only in the addition of certain terms to the quantities

$$a_i', \; b_i', \; \ldots \; A_i', \; \ldots$$

9. We proceed now to the consideration of the forced vibrations. We suppose the matter composing the shell and the contained fluid to be subject to gravitation and to the attraction of some external disturbing bodies, or generally to external disturbing forces which have a potential expressible in spherical harmonic series. Also, for a particular example, we consider the whole mass to be in rotation with an angular velocity ω, which is such that $\frac{\omega^2 a}{g}$ is small, g being the acceleration due to gravity at the surface. Then there will be certain symmetrical displacements which are the same at all points of any concentric sphere and which may be taken to be zero at the internal surface. Superposed on these will be unsymmetrical displacements expressible at any spherical surface by series of spherical harmonics; thus there will be bodily forces depending on the attractions of the harmonic inequalities. There will also be certain surface-tractions due to the attraction exerted by the mass on the harmonic inequalities, since the surface-tractions actually fulfil conditions given at the deformed surfaces and not at the initial spherical surfaces.

10. The bodily forces acting at any point of the shell are now—

(1) Forces arising from the attraction of external matter having a potential W_i.

(2) Forces arising from the attraction of the shell and fluid on itself, and symmetrical with respect to angular space.

(3) Forces arising from the attractions of the harmonic inequalities, and having a potential which may be written $U_i + V_i/r^{2i+1}$. Then, if W_i is periodic with period $2\pi/p$, so also are U_i and V_i.

(4) To these must be added the "centrifugal force" when the shell rotates. This is partly symmetrical, and partly has a potential expressed by a zonal harmonic term of order 2. The symmetrical part of this may be treated along with the forces (2).

In our present problem, of the vibrating elastic shell, the forces (2) and the centrifugal forces can only be of importance as producing certain surface-tractions. We shall throughout suppose the rotation so slow that the square of the angular velocity multiplied by the amplitude of any of the forced harmonic inequalities (*i.e.*, by the height of any tide) is negligible. Then in the surface-tractions the terms which depend on the centrifugal force will disappear. Unless we made this supposition, the ellipticity of the figure of relative equilibrium would not be small enough to allow of our using the solutions of the differential equations within a space bounded by concentric spheres, and we should require the solutions for spheroids. When we are considering the liquid, however, the rotation must be treated differently, as it will appear that the ratio of the period of rotation to that of the disturbing force must be taken into account.

Write $$U_i + V_i/r^{2i+1} + W_i = Y_i \quad \text{.....................} \quad (31).$$

The equations of forced vibration are three such as

$$m\frac{\partial\theta}{\partial x} + n\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2} - \rho\frac{\partial Y_i}{\partial x}$$

or $$m\frac{\partial\theta}{\partial x} + n\nabla^2 u + \rho p^2 u = -\rho\frac{\partial Y_i}{\partial x} \quad \text{.................} \quad (32)$$

To get a particular integral, suppose

$$udx + vdy + wdz = d\phi,$$

then $\theta = \nabla^2\phi$, and $\dfrac{\partial}{\partial x}\left[(m+n)\nabla^2\phi + p^2\rho\phi + \rho Y_i\right] = 0$,

so that a particular integral is $\phi = -Y_i/p^2$.

Thus $$u = -\frac{1}{p^2}\frac{\partial Y_i}{\partial x}, \quad v = -\frac{1}{p^2}\frac{\partial Y_i}{\partial y}, \quad w = -\frac{1}{p^2}\frac{\partial Y_i}{\partial z} \quad \text{...} \quad (33).$$

For the complementary functions, we must take the general solution of the equations when $Y_i = 0$; these are equations (6), and the solutions are those given in (16), the period being $2\pi/p$, that of the forced vibrations. It is plain that we may omit the consideration of the χ, X terms, and indeed these will not be forced by the kind of action here considered.

11. The surface-tractions due to the complementary functions are given by the equations (17). We have here to consider the parts due to the particular integrals (33).

We have $$\theta = \nabla^2\phi = -\frac{1}{p^2}\nabla^2 Y_i = 0.$$

Hence the component surface-traction parallel to x is

$$-\frac{n}{p^2}\left(r\frac{\partial}{\partial r}-1\right)\left(\frac{\partial U_i}{\partial x}+\frac{\partial W_i}{\partial x}\right)-\frac{n}{p^2}\left(r\frac{\partial}{\partial r}-1\right)\frac{\partial}{\partial x}\left(\frac{V_i}{r^{2i+1}}\right)$$

$$-\frac{n}{p^2}\frac{\partial}{\partial x}\left[i(U_i+W_i)-(i+1)\frac{V_i}{r^{2i+1}}\right]$$

$$=-\frac{n}{p^2}\left[\frac{\partial}{\partial x}(i-2)(U_i+W_i)-(i+3)\frac{\partial}{\partial x}\left(\frac{V_i}{r^{2i+1}}\right)\right]$$

$$-\frac{n}{p^2}\left[i\frac{\partial}{\partial x}(U_i+W_i)-(i+1)\frac{\partial}{\partial x}\left(\frac{V_i}{r^{2i+1}}\right)\right]$$

$$=-\frac{2n}{p^2}\left[(i-1)\left(\frac{\partial U_i}{\partial x}+\frac{\partial W_i}{\partial x}\right)-(i+2)\frac{\partial}{\partial x}\left(\frac{V_i}{r^{2i+1}}\right)\right] \quad\ldots\ldots\ldots (34).$$

12. *To find* U_i, V_i. Let σ be the density of the contained fluid, then $U_i =$ potential within a sphere of radius a of a surface distribution of density $\rho\left(\frac{ux+vy+wz}{r}\right)_a$, and $\frac{V_i}{r^{2i+1}}$ is the potential outside a sphere of radius b of a surface distribution of density

$$(\sigma-\rho)\left(\frac{ux+vy+wz}{r}\right)_b.$$

The surface-density for U_i is

$$\frac{\rho}{a}\left\{\begin{array}{c}\frac{i+1}{p^2}\frac{V_i}{r^i}\frac{1}{a^{i+1}}-\frac{i}{p^2}\frac{U_i+W_i}{r^i}a^i+\frac{i}{a}\left(\psi_i(ka)\frac{\phi_i}{r^i}a^i+\Psi_i(ka)\frac{\Phi_i}{r^i}a^i\right)\\ \left(e_i\frac{\omega_i}{r^i}+E_i\frac{\Omega_i}{r^i}\right)a^i\end{array}\right\}.$$

Hence $U_i = \frac{4\pi\gamma}{a^{i-1}.2i+1}$* times this, so that

$$U_i\left(1+\frac{4\pi\gamma\rho}{p^2}\frac{i}{2i+1}\right)-V_i\frac{4\pi\gamma\rho}{p^2}\frac{i+1}{2i+1.a^{2i+1}}$$

$$=\frac{4\pi\gamma\rho}{2i+1}\left[\frac{i}{a}\{\psi_i(ka)\phi_i+\Psi_i(ka)\Phi_i\}-(e_i\omega_i+E_i\Omega_i)-\frac{i}{p^2}W_i\right].$$

The surface-density for $\frac{V_i}{r^{2i+1}}$ is

$$\frac{\sigma-\rho}{b}\left\{\begin{array}{c}\frac{i+1}{p^2}\frac{V_i}{r_i}\frac{1}{b^{i+1}}-\frac{i}{p^2}\frac{U_i+W_i}{r^i}b^i+\frac{i}{b}\left(\psi_i(kb)\frac{\phi_i}{r^i}+\Psi_i(kb)\frac{\Phi_i}{r^i}\right)b^i\\ -\left(e_i'\frac{\omega_i}{r^i}+E_i'\frac{\Omega_i}{r^i}\right)b^i\end{array}\right\}$$

$$\ldots\ldots\ldots (35).$$

* γ is the "constant of gravitation."

And $\frac{V_i}{r^{2i+1}} = \frac{4\pi\gamma b^{i+2}}{r^{2i+1}(2i+1)}$ times this, so that

$$V_i\left(1-\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i+1}{2i+1}\right)+U_i\,\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i}{2i+1}\,b^{2i+1}$$

$$=\frac{4\pi\gamma(\sigma-\rho)\,b^{2i+1}}{2i+1}\left[\frac{i}{b}\left\{\psi_i(kb)\,\phi_i+\Psi_i(kb)\,\Phi_i\right\}-(e_i'\omega_i+E_i'\Omega_i)-\frac{i}{p^2}W_i\right].$$

From (35) it appears that the radial velocity at $r = b$ is

$$\frac{\iota p}{b}\left[\frac{i+1}{p^2}\,\frac{V_i}{r^i}\,\frac{1}{b^{i+1}}-\frac{i}{p^2}\,\frac{U_i+W_i}{r^i}\,b^i\right.$$

$$\left.+\frac{i}{b}\left(\psi_i(kb)\,\frac{\phi_i}{r^i}+\Psi_i(kb)\,\frac{\Phi_i}{r^i}\right)b^i-\left(e_i'\,\frac{\omega_i}{r^i}+E_i'\,\frac{\Omega_i}{r^i}\right)b^i\right] \;\ldots\ldots\; (36).$$

This will be useful when we come to treat the motion of the liquid.

Solving for U_i, V_i, we have

$$U_i\left[1+\frac{4\pi\gamma}{p^2}\left(\rho+\frac{i+1}{2i+1}\,\sigma\right)-\frac{4\pi\gamma\rho\,.\,4\pi\gamma(\sigma-\rho)}{p^4}\,\frac{i(i+1)}{(2i+1)^2}\left(1-\frac{b^{2i+1}}{a^{2i-1}}\right)\right]$$

$$=\frac{4\pi\gamma i}{2i+1}\left[\frac{\rho}{a}\left(1-\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i+1}{2i+1}\right)\left\{\psi_i(ka)\,\phi_i+\Psi_i(ka)\,\Phi_i\right\}\right.$$

$$\left.+\frac{\sigma-\rho}{b}\,\frac{4\pi\gamma\rho}{p^2}\,\frac{i+1}{2i+1}\,\frac{b^{2i+1}}{a^{2i+1}}\left\{\psi_i(kb)\,\phi_i+\Psi_i(kb)\,\Phi_i\right\}\right]$$

$$-\frac{4\pi\gamma}{2i+1}\left[\rho\left(1-\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i+1}{2i+1}\right)(e_i\omega_i+E_i\Omega_i)\right.$$

$$\left.+(\sigma-\rho)\,\frac{4\pi\gamma\rho}{p^2}\,\frac{i+1}{2i+1}\,\frac{b^{2i+1}}{a^{2i+1}}\,(e_i'\omega_i+E_i'\Omega_i)\right]$$

$$-\frac{4\pi\gamma\rho}{p^2}\,\frac{i}{2i+1}\left[1-\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i+1}{2i+1}\left(1-\frac{b^{2i+1}}{a^{2i+1}}\right)\right]W_i \;\ldots\ldots\ldots\ldots (37),$$

and

$$V_i\left[1+\frac{4\pi\gamma}{p^2}\left(\rho+\frac{i+1}{2i+1}\,\sigma\right)-\frac{4\pi\gamma\rho\,.\,4\pi\gamma(\sigma-\rho)}{p^4}\,\frac{i(i+1)}{(2i+1)^2}\left(1-\frac{b^{2i+1}}{a^{2i+1}}\right)\right]\frac{1}{b^{2i+1}}$$

$$=\frac{4\pi\gamma i}{2i+1}\left[\frac{\sigma-\rho}{b}\left(1+\frac{4\pi\gamma\rho}{p^2}\,\frac{i}{2i+1}\right)\left\{\psi_i(kb)\,\phi_i+\Psi_i(kb)\,\Phi_i\right\}\right.$$

$$\left.-\frac{\rho}{a}\left(\frac{4\pi\gamma(\sigma-\rho)}{p^2}\,\frac{i}{2i+1}\right)\left\{\psi_i(ka)\phi_i+\Psi_i(ka)\,\Phi_i\right\}\right]$$

$$-\frac{4\pi\gamma}{2i+1}\left[(\sigma-\rho)\left(1+\frac{4\pi\gamma\rho}{p^2}\frac{i}{2i+1}\right)(e_i'\omega_i+E_i'\Omega_i)\right.$$

$$\left.-\rho\frac{4\pi\gamma(\sigma-\rho)}{p^2}\frac{i}{2i+1}(e_i\omega_i+E_i\Omega_i)\right]$$

$$-\frac{4\pi\gamma(\sigma-\rho)}{p^2}\frac{i}{2i+1}W_i \qquad (38).$$

Write these

$$\left.\begin{aligned}U_i &= g_i\phi_i+h_i\omega_i+G_i\Phi_i+H_i\Omega_i+\alpha_i W_i\\ V_i &= g_i'\phi_i+h_i'\omega_i+G_i'\Phi_i+H_i'\Omega_i+\beta_i W_i\end{aligned}\right\} \qquad (39).$$

Then (34) contributes to the surface-tractions parallel to x

$$\left.\begin{aligned}&-\frac{2n}{p^2}(i-1)\left[(\alpha_i+1)\frac{\partial W_i}{\partial x}+g_i\frac{\partial\phi_i}{\partial x}+h_i\frac{\partial\omega_i}{\partial x}+G_i\frac{\partial\Phi_i}{\partial x}+H_i\frac{\partial\Omega_i}{\partial x}\right]\\ &+\frac{2n}{p^2}(i+2)\left[\beta_i\frac{\partial}{\partial x}\left(\frac{W_i}{r^{2i+1}}\right)+g_i'\frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right)+h_i'\frac{\partial}{\partial x}\left(\frac{\omega_i}{r^{2i+1}}\right)\right.\\ &\qquad\left.+G_i'\frac{\partial}{\partial x}\left(\frac{\Phi_i}{r^{2i+1}}\right)+H_i'\frac{\partial}{\partial x}\left(\frac{\Omega_i}{r^{2i+1}}\right)\right]\end{aligned}\right\} \ldots(40).$$

With the exception of the W_i terms, these are of the same form as the terms of (17), and it is to be noticed that the surface harmonics which occur in these formulæ are the same at both bounding surfaces. The parts of the surface-tractions which depend on the complementary functions are of the same forms as those given by (17) with the χ, X omitted.

13. We have now to consider the surface-tractions arising from the symmetrical parts of the forces acting.

These are $\frac{2}{3}\omega^2 r$, and $-\frac{4}{3}\pi\gamma\left(\rho\frac{r^3-b^3}{r^2}+\sigma\frac{b^3}{r^2}\right)$.

The equation giving the radial displacement U' is

$$(m+n)\left(\frac{d^2U'}{dr^2}+\frac{2}{r}\frac{dU'}{dr}-\frac{2U'}{r^2}\right)+\rho\left[\tfrac{2}{3}\omega^2 r-\tfrac{4}{3}\pi\gamma\left(\rho\frac{r^3-b^3}{r^2}+\sigma\frac{b^3}{r^2}\right)\right]=0,*$$

This may be written

$$(m+n)\left(r\frac{d}{dr}-1\right)\left(r\frac{d}{dr}+2\right)U'+r^2\rho\left[\tfrac{2}{3}\omega^2 r-\tfrac{4}{3}\pi\gamma\left(\rho\frac{r^3-b^3}{r^2}+\sigma\frac{b^3}{r^2}\right)\right]=0,$$

* Webb, "Stress and Strain in Cylindrical and Polar Coordinates" (*Messenger*, 1882).

and the solution is

$$U' = Ar + \frac{B}{r^2} + \frac{1}{10}\,\frac{r^3\rho}{m+n}\left(\tfrac{4}{3}\pi\gamma\rho - \tfrac{2}{3}\omega^2\right) + \tfrac{2}{3}\pi\gamma\,(\sigma-\rho)\,\frac{b^3\rho}{m+n},$$

where A and B are arbitrary constants.

Write this for shortness

$$U' = Ar + \frac{B}{r^2} + Hr^3 + K \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41),$$

where A and B are arbitrary, H and K known constants. The corresponding u, v, w are given by multiplying this by x/r, y/r, z/r respectively, and then for the cubical dilatation and the 6 strains we have

$$\theta = 5Hr^2 + 3A = e + f + g,$$

$$\left.\begin{aligned} e &= H(r^2+2x^2) + K\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + A + B\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) \\ f &= H(r^2+2y^2) + K\left(\frac{1}{r} - \frac{y^2}{r^3}\right) + A + B\left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) \\ g &= H(r^2+2z^2) + K\left(\frac{1}{r} - \frac{z^2}{r^3}\right) + A + B\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \end{aligned}\right\},$$

$$\left.\begin{aligned} a &= 4Hyz - 2K\frac{yz}{r^3} - 6B\frac{yz}{r^5} \\ b &= 4Hzx - 2K\frac{zx}{r^3} - 6B\frac{zx}{r^5} \\ c &= 4Hxy - 2K\frac{xy}{r^3} - 6B\frac{xy}{r^5} \end{aligned}\right\}.$$

The stresses P, Q, R, S, T, U are then given by

$$P = (m-n)\,\theta + 2ne, \;\ldots\; S = na, \;\ldots\,.$$

14. Let

$$\left.\begin{aligned} r &= a + \beta_0 Q_i \text{ be the external surface,} \\ r &= b + \beta_1 Q_i \text{ be the internal surface,} \end{aligned}\right\} Q_i \text{ being a solid harmonic;}$$

then it is at these surfaces that we require the surface-tractions. When the shell contains a given mass of liquid, the condition at $r=b$ is $U'=0$. At a free surface we must make the surface-tractions vanish. Let l', m', n' be the direction-cosines of the outward-drawn normal to the surface $r = a + \beta Q_i$; then parallel to x the component

surface-traction is

$$(l'P+m'U+n'T) = \left(1+\frac{\beta i Q_i}{r}\right)\left(\frac{x}{r}P+\frac{y}{r}U+\frac{z}{r}T\right)$$

$$-\beta\left(P\frac{\partial Q_i}{\partial x}+U\frac{\partial Q_i}{\partial y}+T\frac{\partial Q_i}{\partial z}\right).$$

Here $P\dfrac{x}{r}+U\dfrac{y}{r}+T\dfrac{z}{r}=\left(Hr^2(5m+n)+A(3m-n)-\dfrac{4n}{r^3}B\right)\dfrac{x}{r}$,

and
$$P\frac{\partial Q_i}{\partial x}+U\frac{\partial Q_i}{\partial y}+T\frac{\partial Q_i}{\partial z}$$

$$= Hr^2(5m-3n)\frac{\partial Q_i}{\partial x}+4nHixQ_i+(3m-n)A\frac{\partial Q_i}{\partial x}$$

$$+2n\frac{K}{r}\frac{\partial Q_i}{\partial x}-\frac{2nKx}{r^3}iQ_i+\frac{2nB}{r^3}\frac{\partial Q_i}{\partial x}-\frac{6nBx}{r^5}iQ_i.$$

These and the condition $U'=0$ when $r=b$ give, for A and B,

$$\left.\begin{aligned}&Ab+\frac{B}{b^2}+Hb^3+K=0\\&(5m+n)Ha^2+(3m-n)A-\frac{4nB}{a^3}=0\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots(42),$$

where a is the mean outer radius after the strain is effected, *i.e.*, this is the a of our previous work.

Also the surface-tractions at $r=a+\beta Q_i$ which depend on Q_i are

$$\left[2Ha\beta Q_i(5m+n)-\frac{12nB}{a^4}\beta Q_i\right]\frac{x}{a}$$

$$+\left(-\frac{\beta Q_i}{a}+\frac{i\beta Q_i}{a}\right)\left(Ha^2(5m+n)+(3m-n)A-\frac{4n}{a^3}B\right)\frac{x}{a}$$

$$-\beta\frac{\partial Q_i}{\partial x}\left(Ha^2(5m-3n)+(3m-n)A+\frac{2nK}{a}+\frac{2nB}{a^3}\right)$$

$$-2n\beta xiQ_i\left(2H-\frac{K}{a^3}-\frac{3B}{a^5}\right),$$

parallel to x, with two similar expressions for those parallel to y and z.

The parts contributed by these to $F.r$ are, at the external surface,

$$\left.\begin{aligned}&x\left[2Ha\beta_0Q_i(5m+n)-\frac{12nB}{a^4}\beta_0Q_i\right]\\&-\beta_0\frac{\partial Q_i}{\partial x}\left[\frac{2nK}{a}+\frac{6nB}{a^3}-4nHa^2\right]a\\&-2ni\beta_0xQ_i\left[2H-\frac{K}{a^3}-\frac{3B}{a^5}\right]a\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots(43),$$

and at the internal surface,

$$\left.\begin{aligned}&x\beta_1 Q_i\left[2Hb\,(5m+n)-\frac{12nB}{b^4}\right]\\&+ix\beta_1 Q_i\left[Hb\,(5m+n)+(3m-n)\frac{A}{b}-\frac{4n}{b^4}B\right]\\&-\beta_1\frac{\partial Q_i}{\partial x}\left[Hb^3\,(5m-3n)+(3m-n)\,Ab+2nK+\frac{2nB}{b^3}\right]\\&-2ni\beta_1 xQ_i b\left[2H-\frac{K}{b^3}-\frac{3B}{b^6}\right]\end{aligned}\right\}.$$

Of these, the part at the external surface is

$$\left.\begin{aligned}&2\beta_0\frac{a^3}{2i+1}\frac{\partial Q_i}{\partial x}\left[Ha\,\{5m+(2i+3)\,n\}-\frac{i+1}{a^3}nK+\frac{3n\,(i+3)}{a^6}B\right]\\&-2\beta_0\frac{a^{2i+8}}{2i+1}\frac{\partial}{\partial x}\left(\frac{Q_i}{r^{2i+1}}\right)\left[Ha\,\{5m-(2i-1)\,n\}+\frac{2i}{a^3}nK+\frac{3n\,(i-2)}{a^6}B\right]\end{aligned}\right\} \quad \text{.........}(44),$$

and at the internal surface,

$$\left.\begin{aligned}&\beta_1\frac{b^3}{2i+1}\frac{\partial Q_i}{\partial x}\Big[\{n\,(5+3i)-5m\,(i-1)\}\,Hb\\&\qquad\qquad-2\,(i+1)\,n\frac{K}{b^3}-(3m-n)(i+1)\frac{A}{b}+(6i-10)\frac{nB}{b^6}\Big]\\&-\beta_1\frac{b^{2i+8}}{2i+1}\frac{\partial}{\partial x}\left(\frac{Q_i}{r^{2i+1}}\right)\Big[\{5m\,(i+2)-n\,(3i-2)\}\,Hb\\&\qquad\qquad+2in\frac{K}{b^3}+(3m-n)\,i\frac{A}{b}+(2i-12)\frac{nB}{b^6}\Big]\end{aligned}\right\} \quad \text{..............}(45).$$

In these $\beta_0 Q_i$ is

$$\left.\begin{aligned}&\frac{1}{a}\Big[\frac{i+1}{p^3}\frac{V_i}{a^{2i+1}}-\frac{i}{p^3}(U_i+W_i)\\&\qquad+\frac{i}{a}\{\psi_i(ka)\,\phi_i+\Psi_i(ka)\,\Phi_i\}-(e_i\omega_i+E_i\Omega_i)\Big]\\&\text{and } \beta_1 Q_i \text{ is}\\&\frac{1}{b}\Big[\frac{i+1}{p^3}\frac{V_i}{b^{2i+1}}-\frac{i}{p^3}(U_i+W_i)\\&\qquad+\frac{i}{b}\{\psi_i(kb)\,\phi_i+\Psi_i(kb)\,\Phi_i\}-(e_i'\omega_i+E_i'\Omega_i)\Big]\end{aligned}\right\} \quad \text{...}(46).$$

As $\omega^2\beta$ is of the order neglected, we may suppose A, B, H in (45) simplified by the omission of all terms in ω^2, so that

$$H = (\tfrac{4}{3}\pi\gamma\rho)\frac{1}{5}\frac{\rho}{m+n}, \quad K = \tfrac{4}{3}\pi\gamma(\sigma-\rho)\,b^3\frac{\rho}{m+n},$$

$$A\left(\frac{3m-n}{b^3}+\frac{4n}{a^3}\right)$$

$$= -(\tfrac{4}{3}\pi\gamma\rho)\frac{\rho}{m+n}\left[\frac{1}{5}\frac{a^2}{b^3}(5m+n)+\left(\frac{\sigma}{\rho}-\frac{4}{5}\right)\frac{b^2}{a^3}\,4n\right],$$

$$B\left(\frac{3m-n}{b^3}+\frac{4n}{a^3}\right)$$

$$= -(\tfrac{4}{3}\pi\gamma\rho)\frac{\rho}{m+n}\left[-\frac{1}{5}a^2(5m+n)+\left(\frac{\sigma}{\rho}-\frac{4}{5}\right)b^2(3m-n)\right].$$

15. As a verification consider what these become in case there is no cavity, but the sphere is complete up to the centre and incompressible; in this case we know that the stresses should reduce to a normal stress equal to the weight of the harmonic inequality.*

The constant B is now zero, and A is given by

$$(5m+n)\,Ha^2+(3m-n)\,A = 0.$$

K is also zero, and

$$H = \frac{1}{10}\left(\frac{4}{3}\pi\rho\right)\frac{\rho}{m+n}.$$

Supposing m very great compared with n, we have

$$A = -\tfrac{5}{3}Ha^2,$$

and we need only consider terms in which H or A is multiplied by a factor containing m; thus by (43) the normal stress is

$$\frac{1}{r}\left(Fr\frac{x}{r}+Gr\frac{y}{r}+Hr\frac{z}{r}\right)$$

$$= a\,2H\beta_0 Q_i\,(5m+n) = \tfrac{4}{3}(\pi\gamma\rho a)(\rho\beta_0 Q_i) = g\,\rho\beta_0 Q_i,$$

where $g = \tfrac{4}{3}(\pi\gamma\rho a)$ is the value of gravity at the surface.

* Darwin on the "Bodily Tides of Viscous and semi-elastic Spheroids" (*Phil. Trans.*, 1879, p. 6).

Also the only component stresses which do not vanish are

$$\frac{x}{r}\,2Ha\,Q_i\beta_0(5m+n),\quad \frac{y}{r}\,2Ha\,Q_i\beta_0(5m+n),\quad \frac{z}{r}\,2Ha\,Q_i\beta_0(5m+n),$$

which give no tangential stress.

These calculations are correct to the first order of the small quantity β_0, and we know that the tangential stress is in the case supposed small of the second order. This is no longer true when the sphere is not supposed incompressible.

16. To calculate the fluid pressure we must take account of the rotation. Suppose we know a fluid motion which satisfies the equations of Hydrodynamics and gives no normal flow at a surface S; then if u, v, w be the components of this motion, these satisfy (a) three such equations as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial}{\partial x}\left(V-\frac{p_1}{\sigma}\right),$$

where V is the potential of the acting forces, also (b) the equation of continuity $\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0$, and ($c$) the boundary condition $lu+mv+nw=0$ at S, (l, m, n) being the direction-cosines of the normal to S drawn outwards.

We are going to take for the known motion a steady motion of rotation with angular velocity ω about the axis of z, so that

$$u=-\omega y,\quad v=\omega x,\quad w=0,$$

and to superpose a small oscillatory motion, consisting of an irrotational oscillation combined with periodic changes in the rate and axis of rotation; and we are going to make the boundary change form without changing its volume; *i.e.*, we take for the components of the disturbed motion $u+u'$, $v+v'$, $w+w'$, where u', v', w' are small, and

$$u'=\frac{\partial\phi}{\partial x}-\omega_3 y+\omega_2 z,\quad v'=\frac{\partial\phi}{\partial y}-\omega_1 z+\omega_3 x,\quad w'=\frac{\partial\phi}{\partial z}-\omega_2 x+\omega_1 y,$$

and suppose ϕ satisfies $\nabla^2\phi=0$, while ω_1, ω_2, ω_3 are functions of the time. We suppose that the bounding surface is a sphere deformed into an harmonic spheroid, so that the boundary condition at the surface $r=b$ is of the form

$$r\frac{\partial\phi}{\partial r}=lu'+mv'+nw'=\Sigma S_i,$$

where S_i is a surface harmonic; we shall show that for such a motion to be possible the surface harmonic must be of order 2.

The Helmholtz equations of vortex motion referred to the moving axes* are such as

$$\frac{\partial \xi}{\partial t}-\eta\omega+u'\frac{\partial \xi}{\partial x}+v'\frac{\partial \xi}{\partial y}+w'\frac{\partial \xi}{\partial z}=\xi\frac{\partial(u+u')}{\partial x}+\eta\frac{\partial(u+u')}{\partial y}+\zeta\frac{\partial(u+u')}{\partial z},$$

where $\qquad \xi=\omega_1, \quad \eta=\omega_2, \quad \zeta=\omega+\omega_3.$

Neglecting products of small quantities, these become

$$\left.\begin{aligned}\dot{\omega}_1-\omega\omega_2 &= \omega_2(-\omega)+\omega\frac{\partial^2\phi}{\partial x\partial z}\\ \dot{\omega}_2+\omega\omega_1 &= \omega_1\omega+\omega\frac{\partial^2\phi}{\partial y\partial z}\\ \dot{\omega}_3 \quad\quad &= \omega\frac{\partial^2\phi}{\partial z^2}\end{aligned}\right\} \quad \text{.......................(47)},$$

so that $\qquad \dot{\omega}_1=\omega\frac{\partial^2\phi}{\partial x\,\partial z}, \quad \dot{\omega}_2=\omega\frac{\partial^2\phi}{\partial y\partial z}, \quad \dot{\omega}_3=\omega\frac{\partial^2\phi}{\partial z^2},$

and $\frac{\partial\phi}{\partial z}$ is a linear function $(\dot{\omega}_1x+\dot{\omega}_2y+\dot{\omega}_3z)/\omega$ of x, y, z, which shows that ϕ is a solid harmonic of order 2. Hence $r\frac{\partial\phi}{\partial r}$ contains a surface harmonic of order 2, and the sphere must be deformed into a surface harmonic of order 2. Thus it is only in case the disturbing function is of the second order that this method can be applied.

17. Let V' be the potential of all the forces acting on the liquid, not counting the "centrifugal force." V' contains terms depending on (1) the attraction of the shell and fluid on itself, (2) the attraction of the external disturbing bodies, (3) the attraction of the harmonic inequalities. And the harmonic inequalities are of two sorts, one a function of the time depending on the external disturbing bodies, and the other independent of the time, and due to the rotation.

If, as before, p_1 be the pressure, and σ the density of the fluid, the

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* Greenhill, *Proc. Camb. Phil. Soc.*, iv., p. 6, 1882. This method of proving equations (47) was suggested by the referee.

equations may be written—

$$\frac{\partial}{\partial x}\left(V'-\frac{p_1}{\sigma}\right)=\frac{\partial^2\phi}{\partial x\partial t}-\omega^2x+\left[-\left\{\dot{\omega}_3+\omega\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\right\}\right]y$$

$$+\omega\left(y\frac{\partial^2\phi}{\partial y^2}-\frac{\partial\phi}{\partial y}\right)-2\omega\omega_3x+(\dot{\omega}_2+\omega\omega_1)\,z+\omega x\frac{\partial^2\phi}{\partial x\partial y},$$

$$\frac{\partial}{\partial y}\left(V'-\frac{p_1}{\sigma}\right)=\frac{\partial^2\phi}{\partial y\partial t}-\omega^2y+\left[\dot{\omega}_3+\omega\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\right]x$$

$$-\omega\left(x\frac{\partial^2\phi}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-2\omega\omega_3y+(\omega\omega_2-\dot{\omega}_1)\,z-\omega y\frac{\partial^2\phi}{\partial x\partial y},$$

$$\frac{\partial}{\partial z}\left(V'-\frac{p_1}{\sigma}\right)=\frac{\partial^2\phi}{\partial z\partial t}-\left(\dot{\omega}_2-\omega\frac{\partial^2\phi}{\partial y\partial z}\right)x+\left(\dot{\omega}_1-\omega\frac{\partial^2\phi}{\partial x\partial z}\right)y+\omega\omega_2y+\omega\omega_1x.$$

In these the terms in [ ] vanish, and, since ϕ is homogeneous of order 2,

$$\frac{\partial\phi}{\partial y}=x\frac{\partial^2\phi}{\partial x\partial y}+y\frac{\partial^2\phi}{\partial y^2}+z\frac{\partial^2\phi}{\partial y\partial z},$$

$$\frac{\partial\phi}{\partial x}=x\frac{\partial^2\phi}{\partial x^2}+y\frac{\partial^2\phi}{\partial x\partial y}+z\frac{\partial^2\phi}{\partial x\partial z};$$

hence, using (47), the only terms which remain when we multiply by dx, dy, dz, add, and integrate, are included in

$$V'-\frac{p_1}{\sigma}=\frac{\partial\phi}{\partial t}+f(t)+\int(-\omega^2x-2\omega\omega_3x+\omega\omega_1z)\,dx$$

$$+(-\omega^2y-2\omega\omega_3y+\omega\omega_2z)\,dy+\omega(\omega_2y+\omega_1x)\,dz,$$

so that
$$V'-\frac{p_1}{\sigma}=\frac{\partial\phi}{\partial t}+f(t)-\tfrac{1}{2}\omega^2(x^2+y^2)-\omega\omega_3(x^2+y^2)$$

$$+\omega\omega_1xz+\omega\omega_2yz \quad\ldots\ldots\ldots\ (48),$$

where $f(t)$ is an arbitrary function of t.

If ω be not too great, the solutions depending on the terms of V' which are periodic, and those which are independent of the time, remain separate. The new terms introduced by the rotation into the pressure equation (48) are those in $\omega\omega_1$, $\omega\omega_2$, and $\omega\omega_3$, and it is on the relative magnitude of these and ω^2 that the subsequent discussion turns.

In calculating the pressure from equation (48), we see, by (47), that if p be great compared with ω, the products $\omega\omega_1$, $\omega\omega_2$, $\omega\omega_3$ are small compared with ω^2, and may be neglected; if p be comparable with ω, ω_1 is comparable with β; and if β be small compared with the ellipticity due to the rotation ω, these terms can still be neglected. The term $\omega^2\beta$ is always negligible, so that, unless p be small compared

with ω, the only way in which ω enters into the equations determining the amplitude is by occurring in H, and consequently in A and B. But H is $\frac{1}{10}\frac{1}{m+n}(\frac{4}{3}\pi\gamma\rho-\frac{2}{3}\omega^2)$, and, $\frac{\omega^2}{2\pi\gamma\rho}$ being very small, we see that a rotation ω which is small of this order, does not sensibly alter the amplitude except in the case of oscillations of very long period.

Considering the Earth as a liquid mass contained within a solid elastic shell whose internal and external surfaces are approximately spherical, we see that the amount by which it would yield to the tidal disturbing force of the sun and moon, regarded as having a diurnal or semi-diurnal period, is not sensibly altered by the angular velocity of rotation, since the ratio $\frac{\omega^2 a}{g}=\frac{1}{289}$. If we consider the fortnightly tides we see that $\left(\frac{p}{\omega}\right)=\frac{1}{14}$ nearly, and the terms in (48) hitherto neglected are of the order $14\omega\beta$, which we may take to be small compared with ω^2; so that, with somewhat less accuracy as the period increases, it will be true of the tides of shorter period but not of the semiannual and annual tides, that the "quasi-rigidity" set up in the liquid by the vertex-motion is not such as to interfere sensibly with calculations of the resistance of the Earth to tidal distortion founded on the neglect of the spin. Comparing this result with the investigations of Thomson and Darwin on the rigidity of the Earth, we seem to be justified in concluding that the Earth is not such a shell as is here considered.

18. To consider more generally the problem of the rotating shell, let us refer the motion of the liquid to moving axes which rotate with angular velocity ω about the axis z fixed in space.

The equations of motion are three, such as

$$\frac{\partial u}{\partial t}-\omega v+U\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial z}=\frac{\partial}{\partial x}\left(V'-\frac{p_1}{\sigma}\right),*$$

where U, V, W are the rates at which the coordinates of a fluid particle are changing, so that

$$U=u+\omega y,\quad V=v-\omega x,\quad W=w.$$

* Greenhill, article on "Hydromechanics," in *Encyclopædia Britannica.*

Thus the equations may be written

$$\frac{\partial U}{\partial t}-\omega\,(V+\omega x)+U\frac{\partial U}{\partial x}+V\left(\frac{\partial U}{\partial y}-\omega\right)+w\frac{\partial U}{\partial z}=\frac{\partial}{\partial x}\left(V'-\frac{p_1}{\sigma}\right),$$

$$\frac{\partial V}{\partial t}+\omega\,(U-\omega y)+U\left(\frac{\partial V}{\partial x}+\omega\right)+V\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z}=\frac{\partial}{\partial y}\left(V'-\frac{p_1}{\sigma}\right),$$

$$\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+V\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{\partial}{\partial z}\left(V'-\frac{p_1}{\sigma}\right).$$

Let us suppose the liquid to rotate as if rigid about the axis z, and, superposed on this rotatory motion, let there be a small oscillatory motion, then U, V, w are all small, and we have

$$\left.\begin{aligned}
&\frac{\partial U}{\partial t}-2\omega V=\frac{\partial}{\partial x}\left[V'-\frac{p_1}{\sigma}+\tfrac{1}{2}\omega^2(x^2+y^2)\right]=\frac{\partial\psi}{\partial x}\ \text{say}\\
&\frac{\partial V}{\partial t}+2\omega U=\frac{\partial\psi}{\partial y}\\
&\frac{\partial w}{\partial t}\qquad\quad=\frac{\partial\psi}{\partial z}\\
&\text{and the equation of continuity is}\\
&\qquad\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial w}{\partial z}=0
\end{aligned}\right\}\ldots\ldots(49).$$

Thus
$$\nabla^2\psi=-2\omega\frac{\partial V}{\partial x}+2\omega\frac{\partial U}{\partial y},$$

$$\frac{\partial}{\partial t}\nabla^2\psi=-2\omega\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}-2\omega U\right)+2\omega\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}+2\omega V\right)$$

$$=4\omega^2\left(-\frac{\partial w}{\partial z}\right);$$

hence
$$\frac{\partial^2}{\partial t^2}(\nabla^2\psi)=-4\omega^2\frac{\partial^2\psi}{\partial z^2}\ldots\ldots\ldots\ldots\ldots\ldots(50).*$$

Let us now suppose U, V, $w\propto e^{\iota pt}$, then

$$\nabla^2\psi-\frac{4\omega^2}{p^2}\frac{\partial^2\psi}{\partial z^2}=0,$$

or
$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z'^2}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(51),$$

where
$$z'^2\left(1-\frac{4\omega^2}{p^2}\right)=z^2.$$

* This equation is due to Poincaré, *Acta Mathematica*, Vol. VII., 1885, p. 356.

19. We have to solve the differential equation (50) within a sphere

$$x^2+y^2+z^2 = b^2.$$

This is the same thing as solving the equation (51) within the quadric

$$\frac{x^2+y^2}{b^2}+\frac{z'^2}{b^2}\left(1-\frac{4\omega^2}{p^2}\right) = 1.$$

Let $x^2+y^2 = \rho^2$, and $\tan\phi = \frac{y}{x}$, then the equation is

$$\frac{\partial^2\psi}{\partial z'^2}+\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0 \quad\text{..................(52).}$$

(1) Suppose $1-\frac{4\omega^2}{p^2}$ positive, then the surface within which we have to solve $0 = \nabla^2\psi$ is a spheroid of revolution about its greatest axis.

Take conjugate functions, such that

$$z'+\iota\rho = c\cosh(\alpha+\iota\beta),$$

where $\qquad c^2 = b^2\frac{4\omega^2}{p^2-4\omega^2}$, and write $b_1^2 = \frac{b^2p^2}{p^2-4\omega^2}$.

Then, taking $\psi \propto e^{\iota s\phi}$, we find, for ψ as a function of α, β, the equation

$$\frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}+\coth\alpha\frac{\partial\psi}{\partial\alpha}+\cot\beta\frac{\partial\psi}{\partial\beta}-s^2\psi\left(\frac{1}{\sinh^2\alpha}+\frac{1}{\sin^2\beta}\right) = 0.$$

Let $\psi = S\,.\,T$, where S is a function of α only, and T of β only; then

$$\frac{d^2S}{d\alpha^2}+\coth\alpha\frac{dS}{d\alpha}-\frac{s^2}{\sinh^2\alpha}S-i(i+1)S = 0,$$

and
$$\frac{d^2T}{d\beta^2}+\cot\beta\frac{dT}{d\beta}-\frac{s^2}{\sin^2\beta}T+i(i+1)T = 0,$$

where $i(i+1)$ is some constant.

Put $\cosh\alpha = \nu$, $\cos\beta = \mu$, then

$$\left.\begin{aligned}\frac{d}{d\nu}\left[(1-\nu^2)\frac{dS}{d\nu}\right]+\left[i(i+1)-\frac{s^2}{1-\nu^2}\right]S = 0\\ \frac{d}{d\mu}\left[(1-\mu^2)\frac{dT}{d\mu}\right]+\left[i(i+1)-\frac{s^2}{1-\mu^2}\right]T = 0\end{aligned}\right\} \quad\text{.........(53),}$$

and we can obtain solutions in finite terms by taking i integral. In this case, the equations for S and T are the equations of tesseral

harmonics, the argument ν of S being always finite and greater than unity.

Thus $$\psi = \Sigma_{s<i} \Sigma_i P_i^{(s)}(\nu) P_i^{(s)}(\mu) e^{\iota s\phi} \text{*} \qquad \text{.................(54).}$$

Since $\mu = \dfrac{z'}{b_1} = \dfrac{z}{b}$, the factor $P_i^{(s)}(\mu) e^{\iota s\phi}$ is a spherical surface harmonic on the sphere of radius b. The quantity ν is given at the surface by $c^2(\nu^2-1) = b^2$, so that

$$\nu = \frac{p}{2\omega} \text{ at the surface.}$$

(2) Let $1-\dfrac{4\omega^2}{p^2}$ be negative.

Write $$b_1^2\left(\frac{4\omega^2}{p^2}-1\right) = b^2,$$

then the surface into which we project the sphere is the one-sheeted hyperboloid of revolution

$$\rho^2/b^2 - z'^2/b_1^2 = 1,$$

and taking $$c^2 = b^2+b_1^2 = 4\omega^2 b^2/(4\omega^2-p^2),$$

we write $$\rho + \iota z' = c\cosh(\alpha+\iota\beta),$$

and shall take this to represent a system of conjugate functions. The projection is imaginary, and, in relation to the motion, z' is imaginary. For the purpose of solving equation (50), however, within the real surface of the sphere, we must regard this as solving the equation (51) with z' supposed real for the space on one side of the surface of the hyperboloid also supposed real. We shall then treat the above as a system of conjugate functions, and suppose that we are solving the equation (51) with real variables for the space on one side of this surface.

Supposing $\psi \propto e^{\iota s\phi}$, and multiplying (52) by $\dfrac{\partial(z',\rho)}{\partial(\alpha,\beta)}$, we have

$$\frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}+\tanh\alpha\frac{\partial\psi}{\partial\alpha}-\tan\beta\frac{\partial\psi}{\partial\beta}-s^2\psi(\sec^2\beta-\operatorname{sech}^2\alpha) = 0;$$

---

* I write $P_i^{(s)}(x)$ for the associated function of the first kind, proportional to Ferrers's $T_i^{(s)}(x)$, and to Heine's $P_{(s)}^{(i)}(x)$; the numerical determination is unimportant in this place.

taking $\mu' = \sin\beta$, and $\nu' = \sinh\alpha$, we have $\psi = S.T$, where

$$\frac{d}{d\mu'}\left\{(1-\mu'^2)\frac{dT}{d\mu'}\right\} + \left\{i(i+1) - \frac{s^2}{1-\mu'^2}\right\} T = 0,$$

$$\frac{d}{d\nu'}\left\{(1+\nu'^2)\frac{dS}{d\nu'}\right\} - \left\{i(i+1) - \frac{s^2}{1+\nu'^2}\right\} S = 0.$$

The solutions are $\quad P_i^{(s)}(\sin\beta), \quad P_i^{(s)}(\iota\sinh\alpha).$

These solutions are obtained by supposing α a real variable; but, when z' is imaginary, α becomes imaginary, so that $\iota\sinh\alpha$ is real in the motion, and on the surface

$$\iota\sinh\alpha = \frac{\iota z'}{b_1} = \frac{z}{b} = \mu.$$

Hence, taking μ, μ' new variables defined by the real equations

$$\rho^2 = c^2(1-\mu^2)(1-\mu'^2), \quad z = c\mu\mu',$$

μ will be the ordinary μ at the surface, and the solutions will be

$$T = P_i^{(s)}(\mu'), \quad S = P_i^{(s)}(\mu) \qquad \text{.....................(55)},$$

which remain finite for the space considered. Thus

$$\psi = \Sigma\Sigma P_i^{(s)}(\mu')\, P_i^{(s)}(\mu)\, e^{\iota s\phi} \qquad \text{.....................(56)},$$

where at the surface $c^2(1-\mu'^2) = b^2$, so that $\mu' = p/2\omega$.

20. The normal surface velocity is given by (36); it is also to be found as follows. We have, from (49),

$$\left.\begin{aligned} U\iota p - 2\omega V &= \frac{\partial\psi}{\partial x} \\ V\iota p + 2\omega U &= \frac{\partial\psi}{\partial y} \\ w\iota p \quad &= \frac{\partial\psi}{\partial z} \end{aligned}\right\};$$

hence, if l, m, n be the direction-cosines of the outward-drawn normal to the sphere,

$$(4\omega^2-p^2)(lU+mV+nw)$$

$$= \iota p\left\{l\frac{\partial\psi}{\partial x} + m\frac{\partial\psi}{\partial y} + n\left(1-\frac{4\omega^2}{p^2}\right)\frac{\partial\psi}{\partial z}\right\} + 2\omega\left(l\frac{\partial\psi}{\partial y} - m\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{\iota p}{b}\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y} + z'\frac{b^2}{b_1^2}\frac{\partial\psi}{\partial z'}\right) + \frac{2\omega}{b}\left(x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right)$$

$$= \iota p b\left(\frac{x}{b^2}\frac{\partial\psi}{\partial x}+\frac{y}{b^2}\frac{\partial\psi}{\partial y}+\frac{z'}{b_1^2}\frac{\partial\psi}{\partial z'}\right)+\frac{2\omega}{b}\left(x\frac{\partial\psi}{\partial y}-y\frac{\partial\psi}{\partial x}\right)$$

$$= \iota p b\frac{1}{p'}\frac{\partial\psi}{\partial n'}+\frac{2\omega}{b}\frac{\partial\psi}{\partial\phi} \quad\text{...}(57),$$

where p' is the perpendicular from the centre on the tangent plane to the surface into which we project the sphere, and $\frac{\partial}{\partial n'}$ denotes differentiation along the normal to this surface drawn outwards. The above holds whether the surface into which we project the sphere be a spheroid or an hyperboloid.

The normal velocity at the surface of the sphere can always be expanded in a sum of spherical surface harmonics.

The requisite normal velocity is that of the inner surface of the shell, which is given by equation (36) with (37) and (38).

(1) Take $1-\frac{4\omega^2}{p^2}$ positive.

We have $\qquad p'dn' = c\cosh\alpha\, d\,(c\cosh\alpha) = c^2\nu\, d\nu,$

also $\qquad lU+mV+nw = \Sigma\Sigma\, C_i^{(s)} P_i^{(s)}(\mu)\, e^{\iota s\phi},$

where $C_i^{(s)}$ is a complex constant. This is to be equated to the expression in (36). Also

$$\psi = \Sigma\Sigma\, A_i^{(s)} P_i^{(s)}(\nu)\, P_i^{(s)}(\mu)\, e^{\iota s\phi};$$

hence, by (57),

$$\left[\frac{\iota p b}{c^2}\,\frac{1}{\nu}\,\frac{d}{d\nu}P_i^{(s)}(\nu)+\frac{2\iota s\omega}{b}P_i^{(s)}(\nu)\right]A_i^{(s)} = (4\omega^2-p^2)\, C_i^{(s)},$$

where $\qquad \nu = \frac{p}{2\omega},$

and thus $\qquad \psi = \Sigma\Sigma\,\dfrac{(4\omega^2-p^2)\, C_i^{(s)}\, P_i^{(s)}(\nu)\, P_i^{(s)}(\mu)\, e^{\iota s\phi}\, e^{\iota pt}}{\left[\frac{\iota p b}{c^2}\,\frac{1}{\nu}\,\frac{d}{d\nu}P_i^{(s)}(\nu)+\frac{2\iota s\omega}{b}P_i^{(s)}(\nu)\right]_{\nu=p/2\omega}} \quad\text{...........}(58).$

(2) Take $1-\frac{4\omega^2}{p^2}$ negative.

Then $\qquad p'dn' = -\,c\cos\beta\, d\,(c\cos\beta) = c^2\mu' d\mu',$

since μ' increases as we go outwards from the centre of the sphere.

Taking $\qquad \psi = \Sigma\Sigma\, A_i^{(s)} P_i^{(s)}(\mu')\, P_i^{(s)}(\mu)\, e^{\iota s\phi},$

we have $\left[\frac{\iota p b}{c^2}\,\frac{1}{\mu'}\,\frac{d}{d\mu'}P_i^{(s)}(\mu')+\frac{2\iota s\omega}{b}P_i^{(s)}(\mu')\right]A_i^{(s)} = (4\omega-p^2)\, C_i^{(s)},$

where $$\mu' = \frac{p}{2\omega};$$

thus $$\psi = \Sigma\Sigma \frac{(4\omega^2 - p^2)\, C_i^{(s)}\, P_i^{(s)}(\mu')\, P_i^{(s)}(\mu)\, e^{\iota s\phi}\, e^{\iota pt}}{\left[\frac{\iota pb}{c^2}\frac{1}{\mu'}\frac{d}{d\mu'}P_i^{(s)}(\mu') + \frac{2\iota s\omega}{b}P_i^{(s)}(\mu')\right]_{\mu'=p/2\omega}} \quad \text{.........(59)},$$

where the summation extends to all positive integral values of s which are not greater than i, and to all positive integral values of i.

21. Equation (59) expresses exactly the effect of the rotation, and gives us the means of finding the fluid motion completely, when the inner surface is deformed periodically into an harmonic spheroid. Since, however, the force necessary to keep the amplitude of the inequalities all small is only provided in case $\omega^2 C_i^{(s)}$ is small, we have to evaluate this expression for the different cases that may arise.

(1) Take ω small compared with p, then $\frac{p}{2\omega}$ approaches to an infinite limit, and we have to find the value of

$$\frac{P_i^{(s)}\left(\frac{p}{2\omega}\right)}{\left[\frac{pb}{c^2}\frac{1}{x}\frac{d}{dx}P_i^{(s)}(x) + \frac{2s\omega}{b}P_i^{(s)}(x)\right]_{x=p/2\omega}} = \gamma_i^{(s)} \text{ say,}$$

where $$\frac{p}{2\omega} = \infty.$$

We take for determination of $P_i^{(s)}(x)$,

$$P_i^{(s)}(x) = (x^2-1)^{\frac{1}{2}s}\frac{d^s}{dx^s}P_i(x).$$

Then this is $\frac{2i!}{2^i \,.\, i!\, \overline{i-s}!}$ times Heine's $P_{(s)}^{(i)}(x)$,* whose limit for $x = \infty$ is x^i, and we have

$$\frac{d}{dx}P_i^{(s)}(x) = (x^2-1)^{\frac{1}{2}s}\frac{d^{s+1}}{dx^{s+1}}P_i(x) + sx\,(x^2-1)^{\frac{1}{2}s-1}\frac{d^s}{dx^s}P_i(x)$$
$$= \frac{1}{\sqrt{x^2-1}}P_i^{(s+1)}(x) + \frac{sx}{x^2-1}P_i^{(s)}(x).$$

The limit of this, when x is ∞, is the same as that of

$$\left(\frac{i-s}{\sqrt{x^2-1}} + \frac{sx}{x^2-1}\right)P_i^{(s)}(x).$$

* Heine, *Handbuch der Kugelfunctionen*, Ch. IV., p. 206.

Thus $$\left[\gamma_i^{(s)}\right]_{x=\infty} = \frac{1}{\frac{pb}{b^2}(x^2-1)\frac{1}{x}\left(\frac{i-s}{\sqrt{x^2-1}}+\frac{sx}{x^2-1}\right)+\frac{2s\omega}{b}}$$

$$= \frac{1}{\frac{p}{b}i+\frac{2s\omega}{b}} = \frac{b}{ip} \text{ ultimately.}$$

In this case

$$\psi = \Sigma\Sigma\iota\frac{pb}{i}C_i^{(s)}P_i^{(s)}(\mu)\,e^{\iota s\phi}e^{\iota pt}$$

at the surface $r = b$.

(2) When p and ω are comparable, $(4\omega^2-p^2)\,C_i^{(s)}$ is of the order neglected, $P_i^{(s)}\left(\frac{p}{2\omega}\right)$ and its differential coefficients are finite, and the rotation has no effect.

(3) When ω is great compared with p, we have to evaluate

$$\frac{(4\omega^2-p^2)\,P_i^{(s)}(x)}{\left(\frac{pb}{c^2}\frac{1}{x}\frac{d}{dx}P_i^{(s)}(x)+\frac{2s\omega}{b}P_i^{(s)}(x)\right)},$$

where $x = \frac{p}{2\omega}$ is to be made very small.

There will be two cases according as $P_i^{(s)}(x)$ does or does not vanish with x. If it does not, then $\frac{d}{dx}P_i^{(s)}$ is proportional to x when $x = 0$; otherwise $P_i^{(s)}(x)$ is proportional to x and $\frac{d}{dx}P_i^{(s)}(x)$ does not vanish for $x = 0$.

We take for determination of $P_i^{(s)}(x)$ the form

$$P_i^{(s)}(x) = (1-x^2)^{\frac{1}{2}s}\frac{d^s}{dx^s}P_i(x).$$

The lowest terms are then

$$(-1)^{\frac{1}{2}(i-s)}(1-x^2)^{\frac{1}{2}s}\left\{\frac{2i!}{2^i.i!}\frac{1}{2.4\ldots(i-s)(2i-1)(2i-3)\ldots(i+s+1)}\right.$$
$$\left.+\text{term in } x^2\right\} \text{ when } i-s \text{ is even,}$$

and

$$(-1)^{\frac{1}{2}(i-s-1)}(1-x^2)^{\frac{1}{2}s}\left\{\frac{2i!}{2^i.i!}\frac{x}{2.4\ldots(i-s-1)(2i-1)(2i-3)\ldots(i+s)}\right.$$
$$\left.+\text{term in } x^3\right\} \text{ when } i-s \text{ is odd.}$$

Thus, when $i-s$ is even the limit is $\frac{2\omega b}{s}$, and when $i-s$ is odd it is zero, and

$$\psi = \Sigma\Sigma - \iota \frac{2\omega b}{s} C_i^{(s)} P_i^{(s)}(\mu) e^{\iota s\phi}$$

at the surface $r = b$, when $i-s$ is even.

The case of a zonal harmonic solution requires special attention. If ψ contain a zonal harmonic term, the term of (57) in $\frac{\partial \psi}{\partial \phi}$ will be wanting, and the term containing s in (59) will also vanish; thus

$$\psi = \Sigma \frac{(4\omega^2 - p^2)\, C_i P_i(\mu') P_i(\mu)}{\left(\frac{\iota p b}{c^2} \frac{1}{\mu'} \frac{d}{d\mu'} P_i(\mu')\right)_{\mu' = p/2\omega}},$$

and we have to evaluate this when $p/2\omega = 0$.

The only important case is when i is even, as when i is odd $P_i(\mu')$ vanishes with μ'.

Taking i even, $P_i(\mu') = (-1)^{\frac{1}{2}i} \frac{1.3 \ldots (i-1)}{2.4 \ldots i}$ when $\mu' = 0$,

and $$\frac{1}{\mu'} \frac{d}{d\mu'} P_i(\mu') = (-1)^{\frac{1}{2}i-1}\, 2\, \frac{3.5 \ldots (i+1)}{2.2.4 \ldots (i-2)},$$

so that $$\iota\psi = \Sigma - \frac{4\omega^2 b}{p} \frac{1}{i(i+1)} C_i P_i(\mu) \text{ at the surface,}$$

or $$\psi = \iota \frac{2\omega}{p} \Sigma \frac{2\omega b}{i(i+1)} C_i P_i(\mu),$$

and these terms produce an important effect when $\frac{2\omega}{p}$ is great.

22. We are now in a position to calculate the surface-traction on the inner surface from the fluid pressure.

The pressure equation in the fluid is

$$\frac{p_1}{\sigma} = V' + \tfrac{1}{2}\omega^2 (x^2 + y^2) - \psi,$$

where, at the surface $r = b$,

$$V' = \Sigma \left(U_i + W_i + \frac{V_i}{b^{2i+1}} \right) + \text{non-periodic terms.}$$

The periodic terms in (x^2+y^2) will be proportional to the amplitude of the harmonic inequality, and this multiplied by ω^2 we neglect.

Hence the surface-tractions at the surface $r = b$ are

$$\frac{x}{b}\,\sigma\,(V'-\psi),\quad \frac{y}{b}\,\sigma\,(V'-\psi),\quad \frac{z}{b}\,\sigma\,(V'-\psi)\ldots\ldots\ldots\ldots(60),$$

where the ψ term is to be retained in case it becomes important through the smallness or greatness of the ratio p/ω.

The surface values of $V'-\psi$ are in every case surface harmonics, the same as occur in the typical terms representing the harmonic inequalities, so that these surface values are of the form

$$\frac{\sigma}{b^{i+1}}\,x\,.\,r^i\,(V'-\psi),$$

in which $r^i\,(V'-\psi)$ is a spherical solid harmonic of degree i. Calling this solid harmonic R_i and treating it by the formula

$$xR_i = \frac{1}{2i+1}\left[r^2\frac{dR_i}{dx} - r^{2i+3}\frac{d}{dx}\left(\frac{R_i}{r^{2i+1}}\right)\right],$$

in which r is to be put equal to b after differentiation, we find for this surface-traction a series of terms of the same form as those in (40).

The surface conditions are then to be found as follows:—Fr, Gr, Hr at the inner surface $r = b$ have just been found; at the same surface their values are given by adding the terms of (19), (40), and (45), in which we put $r = b$. These values are to be equated. At the surface $r = a$, the terms contributed by (19), (40), (45), in which r is put $= a$, are to be added and equated to zero. We may then, by a process similar to that in Art. 7, obtain from these two sets of equations four equations between the harmonics ω_i, ϕ_i, Ω_i, Φ_i, and W_i. This will be done by simply dropping the $\frac{\partial}{\partial x}$, ... and powers of r, and taking separately the terms in which occurred $\frac{\partial R_i}{\partial x}$ and those in which occurred $\frac{\partial}{\partial x}\left(\frac{R_i}{r^{2i+1}}\right)$, R_i denoting any one harmonic. These four equations give the ratios of the harmonics ω_i, ϕ_i, Ω_i, Φ_i to W_i, and the process we have gone through shews that any term $P_i^{(s)}(\mu)\,e^{is\phi}$ in W_i yields a term of the same form in each of the harmonics expressing the solution.

The free vibrations under gravity and "centrifugal force" would be found by putting $W_i = 0$, and the frequency equations would not differ in form from those which hold when there are no acting forces.

23. As an example, let us take vibrations under slow rotation only. Then, ω^2 being small, we need not consider the terms of Art. 14, and we have simply to find the forms which have to be substituted for those of Art. 8.

Let us take each of the harmonics ϕ_i, Φ_i, ω_i, Ω_i to consist of a single term proportional to $r^i P_i^{(s)}(\mu)\, e^{\iota s\phi}$, and in future let $\phi_i \dots$ denote the complex constants multiplying this expression in what we before called $\phi_i \dots$. Then

$$\begin{aligned}(lU + mV + nw) &= \iota p\, e^{\iota pt}\left\{ i b^{i-1}\left[\psi_i(kb)\,\phi_i + \Psi_i(kb)\,\Phi_i\right] + b^i\left[e_i'\omega_i + E_i'\Omega_i\right]\right\} \\ &= C_i^{(s)}, \text{ dropping the harmonic and time factors.}\end{aligned}$$

Let $\dfrac{p}{2\omega} > 1$, and take

$$\frac{(4\omega^2 - p^2)\, P_i^{(s)}\left(\dfrac{p}{2\omega}\right) C_i^{(s)}}{\left[\dfrac{\iota p b}{c^2}\,\dfrac{1}{\nu}\,\dfrac{d}{d\nu} P_i^{(s)}(\nu) + \dfrac{2\iota s\omega}{b} P_i^{(s)}(\nu)\right]_{\nu = p/2\omega}} = \delta_i^{(s)} C_i^{(s)},$$

where $c^2 = b^2 \dfrac{4\omega^2}{p^2 - 4\omega^2}$; then the component surface-tractions are

$$-\sigma \delta_i^{(s)} \frac{C_i^{(s)}}{b^i}\,\frac{x}{r}\, r^i P_i^{(s)}(\mu)\, e^{\iota s\phi}, \dots.$$

Calling the solid harmonic Z_i, we have

$$-\sigma \delta_i^{(s)} \frac{C_i^{(s)}}{b^i}\,\frac{x}{r} \cdot Z_i, \dots,$$

where
$$xZ_i = \frac{b^2}{2i+1}\left[\frac{\partial Z_i}{\partial x} - b^{2i+1}\frac{\partial}{\partial x}\left(\frac{Z_i}{r^{2i+1}}\right)\right];$$

therefore
$$\iota Fr / \sigma p\, \delta_i^{(s)} =$$

$$\begin{aligned}&\left[\frac{i}{b}\psi_i(kb)\frac{b^2}{2i+1}\left\{\frac{\partial \phi_i}{\partial x} - b^{2i+1}\frac{\partial}{\partial x}\left(\frac{\phi_i}{r^{2i+1}}\right)\right\}\right. \\ &\quad + \frac{i}{b}\Psi_i(kb)\frac{b^2}{2i+1}\left\{\frac{\partial \Phi_i}{\partial x} - b^{2i+1}\frac{\partial}{\partial x}\left(\frac{\Phi_i}{r^{2i+1}}\right)\right\} \\ &\quad + \frac{b^2}{2i+1} e_i'\left\{\frac{\partial \omega_i}{\partial x} - b^{2i+1}\frac{\partial}{\partial x}\left(\frac{\omega_i}{r^{2i+1}}\right)\right\} \\ &\quad \left. + \frac{b^2}{2i+1} E_i'\left\{\frac{\partial \Omega_i}{\partial x} - b^{2i+1}\frac{\partial}{\partial x}\left(\frac{\Omega_i}{r^{2i+1}}\right)\right\}\right],\end{aligned}$$

with similar equations for $G.r$ and $H.r$, where the $\phi_i \dots$ now stand for solid harmonics.

Hence, just as in Art. 8, we have for $\phi_i \ldots$ the complex constants,

$$\left.\begin{aligned} &\left(a_i' + \sigma\iota p\,\delta_i^{(\sigma)}\frac{b^2}{2i+1}e_i'\right)\omega_i + \left(c_i' + \sigma\iota p\,\delta_i^{(\sigma)}\frac{ib}{2i+1}\psi_i(kb)\right)\phi_i \\ &+\left(A_i' + \sigma\iota p\,\delta_i^{(\sigma)}\frac{b^2}{2i+1}E_i'\right)\Omega_i + \left(C_i' + \sigma\iota p\,\delta_i^{(\sigma)}\frac{ib}{2i+1}\Psi_i(kb)\right)\Phi_i \end{aligned}\right\} = 0,$$

$$\left.\begin{aligned} &\left(b_i' - \sigma\iota p\,\delta_i^{(\sigma)}\frac{b^{2i+3}}{2i+1}e_i'\right)\omega_i + \left(d_i' - \sigma\iota p\,\delta_i^{(\sigma)}\frac{ib^{2i+2}}{2i+1}\psi_i(kb)\right)\phi_i \\ &+\left(B_i' - \sigma\iota p\,\delta_i^{(\sigma)}\frac{b^{2i+3}}{2i+1}E_i'\right)\Omega_i + \left(D_i' - \sigma\iota p\,\delta_i^{(\sigma)}\frac{ib^{2i+2}}{2i+1}\Psi_i(kb)\right)\Phi_i \end{aligned}\right\} = 0,$$

where it is to be observed that $\iota\delta_i^{(\sigma)}$ is a real quantity

$$= \frac{(4\omega^2 - p^2)\,P_i^{(\sigma)}\left(\dfrac{p}{2\omega}\right)}{\left[\dfrac{pb}{c^2}\dfrac{1}{\nu}\dfrac{d}{d\nu}P_i^{(\sigma)}(\nu) + \dfrac{2s\omega}{b}P_i^{(\sigma)}(\nu)\right]_{\nu = p/2\omega}}.$$

The equations just written, together with the first two of equations (24), give the ratios of the amplitudes of the several harmonics ω_i, Ω_i, ϕ_i, Φ_i, and, on elimination of these, give a determinantal equation for the frequency.

The above reduce to the equations (29) when $\omega = 0$, for in this case

$$\iota\delta_i^{(\sigma)} = -\frac{1}{i}pb,$$

as we saw before.

Summary.

The main purpose of this paper is to throw some further light on the question of the rigidity of the earth. One test of the earth's rigidity is the existence of tides. If the bodily tides that would take place in a completely liquid earth exceed sensibly those in the actual earth, there must be sensible ocean tides on the latter. In Thomson and Tait's *Natural Philosophy*, Art. 835 *sq.*, we have an estimate of the earth's rigidity founded on an equilibrium theory of the bodily tides, and the conclusion is that the interior of the earth must be solid, and composed of very rigid material. Sir William Thomson, in an address to Section A of the British Association in 1876, pointed out that this is conclusive against that theory of the internal constitution of the earth, which assumes a solid crust to rest on an inner mass of liquid. It will be so unless the "quasi-rigidity" set up in the liquid by the vortex-motion can interfere with the calculation. In other words,

Sir William's conclusion will be established if an equilibrium theory holds.

We have at our disposal mathematical methods which will enable us to construct a dynamical theory of the bodily tides in an earth whose constitution is that just described. Professor Lamb, in his papers in Vols. XIII. and XIV. of the *Proceedings* of this Society, has taught us what are the modes of vibration of the crust supposed free; and M. Poincaré, in his memoir in the *Acta Mathematica*, Vol. VII., has devised a method for treating the oscillations of a rotating mass of liquid. The problem becomes a special case of that stated at the commencement of this paper. The analysis at our command makes it possible to replace the tidal disturbing forces by any system of bodily forces having a potential expressible in spherical harmonic series.

I have recapitulated, in the first place, the theory of the free vibrations of spheres and spherical shells in order to introduce a modification into Professor Lamb's analysis. The effect of this is to make all the spherical harmonics which occur solid harmonics of positive degree, which facilitates the subsequent applications. This occupies Arts. 1 to 8. The shell is supposed filled with liquid in order to shew in a simple case how the surface-tractions on the inner surface of the shell are to be calculated. The frequency-equations are of the same form as when there is no liquid, coming out in the form of determinants. The equation for the purely tangential vibrations is unaltered, but the elements of two of the rows of the determinant which occurs in the frequency-equation for vibrations of other types are all changed by the addition of certain terms, and the amplitudes of the harmonic disturbances have their ratios altered.

In treating the forced vibrations, we suppose that the external disturbing forces have a potential expressible in spherical harmonic series. The bodily forces arise (1) from the gravitation of the shell and liquid, (2) from the attractions of the harmonic inequalities caused by the external disturbing forces, (3) from the "centrifugal force" when the shell rotates, and (4) include also the external forces themselves. We easily find the particular integrals of the equations of vibration which correspond to forces (2) and (4), and the terms in the surface-tractions on any concentric sphere which are contributed by these particular integrals; this is done in Arts. 10—12. The forces of types (1) and (3) are not periodic. The strain produced by the gravitation included in (1) is purely symmetrical about the centre of the sphere, and the same is true of one part of the forces (3). The effect of the unsymmetrical part of (3) I have not calculated, except for the liquid, as for the problem in hand the rotation is supposed

slow. The symmetrical parts of the forces (1) are of importance because they contribute certain terms to the surface-tractions. The attraction of the shell and liquid is great compared with that of the harmonic inequality, or with the external disturbing force. The consequent strains and stresses are in the same way not of comparable magnitudes. It is sufficient to suppose the surface-tractions arising from the smaller forces to fulfil conditions at the surfaces of the mean inner and outer spheres; it is necessary in the case of the greater forces to remember that the conditions are actually fulfilled at the surfaces of the harmonic inequalities. In the well-known case of incompressible material, the condition that the surface of the harmonic inequality is free can be satisfied by applying to the surface of the mean sphere a normal traction equal to the weight of the harmonic inequality. The formulæ for the tractions that must in general be applied to the surfaces of the mean spheres, in order that the conditions may be fulfilled, are given in Arts. 13 and 14, and in Art. 15 it is verified that the known theory for incompressible material is included in these formulæ. It may be remarked that, in general, the *tangential* stresses are of the same order of magnitude as the *normal* stresses.

We have now only to calculate the fluid pressure in order to form the stress-conditions, and these conditions will determine the amplitudes of the inequalities or tides produced.

Before proceeding to apply Poincaré's general method, it seemed worth while to try under what conditions the oscillations of the fluid could be reduced to an irrotational oscillation combined with periodic changes in the rate and axis of rotation; and it was found that it was necessary and sufficient that the disturbing function should be a spherical solid harmonic of order 2. This is precisely the most important case of tidal forced vibrations. In treating the liquid, "centrifugal force" is eliminated, and the rotation treated kinematically. The pressure-equation for the disturbed motion is formed without difficulty, and makes it possible to answer the question as to the effect of the "quasi-rigidity," due to vortex-motion, on the "tidal effective rigidity." It appears that this effect is negligible when the period of the tide is of the same order of magnitude as the period of rotation, but that it becomes of great importance as the ratio of the former to the latter increases; so that it will vitiate an equilibrium theory for long-period tides, but not for short-period tides. On the other hand, it is only in the case of long-period tides that an equilibrium theory of the ocean tides gives a good result. The very long- and very short-period tides are thus beset with difficulties of a different nature: for the first, the vortex-motion of the supposed in-

ternal liquid produces great tidal effective rigidity; for the second, the inertia of the external liquid covering comes into play. Both theories point to the fortnightly tide as the one which alone can settle the question. So many difficulties have occurred in the work of tide-registering that the Tidal Committee of the British Association appears to be still doubtful whether there really is an appreciable fortnightly tide. If there is, we shall be entitled to say that the tidal effective rigidity of the earth is too great to allow us to suppose it to consist of a liquid mass covered with a thin solid crust.

The simple method of treating the liquid just described will only be permissible in case the disturbing function contains only harmonics of order 2, and for the treatment of the general case we must adopt the method of Poincaré's memoir. It was there shown that the oscillations of a mass of fluid about that state of steady motion in which it rotates as if rigid, can be determined by solving Laplace's equation subject to a condition given over a surface which may be described as a projection of the surface of the liquid. In the case before us, the spherical surface of the liquid is to be projected into a quadric of revolution about the axis of rotation, which is a spheroid or an hyperboloid of one sheet according as the period of rotation is greater or less than twice the period of the forced oscillation. The required solution ψ of Laplace's equation can thus be expressed by means of spheroidal harmonics, and the surface-values of these are proportional to spherical surface harmonics on the sphere of which the spheroid or hyperbolid is a projection. The condition to be satisfied by ψ at the surface of the quadric is obtained thus:—the radial velocity at the surface of the sphere can be expressed in terms of differential coefficients of ψ; it can also be expressed in terms of the harmonics which occur in the complementary and particular solutions of the equations of vibration. These two expressions are to be equated. Now, it appears that if we take one tesseral harmonic term as the typical surface term in ψ, the equation will contain the other tesseral harmonic of the same degree and rank, but no other harmonics. It is, therefore, better to take for the representative harmonic term a complex harmonic containing both the harmonics of the same degree and rank, and after solution retain only the real part of our equations. When this is done it appears that if the disturbing function consist of a series of harmonic terms of the form $r^i P_i^{(s)}(\mu)\, e^{\iota s\phi}$ multiplied by complex constants, then all the harmonics which occur in the solution are of the same form, and each term of this form provokes a motion represented by surface harmonics of this form and of the same degree and rank as the term considered.

When the function ψ is known the whole motion is known and the

pressure. Now, ψ involves linearly the harmonics which occur in the complementary and particular solutions of the equations of vibration, the pressure involves the same harmonics linearly, and the result is the determination of the surface-tractions applied to the inner surface in terms of these harmonics. We have already another expression for these calculated from the strain. Equating the two expressions, we should obtain sufficient equations, with those which hold at the free surface, to express all the unknown harmonics in terms of those occurring in the expression for the disturbing potential.

This completes the analysis for the bodily tides of any order in the system considered.

As an example, I have considered free vibrations of the system supposed rotating slowly, but free from gravitation, and have verified that the solution reduces to that which we already know for free vibrations when the rotation is also annulled.

The results in the general case bear out those arrived at by other methods when the disturbing function is of order 2, shewing how the effect of the rotation in altering an equilibrium solution is null, unless the period of the disturbance is long compared with that of the rotation.

On the Volume generated by a Congruency of Lines.

By R. A. Roberts, M.A.

[*Read Feb. 9th*, 1888.]

In this paper I propose to investigate an expression for the volume generated by a congruency of lines, and to deduce therefrom an extension of Abel's theorem to double integrals.

Taking rectangular coordinates, we may write the equation of a line in the form
$$x = pz + \alpha, \quad y = qz + \beta \quad \text{..........................(1)},$$
and then, if the line belongs to a congruency, that is, if it varies subject to two conditions, we may suppose that α, β are given as functions of p, q. Thus for any point in space, drawing through it a line of the system, we may regard x, y as functions of p, q, and z; so that, if the element of volume $dx\,dy\,dz$ be expressed in terms of the

latter variables, we have

$$dV = \left(\frac{dx}{dp}\frac{dy}{dq} - \frac{dx}{dq}\frac{dy}{dp}\right) dp\,dq\,dz$$

$$= \left\{ \left(z + \frac{da}{dp}\right)\left(z + \frac{d\beta}{dq}\right) - \frac{da}{dq}\frac{d\beta}{dp} \right\} dp\,dq\,dz$$

from (1), by the ordinary transformation for the element of a multiple integral. Now the coefficient of $dp\,dq\,dz$, being equated to zero, is evidently the result which we should obtain if we differentiated (1) with regard to p, q, and eliminated dp/dq; and this gives the two points in which a line of the system is intersected by two other consecutive lines, or is, in general, bitangent to the focal surface. Hence, if z_1, z_2 belong to the points just mentioned, we have

$$dV = (z - z_1)(z - z_2)\,dp\,dq\,dz.$$

But, if $\cos a$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the line (1), we have

$$p = \frac{\cos a}{\cos\gamma}, \quad q = \frac{\cos\beta}{\cos\gamma},$$

whence $$p = \tan\theta\cos\phi, \quad q = \tan\theta\sin\phi,$$

if we put $$\cos a = \sin\theta\cos\phi, \quad \cos\beta = \sin\theta\sin\phi, \quad \gamma = \theta;$$

and therefore $dp\,dq$ is replaced by $\tan\theta\sec^2\theta\,d\theta\,d\phi$. Also, if ρ, ρ_1, ρ_2 are distances measured along the line corresponding to z, z_1, z_2, respectively, we have

$$z - z_1 = (\rho - \rho_1)\cos\theta, \quad z - z_2 = (\rho - \rho_2)\cos\theta,$$

$$dz = d\rho\cos\theta.$$

We thus find

$$dV = (\rho - \rho_1)(\rho - \rho_2)\sin\theta\,d\theta\,d\phi\,d\rho = (\rho - \rho_1)(\rho - \rho_2)\,d\rho\,d\omega \ldots\ldots(2),$$

if $d\omega$ denotes the solid angle of the elementary cone formed by drawing parallels to the lines of the system through a point. Hence, integrating, we get

$$V = \iiint (\rho - \rho_1)(\rho - \rho_2)\,d\rho\,d\omega = \iint \left(\tfrac{1}{3}\rho^3 - \tfrac{1}{2}\rho^2(\rho_1 + \rho_2) + \rho\rho_1\rho_2\right) d\omega \ldots\ldots(3),$$

if the integration with regard to ρ is effected between the limits ρ and 0.

In this expression (3) it is evident that the volume described is

included between a line of length ρ one of whose extremities is situated at the distances ρ_1, ρ_2 from the points of contact with the surface; and, when the integration is effected, the boundary of the space is formed by the loci of the extremities just mentioned, and a ruled surface described by the lines of the system subject to a relation connecting p, q, or θ, ϕ.

As a particular case of (3), let us consider the volume enclosed within the lines joining the points of contact on the focal surface. Putting $\rho_1 = \delta$, $\rho_2 = 0$, $\rho = \delta$, we get from (3)

$$V = \tfrac{1}{6} \iint \delta^3 d\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

that is, the volume sought is equal to half the volume enclosed within a parallel cone, and the surface formed by measuring equal lengths on the parallel radii vectores through the vertex of the cone, as readily follows from the expression for the volume in polar coordinates. If we seek the volume described by either half of the line joining the foci, we find

$$V = \tfrac{1}{12} \iint \delta^3 d\omega,$$

which is equal to half (4); that is, the locus of the middle point of the foci bisects the volume described by the entire line.

Again, suppose that the lines of the system are inflexional tangents of a surface; then, putting $\rho_1 = \rho_2 = 0$, $\rho = \delta$, we have from (3)

$$V = \tfrac{1}{3} \iint \delta^3 d\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence the volume described by the inflexional tangents of a surface measured from the points of contact is equal to the volume enclosed within the extremities of parallel lines of equal length drawn through a point. Applying (3) to the case of the normals to a surface, we get

$$V = \iiint (r-R_1)(r-R_2)\, dr\, d\omega = \iint \left\{\tfrac{1}{3}r^3 - \tfrac{1}{2}r^2(R_1+R_2) + rR_1R_2\right\} d\omega,$$

where R_1, R_2 are the principal radii of curvature, and r is the length measured out on the normal.

Hence for the volume enclosed within a surface, a system of normals, and a parallel surface, we get

$$V = \tfrac{1}{3}r^3 \iint d\omega - \tfrac{1}{2}r^2 \iint \left(\frac{1}{R_1}+\frac{1}{R_2}\right) dS + rS,$$

where dS is an element of area.

We now proceed to apply the formula (3) so as to obtain a result which may be considered as an extension of Abel's theorem concerning transcendents to double integrals. Suppose a line to meet the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where ϕ_n, ϕ_{n-3} are general surfaces of the degrees n and $n-3$ respectively; then for the points of intersection, transforming to polar coordinates at a point on the line, and writing

$$\phi_n = A_0 r^n + A_1 r^{n-1} + \ldots + A_n = 0,$$

$$\phi_{n-3} = B_0 r^{n-3} + B_1 r^{n-4} + \ldots + B_{n-3} = 0,$$

we evidently get, by the theory of algebraic equations,

$$\Sigma r = -\frac{A_1}{A_0} = \Sigma r',$$

$$\Sigma r^2 = \frac{A_1^2 - 2A_0A_2}{A_0^2} = \Sigma r'^2,$$

$$\Sigma r^3 = -\frac{A_1^3 + 3A_1A_2A_0 - 3A_3A_0^2}{A_0^3},$$

$$\Sigma r'^3 = -\frac{A_1^3 + 3A_1A_2A_0 - 3A_3A_0^2}{A_0^3} - \frac{3kB_0}{A_0},$$

where the unaccented and accented letters refer to the two surfaces (6), respectively.

Hence
$$\Sigma r - \Sigma r' = 0, \quad \Sigma r^2 - \Sigma r'^2 = 0,$$

$$\Sigma r^3 - \Sigma r'^3 = \frac{3kB_0}{A_0},$$

so that, if a congruency of lines intersect the surfaces (6), we get from (3)

$$\Sigma V - \Sigma V' = \iint \left\{ \tfrac{1}{3}(\Sigma r^3 - \Sigma r'^3) - \tfrac{1}{2}(\rho_1 + \rho_2)(\Sigma r^2 - \Sigma r'^2) + \rho_1\rho_2(\Sigma r - \Sigma r') \right\} d\omega$$

$$= k \iint \frac{B_0}{A_0}\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Now, since A_0 and B_0 are functions of the direction of a line of the system, that is, of θ and ϕ, we see that the integration can be effected in (7). This result thus shows that the algebraic sum of the volumes intercepted between the surfaces (6) by a congruency of lines, can be expressed by a double definite integral which depends upon the direction of the lines only.

If B_0 vanishes, the surface ϕ_{n-3} reduces to the degree $n-4$, and the double integral in (7) disappears. This result may be stated as follows: If a congruency of lines intersect the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-4} = 0,$$

the algebraic sum of the intercepted volumes is equal to zero.

By taking k indefinitely small in the preceding results, we can arrive at the relation connecting double integrals referred to above.

The volume intercepted between the surfaces

$$\phi_n = 0, \quad \phi_n + k\phi_{n-3} = 0,$$

at any point of the former, is evidently $dp\,dS$, where dS is an element of area, and dp is the portion of the normal intercepted between the two surfaces. Now, if the point x, y, z lie on the surface $\phi_n = 0$, and the consecutive point $x+\delta x$, $y+\delta y$, $z+\delta z$ on the normal to ϕ_n satisfy the equation of the surface

$$\phi_n + k\phi_{n-3} = 0,$$

we have evidently

$$\frac{d\phi_n}{dx}\delta x + \frac{d\phi_n}{dy}\delta y + \frac{d\phi_n}{dz}\delta z + k\phi_{n-3} = 0,$$

where k is regarded as an infinitesimal of the first order.

But $$\delta x = \frac{dp}{G}\frac{d\phi_n}{dx}, \quad \delta y = \frac{dp}{G}\frac{d\phi_n}{dy}, \quad \delta z = \frac{dp}{G}\frac{d\phi_n}{dz},$$

where $$G = \sqrt{\left\{\left(\frac{d\phi_n}{dx}\right)^2 + \left(\frac{d\phi_n}{dy}\right)^2 + \left(\frac{d\phi_n}{dz}\right)^2\right\}}.$$

We thus find $$dp = -\frac{k\phi_{n-3}}{G};$$

hence, since $$dS = \frac{G\,dx\,dy}{\dfrac{d\phi_n}{dz}},$$

we have $$dp\,dS = -k\phi_{n-3}\,du,$$

where we have written du for $\dfrac{dx\,dy}{\dfrac{d\phi_n}{dz}}$. We get, therefore, substituting this value for the difference of the volumes in (7), and dividing by k,

$$\Sigma\phi_{n-3}\,du = -\frac{B_0}{A_0}\,d\omega \qquad \text{(8)},$$

and, therefore, $$\Sigma\iint\phi_{n-3}\,du = -\iint\frac{B_0}{A_0}\,d\omega \qquad \text{(9)}.$$

P 2

This result is an extension of a theorem given already by me in a paper published in the *Proceedings*, Vol. XVI., p. 238. It is an exact analogue of Abel's theorem for double integrals. And it is evident that there are precisely similar results for multiple integrals which can be readily obtained by considerations of space of n dimensions. Of course, these relations connecting the multiple integrals and the algebraic systems of equations are merely consistent with each other, whereas, in the case of Abel's theorem, the algebraic conditions follow necessarily from the transcendental equations.

As a particular case, let us consider the application of (9) to the cubic surface. Denoting $\iint du_r$ by u_r, we get for the three systems of points in which a congruency meets the surface

$$u_1+u_2+u_3+\iint\frac{d\omega}{A_0}=0 \qquad\text{.......................(10)},$$

where we have put $\phi_{n-3}=B_0=1.$

It may be observed that the integral u_r vanishes at any point of the surface which describes a curve. For instance, if a chord of a curve lying on the cubic meet the surface again at a point the integral corresponding to which is u, we get

$$u=-\iint\frac{d\omega}{A_0} \qquad\text{..............................(11)}.$$

As *in planò* for the cubic curve, we can readily find an expression for du in the case of the cubic surface by taking rectangular axes such that the axis of z passes through a point at infinity on the surface. We may write, then,

$$U=v_1z^2+v_2z+v_3=0,$$

where v_1, v_2, v_3 are expressions in x, y of the first, second, and third degrees, respectively. Hence

$$du=\frac{dx\,dy}{\dfrac{dU}{dz}}=\frac{dx\,dy}{2v_1z+v_2}=\frac{dx\,dy}{\sqrt{(v_2^2-4v_1v_3)}},$$

and
$$u=\iint\frac{dx\,dy}{\sqrt{(v_2^2-4v_1v_3)}} \qquad\text{......................(12)},$$

where it may be observed that the expression under the radical is the most general rational integral function of the fourth degree in x, y; for $v_2^2-4v_1v_3$ being equated to zero represents a cylinder circumscribed about the surface parallel to the axis of z, that is, a

circumscribed cone having its vertex on the surface; and we know that the tangent cone of a cubic surface drawn from any point of itself is the most general cone of the fourth degree.

This result gives a transformation which seems of some importance in the theory of double integrals. It shows that any double integral of the form

$$\iint \frac{dx\,dy}{\sqrt{\Phi}} \quad \text{..............................(13)},$$

where Φ is the most general rational integral expression in x, y of the fourth degree, can be transformed so as to become the rational integral

$$\iint \frac{dp\,dq}{V} \quad \text{..............................(14)},$$

where V is a rational integral expression in p, q of the third degree.

For, by (11) and (12), the first integral can be transformed into

$$\iint \frac{d\omega}{A_0},$$

which, by putting

$$\cos\alpha = \sin\theta\cos\phi, \quad \cos\beta = \sin\theta\sin\phi, \quad \cos\gamma = \cos\theta,$$

and then $\quad \tan\theta\cos\phi = p, \quad \tan\theta\sin\phi = q,$

becomes of the form (14).

The simplest curves which we could take on the cubic would be two non-intersecting right lines. Hence, to transform (13) to the form (14), we describe a cubic surface S so as to be inscribed in the cylinder Φ, when V is immediately known, and the limiting values of p, q are given by the directions of the lines drawn through the limiting curve on S to intersect two non-intersecting lines lying on S.

For the four systems of points in which a congruency meets a surface of the fourth degree, we find

$$\left.\begin{aligned} \Sigma\iint du &= 0, \\ \Sigma\iint x\,du + \iint \frac{\cos\alpha\,d\omega}{A_0} &= 0 \\ \Sigma\iint y\,du + \iint \frac{\cos\beta\,d\omega}{A_0} &= 0 \\ \Sigma\iint z\,du + \iint \frac{\cos\gamma\,d\omega}{A_0} &= 0 \end{aligned}\right\} \quad \text{....................(15)},$$

where A_0 is now a rational integral homogeneous function of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the fourth degree.

As to the geometrical meaning of the preceding results, we may notice that the integral u, taken over a portion of the surface U, is proportional to the mass of the shell formed by U and the consecutive surface $U+k$, where k is indefinitely small. It is to be observed that, for two different branches of U, the surface $U+k$ may lie on the inner and outer sides, so that, to represent the mass of the shell, u must be taken with the proper sign. As, for instance, if a quartic consist of two oval surfaces, two values of u on one oval should be taken positively, and the other two on the second oval negatively.

Again, it is evident that the relations connecting the integrals $\iint x\,du$, &c. will give theorems concerning the centres of gravity of the shells; thus, in the case of the quintic surface which is intersected by a congruency in five shells, the centre of gravity of two of the shells must coincide with that of the three others. Similarly, the relations connecting the integrals $\iint x^2\,du$ will give theorems concerning the moments of inertia of the shells.

In the case of particular surfaces, the integral u has a simple geometrical meaning. For instance, for the surface

$$z^n U = \text{a constant},$$

where U is a rational integral function of x, y, u is proportional to $\iint z\,dx\,dy$, that is, the volume of the cylinder included between the surface and the plane of xy. As a further particular case, if the surface is of the fourth order, chords of a curve lying on the surface intersect it again in two areas which are such that the volumes of the cylinders between them and the plane of xy are equal.

Again, for the surface whose equation is

$$V = \text{a constant},$$

where V is a rational integral homogeneous expression in x, y, z, we find, by transformation to polar coordinates, that u is proportional to $\iint r^3\,d\omega$, namely, the volume of the cone having its vertex at the origin and standing on the boundary of a superficial area.

The formula for the volume in the case in which the lines of the congruency are chords of a curve may be noticed here.

For any point x, y, z of the line joining the points x_1, y_1, z_1; x_2, y_2, z_2,

we may obviously put

$$x = \theta x_1 + (1-\theta)\, x_2, \quad y = \theta y_1 + (1-\theta)\, y_2, \quad z = \theta z_1 + (1-\theta)\, z_2.$$

Hence, if we consider x_1, y_1, z_1 as functions of s_1, and x_2, y_2, z_2 as functions of s_2, where s is the length of the arc, x, y, z will be functions of the three variables θ, s_1, s_2, and $dx\,dy\,dz$ can be replaced by

$$\begin{vmatrix} x_1 - x_2, & y_1 - y_2, & z_1 - z_2 \\ \dfrac{dx_1}{ds_1}, & \dfrac{dy_1}{ds_1}, & \dfrac{dz_1}{ds_1} \\ \dfrac{dx_2}{ds_2}, & \dfrac{dy_2}{ds_2}, & \dfrac{dz_2}{ds_2} \end{vmatrix} \theta\,(1-\theta)\, d\theta\, ds_1\, ds_2,$$

by the formula for the transformation of a triple integral. Now, by solid geometry, the determinant is equal to $D \sin\phi$, where D is the shortest distance between the tangents to the curve at the extremities of the chord, and ϕ is the angle between the same lines. Also

$$\theta = (\rho - r_1) / (r_2 - r_1),$$

so that we get $$dV = \frac{(\rho - r_1)(\rho - r_2)\, D \sin\phi\, ds_1\, ds_2\, d\rho}{(r_2 - r_1)^3}.$$

Thursday, March 8th, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. R. W. D. Christie, Carlton School, near Selby, Yorkshire, was elected a member.

The following communications were made :—

Supplementary Remarks on the Theory of Distributions : Captain P. A. MacMahon, R.A.

Complex Multiplication Moduli : Prof. A. G. Greenhill, M.A.

Geometrical Proof of Feuerbach's Nine-Point-Circle Theorem : Prof. Genese, M.A.

Isostereans : R. Tucker, M.A.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XLIII., Nos. 261 and 262.

"Educational Times," for March, 1888.

"An Elementary Treatise on the Integral Calculus," by B. Williamson, M.A., F.R.S., Fifth Edition, 8vo; London, 1888,

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Pt. III.; 1888.

"Royal Irish Academy:—Transactions," Vol. XXIX., Parts I. & II.—List of Papers published between 1786 and 1886.

"Royal Irish Academy:—Cunningham Memoirs,—Dynamics and Modern Geometry," by Sir R. S. Ball; Dublin, June, 1887.

"Royal Irish Academy:—Proceedings—Science," Vol. IV., No. 6; "Polite Literature and Antiquities," Vol. II., No. 8.

'Œuvres de Fourier," publiées par les soins de M. Gaston Darboux, Tome I., 4to; Paris, 1888.

"Bulletin des Sciences Mathématiques," Feb., 1888.

"Journal für die reine und angewandte Mathematik," Band 102, Heft IV.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 2.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 9.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," No. 51.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXXII., Heft 2 and 3.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo I., Nos. 6 and 7.

Geometrical Demonstration of Feuerbach's Theorem concerning the Nine-Point Circle. *By* Professor R. W. Genese, M.A.

[*Read March 8th,* 1888.]

Let A', B', C' be the mid-points of the sides of the triangle ABC; D, E, F the feet of the perpendiculars; O the circumcentre, I the in-centre, X, Y, Z the points of contact of the in-circle with the sides.

Let OA' meet the circumcircle at U, OC' at W. Then AU bisects $B\widehat{A}C$, and therefore also $O\widehat{A}D$.

Let P, Q, R be the mid-points of the arcs of the nine-point circle exterior to the triangle. Then the tangent at P is parallel to BXD; therefore PX produced will pass through the external centre of similitude T of the nine-point circle and in-circle, and so will RZ.

Feuerbach's theorem will be proved if we show that T is on either the in-circle or the nine-point circle.

Let $A'A''$ be the diameter of the nine-point circle through A'.
Then AA'' is equal and parallel to OA';
therefore AA'' is equal and parallel to OA;

therefore $\qquad P\widehat{A'}D = P\widehat{D}A' = \tfrac{1}{2}A'\widehat{A''}D$

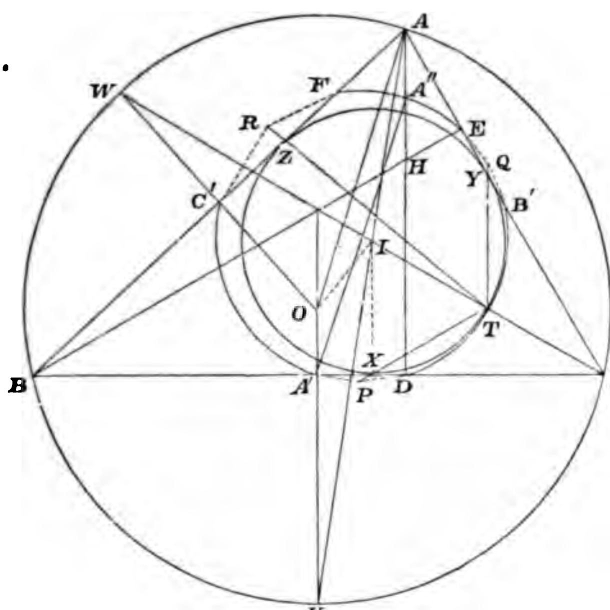
$$= \tfrac{1}{2}O\widehat{A}D$$

$$= O\widehat{A}U = O\widehat{U}A;$$

therefore triangles $PA'D$, OUA are similar.

Also OU, IX, AD are parallel,

therefore $\qquad A'X : XD :: UI : IA.$



Therefore PX, OI divide the triangles $PA'D$, OUA similarly, and

$$P\widehat{X}A' = O\widehat{I}U.$$

Similarly $\qquad R\widehat{Z}C' = O\widehat{I}W,$

therefore $\qquad Z\widehat{T}X = P\widehat{X}A' + R\widehat{Z}C' - \widehat{B}$

$$= U\widehat{I}W - \widehat{B}$$

$$= A\widehat{I}C - \widehat{B}$$

$$= \frac{A}{2} + \frac{C}{2} = \text{complement of } \frac{B}{2}.$$

---

* In the diagram, CW appears accidentally to pass through T; the condition for this is easily found to be $a^2 + b^2 = c\,(a+b)$.

But this is the angle in the segment ZYX of the in-circle. Therefore T is on the in-circle.

Or, we may show that PX meets the nine-point circle in the centre of similitude of the circles, thus:

Let PX meet the nine-point circle at T.

Then $$\frac{PX}{XT} = \frac{PX^2}{PX.XT} = \frac{PX^2}{A'X.XD}$$

$$= \frac{OI^2}{UI.IA}$$

$$= \frac{R^2-2Rr}{2Rr};$$

therefore $$\frac{PT}{XT} = \frac{R^2}{2Rr} = \frac{R}{2} : r.$$

The demonstrations may be easily modified to show that the nine-point circle touches the ex-circles.

A Group of Isostereans. *By* R. Tucker, M.A.

[*Abstract read March* 8*th*, 1888.]

Transversals $\beta_2\gamma_1$, $\beta_2'\gamma_1'$; $\gamma_2\alpha_1$, $\gamma_2'\alpha_1'$; $\alpha_2\beta_1$, $\alpha_2'\beta_1'$, are drawn parallel to the sides BC, CA, AB of a triangle in order, so that *isoclinals* to them from the angles are equal in length; *i.e.*, to take two cases for illustration,

$$KAB = LBC = MCA = \theta,\quad AK = BL = CM = l;$$

and $$K'AC = L'BA = M'CB = \theta',\quad AK' = BL' = CM' = l'.$$

Then, for l, m, n, we have $\theta = \frac{C}{2}, \frac{A}{2}, \frac{B}{2}$ respectively,

and for l', m', n', we have $\theta' = \frac{B}{2}, \frac{C}{2}, \frac{A}{2}$ respectively,

with $$l = c\sin\frac{B}{2}\Big/\cos\frac{A}{2},\quad l' = b\sin\frac{C}{2}\Big/\cos\frac{A}{2};$$

hence $$lmn = abc\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = l'm'n' \quad\text{.................}(1).$$

The unaccented and accented transversals cointersect in

$$T\,(bc^2 : ca^2 : ab^2 \text{ to mod. } r/abc)$$

and $$T'\,(b^2c : c^2a : a^2b \text{ to same modulus})\,;$$

i.e., in trilinear coordinates, $\alpha\beta\gamma = r^3 = \alpha'\beta'\gamma'$.

The following results are readily obtained—

$$\left.\begin{array}{l} B\alpha_2 : \alpha_2\alpha_1 : \alpha_1 C = b : c : a = C\beta'_2 : \beta'_2\beta'_1 : \beta'_1 A \\ C\beta_2 : \beta_2\beta_1 : \beta_1 A = c : a : b = A\gamma'_2 : \gamma'_2\gamma'_1 : \gamma'_1 B \\ A\gamma_2 : \gamma_2\gamma_1 : \gamma_1 B = a : b : c = B\alpha'_2 : \alpha'_2\alpha'_1 : C\alpha'_1 \end{array}\right\} \;\ldots\ldots\ldots(2).$$

$$\left.\begin{array}{l} \beta_2\gamma_1 : \gamma_2\alpha_1 : \alpha_2\beta_1 = a\,(a+b) : b\,(b+c) : c\,(c+a) \\ \beta'_2\gamma'_1 : \gamma'_2\alpha'_1 : \alpha'_2\beta'_1 = a\,(a+c) : b\,(b+a) : c\,(c+b) \end{array}\right\} \;\ldots\ldots\ldots(3).$$

$T\beta_2 : T\gamma_2 : T\alpha_2 = a^2 : b^2 : c^2$ and $T\gamma_1 : T\alpha_1 : T\beta_1 = ab : bc : ca$ ...(4).

From (3) we have

$$\alpha_2\beta_1 \,.\, \beta_2\gamma_1 \,.\, \gamma_2\alpha_1 = abc \,.\, a+b \,.\, b+c \,.\, c+a/8s^3 = \alpha'_2\beta'_1 \,.\, \beta'_2\gamma'_1 \,.\, \gamma'_2\alpha'_1 \ldots(5).$$

It is from this property that I call the two triplets *Isostereans.*

$C\alpha_2 = \beta'_2\gamma'_1$, therefore $C\gamma'_1$ is a parallelogram, and so on.

The perimeters of $T\gamma_1\gamma_2$, $T\beta_1\beta_2$, $T\alpha_1\alpha_2 = b, a, c$ respectively; hence sums of their in- and circum-radii equal in- and circum-radius of ABC, and so for T'.

If $\left.\begin{array}{l} AT,\ BT,\ CT \\ AT',\ BT',\ CT' \end{array}\right\}$ $\begin{array}{l} \text{cut opposite sides in } P,\ Q,\ R \\ \text{cut opposite sides in } P',\ Q',\ R' \end{array}$ respectively,

then $$BP : CP = b : a\,;\quad BP' : CP' = a : c,$$

and $$\Delta PQR = \frac{abc}{a+b \,.\, b+c \,.\, c+a} \,.\, \Delta ABC = \Delta P'Q'R' \ldots\ldots\ldots(6).$$

$$\left.\begin{array}{l} CT,\ BT' \text{ intersect in } p \text{ on the bisector of } A \\ AT,\ CT' \text{ intersect in } q \text{ on the bisector of } B \\ BT,\ AT' \text{ intersect in } r \text{ on the bisector of } C \\ \text{and the centroid of } \Delta pqr \text{ is centroid of } ABC \end{array}\right\} \;\ldots\ldots\ldots\ldots(7).$$

CT', BT; AT', CT; BT', AT intersect on the connectors of A, B, C with the Kiepertian point $(a^{-3} : b^{-3} : c^{-3})$..................................(8).

Additional results are—

$C\Omega$, BT; $A\Omega$, CT; $B\Omega$, AT intersect, respectively, on the bisectors of A, B, C; as do also $B\Omega'$, CT'; $C\Omega'$, AT'; $A\Omega'$, BT'.
$B\beta_2$, $C\gamma_1$ intersect on median through A; and so on.
$B\beta'_2$, $C\gamma'_1$ intersect on symmedian through A; and so on. ...(9).

And ΩT, $\Omega' T'$ intersect on the join of O (the incentre) and the Kiepertian point.

The equation to TT' is

$$aa(a^2-bc)+\ldots+\ldots=0;$$

to the join of O and mid-point of TT' is

$$aa(b-c)+\ldots+\ldots=0,$$

which passes through G; hence G is centroid of $\Delta OTT'$.

The Isostereans are easily obtained thus: let l line from A, meet $\beta_2 T\gamma_1$ in h; then we have

$$\gamma_1 h = c^2/2s = B\gamma_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

and therefore Bh is bisector of $\angle B$.

[*Note.*—I began the study of these lines with considering the general case of the transversals obliquely drawn, but I have not obtained many simple results. I hope, however, to return to this view of the question as it was suggested in Mr. Kempe's remarks when my communication was made to the Society.]

Symmetric Functions and the Theory of Distributions.

By Captain P. A. MacMahon, R.A.

[*Read March 8th*, 1888.]

The theory of distributions is discussed in an elementary manner in Whitworth's *Choice and Chance*, Third Edition, Ch. III. The subject is studied in France under the title *L'Analyse Combinatoire.* There have been very few researches during recent years, and none, so far as my knowledge extends, which proceed by the method employed in this paper. This method is essentially constructive in its nature.

The investigation has for its object the bringing forward of the theory as an analytical weapon of considerable power in algebraical research. The notation employed is new, and possesses the advantage of being the simplest that it is possible to use.

Among results of minor importance and interest, four important and very general purely algebraical theorems are established. These are—

(1) A comprehensive law of algebraic reciprocity.

(2) A cardinal theorem of symmetric function expressibility.

(3) A generalisation of Vandermonde's (or Waring's) formula in symmetric functions.

(4) The formation of symmetrical symmetric-function tables corresponding to every partition of every number.

The research is continued, from the point of view of the Algebra of Symmetric Functions, in a paper by the author ("Memoir on a New Theory of Symmetric Functions"), which will shortly appear in No. 4, Vol. x. of the *American Journal of Mathematics.*

The notation employed throughout is that of partitions.

The Theory of Partitions, from the point of view of the Theory of Numbers, has been studied chiefly by Cayley, Sylvester, Glaisher, Franklin, and Hammond. These researches have appeared principally in the *Philosophical Transactions of the Royal Society*, the *American Journal of Mathematics*, the *Quarterly Journal of Mathematics*, and the *Messenger of Mathematics.*

An important reference is "A constructive Theory of Partitions," by J. J. Sylvester, *American Journal of Mathematics*, Vol. v., p. 251.

The first mathematician who employed the notation of a partition in ordinary algebra was Meyer Hirsch in his Algebra published in 1812; since then the idea has been further developed by Cayley, Hammond, the author of this paper, and probably a few others. The following memoirs may be consulted :—

Cayley: "A Memoir on the Symmetric Functions," *Phil. Trans. R. S.*, 1857.

The Author: "Seminvariants and Symmetric Functions," *Amer. Jour. of Math.*, Vol. vi.; the Author: "On Perpetuants," *Amer. Jour. of Math.*, Vol. vii.; the Author: "Memoir on Seminvariants," *Amer. Jour. of Math.*, Vol. viii.; Hammond: "On Perpetuants," *Amer. Jour. of Math.*, Vol. viii.; the Author: "The Expression of Syzygies," &c., *Amer. Jour. of Math.*, Vol. x.

Preliminary.

As defined by Whitworth (*loc. cit.*), "Distribution" is the separation of a series of elements into a series of classes; in the general problem, the things to be distributed may be of any species, viz., there may be n things, of which p are of one kind, q of a second kind, r of a third, &c. ..., where $p+q+r+\ldots=n$; it is then convenient to speak of things or objects $(pqr\ldots)$ where, in this particular connection, the partition $(pqr\ldots)$ is to be regarded as defining the objects in regard to species; again, the classes into which the objects are to be distributed may be of any species, and this leads us to speak of classes $(p_1q_1r_1\ldots)$, where $p_1+q_1+r_1+\ldots=n_1=$ the number of classes; the

partition $(p_1 q_1 r_1 \ldots)$ here defines the classes in regard to species, indicating p_1 classes of one description, q_1 of a second, r_1 of a third, and so forth.

It should be observed that, in the use of partitions, repetitions of the same part are indicated by an index; for instance

$$(pppqqr \ldots) \text{ is written } (p^3 q^2 r \ldots).$$

If no attention is paid to the order of the objects (whatever be their species) in a class, the distribution may be described as one into "parcels"; each parcel is a class of unarranged objects.

If, however, permutations are permissible amongst objects in the same class, the distribution is said to be one into "groups"; each group is a class of arranged objects.

Two chief problems may be enunciated as follows:—

"To determine the number of distributions of objects $(pqr \ldots)$ into parcels $(p_1 q_1 r_1 \ldots)$."

"To determine the number of distributions of objects $(pqr \ldots)$ into groups $(p_1 q_1 r_1 \ldots)$."

Further, we may discuss each of these problems when the distributions are subject to certain restrictions; it is from the consideration of restricted distributions that most of the analytical results of this paper are evolved.

Section 1.

The Distribution Function.

Let $\alpha, \beta, \gamma, \ldots$ be the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0.$$

The symmetric function $\Sigma \alpha^p \beta^q \gamma^r \ldots$, where $p+q+r+\ldots = n$, is, in the partition notation, written

$$(pqr \ldots).$$

Let
$$A_{(pqr \ldots), (p_1 q_1 r_1 \ldots)}$$

denote the number of ways of distributing objects, defined by the partition $(pqr \ldots)$, into parcels, defined by the partition $(p_1 q_1 r_1 \ldots)$. I suppose there to be m parcels, so that $p_1 + q_1 + r_1 + \ldots = m$.

It will be convenient henceforward to speak simply of the distribution of objects $(pqr \ldots)$ into parcels $(p_1 q_1 r_1 \ldots)$.

I attach the number

$$A_{(pqr \ldots), (p_1 q_1 r_1 \ldots)}$$

to the symmetric function

$$(pqr \ldots),$$

and construct the expression

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

by taking the summation over every partition $(pqr\ldots)$ of the number n.

Definition. The Distribution Function of n objects into parcels $(p_1\,q_1\,r_1\ldots)$ is the expression

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

where $$p+q+r+\ldots = n.$$

I write also

$$\Sigma A_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots) = Dp\,(n),\ (p_1\,q_1\,r_1\ldots).$$

Let, also,

$$B_{(pqr\ldots),(p_1q_1r_1\ldots)}$$

denote the number of ways of distributing objects $(pqr\ldots)$ into groups $(p_1q_1r_1\ldots)$.

Definition. The Distribution Function of n objects into groups $(p_1\,q_1\,r_1\ldots)$ is the expression

$$\Sigma B_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots),$$

where $$p+q+r+\ldots = n.$$

In this case I write

$$\Sigma B_{(pqr\ldots),(p_1q_1r_1\ldots)}(pqr\ldots) = Dg\,(n),\ (p_1\,q_1\,r_1\ldots).$$

My present purpose is the study of these two Distribution Functions.

SECTION 2.

Parcels, m in number (i.e., $m = n$).

Let h_s be the homogeneous product-sum, of degree s, of the n quantities $\alpha, \beta, \gamma, \ldots$; so that

$$\begin{aligned} h_1 &= \Sigma\alpha = (1), \\ h_2 &= \Sigma\alpha^2+\Sigma\alpha\beta = (2)+(1^2), \\ h_3 &= \Sigma\alpha^3+\Sigma\alpha^2\beta+\Sigma\alpha\beta\gamma = (3)+(21)+(1^3), \\ \&c. &= \&c. \end{aligned}$$

Consider the product $$h_{p_1}\,h_{q_1}\,h_{r_1}\ldots.$$

The symmetric function $(pqr\ldots)$ will, on performing the multiplication, be produced a certain number of times. In the factor h_{p_1} every term is of degree p_1 in the quantities. Taking any particular term, write down the p_1 quantities occurring therein in any order with a dot between each pair of consecutive quantities. We may consider these p_1 quantities as distributed into p_1 similar parcels, one quantity into each parcel. In the same way, any q_1 quantities which occur in any term of h_{q_1} may be considered to be q_1 quantities distributed into q_1 parcels, similar to one another, but different from the former. Hence it is clear that the number of times that the symmetric function $(pqr\ldots)$ occurs in the development of the product $h_{p_1}h_{q_1}h_{r_1}\ldots$ is precisely the number of ways that it is possible to distribute objects $(pqr\ldots)$ into parcels $(p_1q_1r_1\ldots)$, one object in each parcel. Hence, when $m = n$, and no parcel is empty,

$$Df(n),\ (p_1q_1r_1\ldots) = \Sigma A_{(pqr\ldots),\,(p_1q_1r_1\ldots)}(pqr\ldots) = h_{p_1}h_{q_1}h_{r_1}\ldots.$$

Consider, for a moment, the distribution of objects (43) into parcels (52), and represent objects and parcels by small and capital letters respectively. One distribution is represented by the scheme

$$\begin{matrix} A & A & A & A & A & B & B \\ a & a & a & a & b & b & b \end{matrix}$$

wherein an object denoted by a small letter is placed in a parcel denoted by the capital letter directly above it. Corresponding to this distribution of objects (43) into parcels (52), we have a distribution of objects (52) into parcels (43), given by the scheme

$$\begin{matrix} A & A & A & A & B & B & B \\ a & a & a & a & a & b & b \end{matrix}$$

derived from the former by interchanging rows as well as small and capital letters. The process is clearly general and exhibits a one-to-one correspondence between the distributions of objects $(pqr\ldots)$ into parcels $(p_1q_1r_1\ldots)$, and the distributions of objects $(p_1q_1r_1\ldots)$ into parcels $(pqr\ldots)$. It is, in fact, an intuitive observation, that we may either consider an object placed in or attached to a parcel, *or* a parcel placed in or attached to an object.

Hence the very important theorem

$$A_{(pqr\ldots),\,(p_1q_1r_1\ldots)} = A_{(p_1q_1r_1\ldots),\,(pqr\ldots)}.$$

Analytically this result leads to a law of algebraic symmetry which I now enunciate.

Theorem.—" The coefficient of symmetric function $(pqr\ldots)$ in the

development of the product $h_{p_1} h_{q_1} h_{r_1} \ldots$ is equal to the coefficient of symmetric function $(p_1 q_1 r_1 \ldots)$ in the development of the product $h_p h_q h_r \ldots$."

This law of symmetry I established in the *Quarterly Journal of Mathematics*.

The problem of the distribution of n objects into n parcels, one object into each parcel, is thus completely solved by means of a table of symmetric functions which expresses the h-products as linear functions of the single partition forms. (*Vide* the Tables at the end of the paper.)

Section 3.

Parcels of species (1^m), *where* $m < n$.

I now discuss the distributions of n objects into m parcels, no two of which are similar. Whitworth would describe the problem as a distribution into m DIFFERENT parcels.

Let $$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots), \qquad [\Sigma \pi = m,\ \Sigma \pi p = n],$$

be any partition of n into m parts.

Of the whole number of distributions, there will be a certain number such that π_s parcels each contain p_s objects,

$$(s = 1, 2, 3, \ldots).$$

The distribution function of this particular case of the distribution is

$$\frac{m!}{\pi_1!\, \pi_2!\, \pi_3! \ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots .$$

To see how this is, observe that the product $h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots$ is susceptible of $\dfrac{m!}{\pi_1!\, \pi_2!\, \pi_3! \ldots}$ permutations. The parcels are all different, and hence there are distributions corresponding to each of these permutations. By the last section, for each of these permutations there will be a distribution function

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots ,$$

and for the aggregate of permutations a distribution function

$$\frac{m!}{\pi_1!\, \pi_2!\, \pi_3! \ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots .$$

Hence the distribution function of n objects into parcels (1^m) is

$$\Sigma \frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\,\ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots, \qquad [\Sigma\pi = m,\ \Sigma\pi p = n],$$

that is, it is the coefficient of x^n in the expansion of

$$(h_1 x + h_2 x^2 + h_3 x^3 + \ldots)^m.$$

We may write this result

$$Dp\,(n),\,(1^m) = \Sigma A_{(pqr\ldots),\,(1^m)}\,(pqr\ldots) = \Sigma \frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\,\ldots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots,$$

where $\qquad \Sigma\pi = m, \quad \Sigma p\pi = n.$

Section 4.

General value of $A_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),\,(1^m)}$.

We require the coefficient of x^n in the expansion of

$$(h_1 x + h_2 x^2 + h_3 x^3 + \ldots)^m = u^m, \text{ suppose.}$$

Put $\qquad f(x) = 1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots = 1 + u,$

then $\qquad (1+u)^m = (1-\alpha x)^{-m}(1-\beta x)^{-m}(1-\gamma x)^{-m}\ldots.$

and $\qquad u^m = (1+u-1)^m$

$$= (1+u)^m - m\,(1+u)^{m-1} + \frac{m\,(m-1)}{2!}(1+u)^{m-2} - \ldots + (-)^m\,1.$$

Now, the coefficient of $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)\,x^n$ in

$$(1+u)^s = (1-\alpha x)^{-s}(1-\beta x)^{-s}(1-\gamma x)^{-s}\ldots$$

is
$$\left\{\frac{(s+p_1-1)!}{p_1!\,(s-1)!}\right\}^{\pi_1} \left\{\frac{(s+p_2-1)!}{p_2!\,(s-1)!}\right\}^{\pi_2} \left\{\frac{(s+p_3-1)!}{p_3!\,(s-1)!}\right\}^{\pi_3} \ldots.$$

Hence $\qquad A_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),\,(1^m)}$

$$= \left\{\frac{(m+p_1-1)!}{p_1!\,(m-1)!}\right\}^{\pi_1} \left\{\frac{(m+p_2-1)!}{p_2!\,(m-1)!}\right\}^{\pi_2} \left\{\frac{(m+p_3-1)!}{p_3!\,(m-1)!}\right\}^{\pi_3} \ldots$$

$$-m \left\{\frac{(m+p_1-2)!}{p_1!\,(m-2)!}\right\}^{\pi_1} \left\{\frac{(m+p_2-2)!}{p_2!\,(m-2)!}\right\}^{\pi_2} \left\{\frac{(m+p_3-2)!}{p_3!\,(m-2)!}\right\}^{\pi_3} \ldots$$

$$+ \frac{m\,(m-1)}{2!} \left\{\frac{(m+p_1-3)!}{p_1!\,(m-3)!}\right\}^{\pi_1} \left\{\frac{(m+p_2-3)!}{p_2!\,(m-3)!}\right\}^{\pi_2} \left\{\frac{(m+p_3-3)!}{p_3!\,(m-3)!}\right\}^{\pi_3} \ldots$$

$-\ldots$ *to* $m+1$ *terms.*

Observe that, when

$$p_1 = p_2 = \ldots = \pi_1 = \pi_2 = \ldots = 1,$$

this expression reduces to the m^{th} divided difference of 0^n.

SECTION 5.

Parcels of species (m).

We now discuss what is commonly known as the distribution of n objects into m indifferent parcels, but here the objects are of type

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots).$$

We may separate the distribution function into portions corresponding to every partition of the number n into exactly m parts. First, consider such a partition which consists wholly of *unrepeated* parts, say

$$(r_1 r_2 r_3 \ldots r_m), \qquad [\Sigma r = n],$$

the corresponding distribution function is necessarily

$$h_{r_1} h_{r_2} h_{r_3} \ldots h_{r_m},$$

but in any other case of distribution the function is much less simple.

For clearness first take $n = 4$, $m = 2$, and let us examine the distribution function corresponding to two objects in each parcel.

We have $h_2^2 = (\alpha^2 + \beta^2 + \gamma^2 + \ldots + \alpha\beta + \beta\gamma + \gamma\alpha + \ldots)^2$,

and here the distribution, $\alpha\alpha$ in one parcel, $\beta\beta$ in the other, occurs twice instead of once, as would have to be the case if this were really the distribution function.

Take the expression

$$\frac{1}{(1-\alpha^2x^2)(1-\beta^2x^2)(1-\gamma^2x^2)\ldots(1-\alpha\beta x^2)(1-\beta\gamma x^2)(1-\gamma\alpha x^2)\ldots},$$

and expand it in ascending powers of x; herein the coefficient of x^{2s} will be the sum of order s of the homogeneous products of the quantities

$$\alpha^2, \beta^2, \gamma^2, \ldots \alpha\beta, \beta\gamma, \gamma\alpha, \ldots,$$

which compose the function h_2. This homogeneous product sum consists of a number of terms each of which is obtained by multiplying together s of the quantities

$$\alpha^2, \beta^2, \gamma^2, \ldots \alpha\beta, \beta\gamma, \gamma\alpha, \ldots,$$

repeated or unrepeated; clearly then, in this homogeneous product sum, the symmetric function

$$(p_1 p_2 p_3 \dots), \qquad [\Sigma p = 2s],$$

will occur just as many times as it is possible to distribute $2s$ objects $(p_1 p_2 p_3 \dots)$ into parcels (s), two objects being in each parcel.

If then we write

$$\frac{1}{(1-\alpha^2x^2)(1-\beta^2x^2)(1-\gamma^2x^2)\dots(1-\alpha\beta x^2)(1-\beta\gamma x^2)(1-\gamma\alpha x^2)\dots}$$
$$= 1 + h_{2^1}x^2 + h_{2^2}x^4 + h_{2^3}x^6 + \dots,$$

h_{2^s} will be the distribution function corresponding to the particular case of $2s$ objects in parcels (s), each parcel containing 2 objects.

Now take rs objects in parcels (s), each parcel containing r objects.

Form a fraction whose denominator contains a factor corresponding to each component member of h_r, and then suppose

$$\frac{1}{\left[\begin{array}{l}(1-\alpha^rx^r)(1-\beta^rx^r)\dots(1-\alpha^{r-1}\beta x^r)(1-\alpha\beta^{r-1}x)\\ \qquad\qquad \dots(1-\alpha^{r-2}\beta^2x^r)\dots(1-\alpha\beta\gamma\dots x^r)\end{array}\right]}$$
$$= 1 + h_{r^1}x^r + h_{r^2}x^{2r} + h_{r^3}x^{3r} + \dots.$$

Previous reasoning shows that the distribution function is

$$h_{r^s}.$$

Reserving for the present the particular examination of this important symmetric function, I continue the general discussion.

We have already considered the distribution function corresponding to the particular case of the partition of n into unrepeated parts; we are now in a position to determine the function corresponding to the case of r_1 parcels each containing t_1 objects, r_2 parcels each containing t_2 objects, &c., or say, corresponding to the partition of n,

$$(t_1^{r_1} t_2^{r_2} t_3^{r_3} \dots t_v^{r_v}) \qquad [\text{where } \Sigma r = m].$$

For, form the symmetric function

$$h_{t_1^{r_1}} h_{t_2^{r_2}} h_{t_3^{r_3}} \dots h_{t_v^{r_v}},$$

and observe the meaning of the coefficient of the symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots),$$

which will appear when the symmetric function product is developed.

The function $h_{r_1}^{t_1}$ contains terms corresponding to every selection of $t_1 r_1$ objects of the total number n, and corresponding to every distribution of each of these selections into parcels (r_1), each parcel containing exactly t_1 objects. Hence in the product

$$h_{t_1}^{r_1} h_{t_2}^{r_2} h_{t_3}^{r_3} \ldots h_{t_\nu}^{r_\nu},$$

the symmetric function $\quad (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$

will occur just so many times as it is possible to distribute objects $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$ into m parcels, of which r_1 contain exactly t_1 objects, r_2 exactly t_2, r_3 exactly t_3, &c., and r_ν parcels exactly t_ν objects, &c.

Hence the particular distribution function sought for is

$$h_{t_1}^{r_1} \ldots h_{t_\nu}^{r_\nu} \ldots .$$

Finally, noticing that $h_{r_1}^{t_1}$ and $h_{t_1}^{r_1}$ are identical, we see that the distribution function of n objects into parcels (m) is

$$\Sigma\, h_{t_1}^{r_1} h_{t_2}^{r_2} h_{t_3}^{r_3} \ldots,$$

the summation taking place over every partition

$$(t_1^{r_1} t_2^{r_2} t_3^{r_3} \ldots)$$

of n which contains exactly $m\,[= \Sigma r]$ parts.

We may write this theorem in the form—

$$Dp\,(n), (m) = \Sigma A_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),\,(m)} (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$$

$$= \Sigma h_{t_1}^{r_1} h_{t_2}^{r_2} h_{t_3}^{r_3} \ldots,$$

where $\qquad \Sigma r = m, \quad \Sigma rt = n.$

It is now clear that $Df\,(n)\,(m)$ is the coefficient of $x^n a^m$ in the expression

$$(1 + h_{1^1} xa + h_{1^2} x^2 a^2 + \ldots)(1 + h_{2^1} x^2 a + h_{2^2} x^4 a^2 + \ldots)(1 + h_{3^1} x^3 a + h_{3^2} x^6 a^2 + \ldots) \ldots$$

$$\ldots (1 + h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \ldots) \ldots$$

$$\equiv \prod_{s=1}^{s=\infty} (1 + h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \ldots),$$

which is, therefore, its generating function.

Section 6.

Parcels of type $(m_1 m_2)$.

In this case, we are concerned with m_1 similar parcels of one kind and m_2 similar parcels of another kind. Of the n objects we may have m_1 objects only distributed amongst the m_1 similar parcels, and the remaining $n-m_1$ objects distributed amongst the m_2 similar parcels; or we may have m_1+s objects distributed amongst the m_1 similar parcels, and the remaining $n-m_1-s$ objects amongst the m_2 similar parcels, where $s \not> n-m_1-m_2$.

Hence

$$Dp\,(n), (m_1 m_2) = \sum_{s=0}^{n-m_1-m_2} Dp\,(m_1+s), (m_1) \,.\, Dp\,(n-m_1-s), (m_2),$$

and $Dp\,(n), (m_1 m_2)$ is the coefficient of $x^n a^{m_1} b^{m_2}$ in the product

$$\prod_{s=1}^{s=\infty} (1+h_{s^1} x^s a + h_{s^2} x^{2s} a^2 + \ldots)(1+h_{s^1} x^s b + h_{s^2} x^{2s} b^2 + \ldots),$$

which is its generating function.

Section 7.

Parcels of type $(m_1 m_2 m_3 \ldots)$.

By similar reasoning we find :—

$$Dp\,(n), (m_1 m_2 m_3) = \sum_{s=0}^{s=n-m_1-m_2-m_3} Dp(m_1+s), (m_1) . Dp\,(n-m_1-s), (m_2 m_3)$$

$$= \sum_{s=0}^{s=n-m_1-m_2-m_3} \Big[Dp\,(m_1+s), (m_1) \sum_{t=0}^{t=n-m_1-m_2-m_3-s}$$

$$Dp\,(m_2+t), (m_2) \,.\, Dp\,(n-m_1-m_2-s-t), (m_3) \Big]$$

$$= \sum_{s=0}^{s=n-\Sigma m} \sum_{t=0}^{t=n-\Sigma m-s} Dp\,(m_1+s), (m_1) \,.\, Dp\,(m_2+t), (m_2)$$

$$\times\, Dp\,(n+m_3-\Sigma m-s-t), (m_3),$$

and generally,

$$Dp\,(n), (m_1 m_2 m_3 \ldots m_r) = \sum_{s_1=0}^{s_1=n-\Sigma m,} \sum_{s_2=0}^{s_2=n-\Sigma m-s_1} \sum_{s_3=0}^{s_3=n-\Sigma m-s_1-s_2} \ldots$$

$$Dp\,(m_1+s_1), (m_1) \,.\, Dp\,(m_2+s_2), (m_2) \,\ldots\, Dp\,(n+m_r-\Sigma m-\Sigma s), (m_r)\,;$$

which is the coefficient of $x^n \mu_1^{m_1} \mu_2^{m_2} \ldots \mu_r^{m_r}$ in the product

$$\prod_{s=1}^{s=\infty} \prod_{t=1}^{t=\infty} (1+h_{s^1} x^s \mu_t + h_{s^2} x^{2s} \mu_t^2 + h_{s^3} x^{3s} \mu_t^3 + \ldots),$$

the generating function.

This determination completes analytically the solution of the problem of the distribution of objects $(p_1^{\pi_1} p_2^{\pi_2} \dots)$ into parcels $(m_1 m_2 \dots m_r)$.

Before proceeding to the subject of distributions, involving restrictions, I will draw up a list of some of the simpler results.

SECTION 8.

The simplest cases of Distribution into Parcels.

No. of Objects.	No. of Parcels.	Type of Parcels.	Distribution Function.
1	1	(1)	h_1,
2	1	(1)	h_2,
2	2	(2)	h_2,
2	2	(1^2)	h_1^2,
3	1	(1)	h_3,
3	2	(2)	$h_2 h_1$,
3	2	(1^2)	$2h_2 h_1$,
3	3	(3)	h_3,
3	3	(21)	$h_2 h_1$,
3	3	(1^3)	h_1^3,
4	1	(1)	h_4,
4	2	(2)	$h_4 + h_2^2$,
4	2	(1^2)	$2h_3 h_1 + h_2^2$,
4	3	(3)	h_2^2,
4	3	(21)	$h_2^2 + h_2 h_1^2$,
4	3	(1^3)	$3h_2 h_1^2$,
4	4	(4)	h_4,
4	4	(31)	$h_3 h_1$,
4	4	(2^2)	h_2^2,
4	4	(21^2)	$h_2 h_1^2$,
4	4	(1^4)	h_1^4,
5	1	(1)	h_5,
5	2	(2)	$h_4 h_1 + h_3 h_2$,
5	2	(1^2)	$2h_4 h_1 + 2h_3 h_2$,
5	3	(3)	$h_4 h_1 + h_3 h_2 - h_3 h_1^2 + h_2^2 h_1$,
5	3	(21)	$h_4 h_1 + h_3 h_2 + 2h_2^2 h_1$,
5	3	(1^3)	$3h_3 h_1^2 + 3h_2^2 h_1$,

No. of Objects.	No. of Parcels.	Type of Parcels.	Distribution Function.
5	4	(4)	h_3h_2,
5	4	(31)	$h_3h_2+h^2h_1$,
5	4	(2^2)	$2h_2^2h_1$,
5	4	(21^2)	$2h_2^2h_1+h_2h_1^3$,
5	4	(1^4)	$4h_2h_1^3$,
6	1	(1)	h_6,
6	2	(2)	$h_6+2h_4h_2$,
6	2	(1^2)	$2h_5h_1+2h_4h_2+h_3^2$,
6	3	(3)	$h_6-h_5h_1+h_4h_2+h_4h_1^2+h_3^2-h_3h_2h_1+h_2^3$,
6	3	(21)	$2h_4h_2+h_4h_1^2+2h_3h_2h_1+h_2^3$,
6	3	(1^3)	$3h_4h_1^2+6h_3h_2h_1+h_2^3$,
6	4	(4)	$h_4h_2+h_3^2-h_3h_2h_1+h_2^3$,
6	4	(31)	$h_4h_1^2+h_3^2+h_3h_2h_1-h_3h_1^3+h_2^3+h_2^2h_1^2$,
6	4	(2^2)	$2h_4h_2+2h_2^3+h_2^2h_1^2$,
6	4	(21^2)	$h_4h_1^2+2h_3h_2h_1+h_2^3+3h_2^2h_1^2$,
6	4	(1^4)	$4h_3h_1^3+6h_2^2h_1^2$,
6	5	(5)	h_4h_2,
6	5	(41)	$h_4h_2+h_3h_2h_1$,
6	5	(32)	$h_3h_2h_1+h_2^3$,
6	5	(31^2)	$2h_3h_2h_1+h_2^2h_1^2$,
6	5	(21^2)	$h_2^3+2h_2^2h_1^2$,
6	5	(21^3)	$3h_2^2h_1^2+h_2h_1^4$,
6	5	(1^5)	$5h_2h_1^4$.

This table may be continued with little labour, the distribution functions being derived from those corresponding to a lesser number of objects whenever the parcel is of such a type that its partition contains more than a single part. For instance, we may employ either of the two formulæ

$$Dp\,(6),\,(31^2) = Dp\,(3),\,(3)\;Dp\,(3),\,(1^2)+Dp\,(4),\,(3)\;Dp\,(2),\,(1^2),$$

$$Dp\,(6),\,(31^2) = Dp\,(4),\,(31)\;Dp\,(2),\,(1)+Dp\,(5),\,(31)\;Dp\,(1),\,(1),$$

for the calculation of $Dp\,(6),\,(31^2)$.

The Distribution Functions can then be evaluated in terms of single partition forms by means of the tables subsequently given.

I proceed now to show how to express the symmetric function

$$h_{r^s}$$

in terms of $h_1, h_2, h_3, \ldots$ so as to obtain the expression generally of $Df(n)(m)$.

SECTION 9.

The symmetric function h_{r^s}.

This function is a homogeneous product sum, formed by taking s and s together the terms which compose the homogeneous product sum h_r. h_r is the homogeneous product sum of the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}.$$

Form the equation whose roots are the several terms of h_r, viz.,

$$x^p - j_1 x^{p-1} + j_2 x^{p-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

where $$p = \frac{(n+r-1)!}{(n-1)!\,r!}, \quad \text{and} \quad j_1 = h_{r^s} = h_r.$$

Form also the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

Let partitions in ( ) and [ ] denote respectively the symmetric functions of the roots of (i.) and (iii.), and σ_κ the sum of the κ^{th} powers of the roots of (ii.).

We may easily establish the two results

$$(\kappa) = (-)^{\kappa+1}[\kappa],$$
$$[\kappa^r] = (-)^{r(\kappa+1)}\sigma_\kappa\,;\text{*}$$

---

* We have in fact

$$\frac{1}{1-\frac{a_1}{x}+\frac{a_2}{x^2}-\ldots+(-)^n\frac{a_n}{x^n}} = 1+\frac{h_1}{x}+\frac{h_2}{x^2}+\ldots+\frac{h_n}{x^n}+\ldots,$$

which may be written

$$\frac{1-\frac{\phi}{x^{n+1}}-\frac{\psi}{x^{n+2}}-\ldots}{1-\frac{a_1}{x}+\frac{a_2}{x^2}-\ldots+(-)^n\frac{a_n}{x^n}} = 1+\frac{h_1}{x}+\frac{h_2}{x^2}+\ldots+\frac{h_n}{x^n},$$

or $$\frac{1-\frac{\phi}{x^{n+1}}-\frac{\psi}{x^{n+2}}-\ldots}{\left(1-\frac{a}{x}\right)\left(1-\frac{\beta}{x}\right)\ldots} = \left(1+\frac{a'}{x}\right)\left(1+\frac{\beta'}{x}\right)\ldots,$$

wherein $a, \beta, \ldots$ are the roots of (i.), and $a', \beta', \ldots$ the roots of (iii.).

whence $h_{r^1} = j_1 = \sigma_1 = [1^r]$,

$$h_{r^2} = j_1^2 - j_2 = \tfrac{1}{2}(\sigma + \sigma_2)$$
$$= \frac{\sigma_1^2}{2!} + \frac{\sigma_2}{2}$$
$$= \frac{[1^r]^2}{.2!} + (-)^r \frac{[2^r]}{2};$$

Hence $\log\left(1 - \frac{\phi}{x^{n+1}} - \frac{\psi}{x^{n+2}} - \dots\right) + \frac{(1)}{x} + \frac{1}{2}\frac{(2)}{x^2} + \dots + \frac{1}{n}\frac{(n)}{x^n} + \dots$

$$= \frac{[1]}{x} - \frac{1}{2}\frac{[2]}{x^2} + \frac{1}{3}\frac{[3]}{x^3} - \dots + (-)^{n+1}\frac{[n]}{x^n} + \dots,$$

leading to the result

$$(\kappa) = (-)^{\kappa+1}[\kappa], \quad \text{where } \kappa \not> n.$$

Next, consider the identity

$$\frac{1}{\left(1-\frac{a}{x}\right)\left(1-\frac{\beta}{x}\right)\left(1-\frac{\gamma}{x}\right)\dots} = \left(1+\frac{u}{x}\right)\left(1+\frac{v}{x}\right)\left(1+\frac{w}{x}\right)\dots,$$

wherein $u, v, w, \dots$ are the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \dots = 0,$$

and n is supposed indefinitely great.

Let the κ, κ^{th} roots of unity be denoted by

$$\epsilon_1, \epsilon_2, \epsilon_3, \dots \epsilon_\kappa;$$

then $$\prod_{s=1}^{s=\kappa} \frac{1}{\left(1-\frac{\epsilon_s a}{x}\right)\left(1-\frac{\epsilon_s \beta}{x}\right)\left(1-\frac{\epsilon_s \gamma}{x}\right)\dots} = \prod_{s=1}^{s=\kappa}\left(1+\frac{\epsilon_s u}{x}\right)\left(1+\frac{\epsilon_s v}{x}\right)\left(1+\frac{\epsilon_s w}{x}\right)\dots,$$

or $$\frac{1}{\left(1-\frac{a^\kappa}{x^\kappa}\right)\left(1-\frac{\beta^\kappa}{x^\kappa}\right)\left(1-\frac{\gamma^\kappa}{x^\kappa}\right)\dots}$$

$$= \left\{1+(-)^{\kappa+1}\frac{u^\kappa}{x^\kappa}\right\}\left\{1+(-)^{\kappa+1}\frac{v^\kappa}{x^\kappa}\right\}\left\{1+(-)^{\kappa+1}\frac{w^\kappa}{x^\kappa}\right\}\dots,$$

which is

$$\frac{1}{1-\frac{(\kappa)}{x^\kappa}+\frac{(\kappa^2)}{x^{2\kappa}}-\frac{(\kappa^3)}{x^{3\kappa}}+\dots} = 1+(-)^{\kappa+1}\frac{[\kappa]}{x^\kappa}+(-)^{2(\kappa+1)}\frac{[\kappa^2]}{x^{2\kappa}}+\dots+(-)^{r(\kappa+1)}\frac{[\kappa^r]}{x^{r\kappa}}+\dots.$$

The coefficient of $\frac{1}{x^{r\kappa}}$ in the development of the sinister of this identity, according to ascending powers of $\frac{1}{x^\kappa}$, is the homogeneous product sum of order r of the quantities $a^\kappa, \beta^\kappa, \gamma^\kappa, \dots$; it is thus equal to σ_κ, the sum of the κ^{th} powers of the roots of (ii.).

Hence $$\sigma_\kappa = (-)^{r(\kappa+1)}[\kappa^r],$$

which is equivalent to the second of the two results.

$$h_{r^3} = j_1^3 - 2j_1j_2 + j_3$$
$$= \frac{\sigma_1^3}{3!} + \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3}$$
$$= \frac{[1^r]^3}{3!} + (-)^r \frac{[1^r][2^r]}{2} + \frac{[3^r]}{3};$$

$$h_{r^4} = j_1^4 - 3j_1^2j_2 + j_2^2 + 2j_1j_3 - j_4$$
$$= \frac{\sigma_1^4}{4!} + \frac{\sigma_1^2\sigma_2}{2!\,2} + \frac{\sigma_1\sigma_3}{3} + \frac{\sigma_2^2}{2^2.\,2!} + \frac{\sigma_4}{4}$$
$$= \frac{[1^r]^4}{4!} + (-)^r \frac{[1^r]^2[2^r]}{2!\,2} + \frac{[1^r][3^r]}{3} + \frac{[2^r]^2}{2^2.\,2!} + (-)^r \frac{[4^r]}{4},$$

and so forth.

The law is identical with that which obtains in the expression of the elementary symmetric functions in terms of the sums of powers, with the exception that the signs are all positive when r is even.

Hence we can express h_{r^s} in terms of $h_1, h_2, h_3, \ldots$.

In particular we thus find

$$h_{1^1} = h_1,$$

and generally

$$h_{1^r} = h_{r^1} = h_r,$$

$$h_{2^2} = \frac{1}{2!}\left\{h_2^2 + h_2^2 - 2h_1h_3 + 2h_4\right\} = h_2^2 - h_1h_3 + h_4,$$

$$h_{2^3} = \frac{1}{3!}\left\{h_2^3 + 3h_2(h_2^2 - 2h_1h_3 + 2h_4)\right.$$
$$\left. + 2(h_2^3 - 3h_3h_2h_1 + 3h_3^2 + 3h_4h_1^2 - 3h_4h_2 - 3h_5h_1 + 3h_6)\right\}$$
$$= h_6 - h_5h_1 + h_4h_1^2 + h_3^2 - 2h_3h_2h_1 + h_2^3,$$

$$h_{3^2} = \frac{1}{2!}\left\{h_3^2 - h_3^2 + 2h_4h_2 - 2h_5h_1 + 2h_6\right\} = h_6 - h_5h_1 + h_4h_2,$$

$$h_{2^4} = \frac{1}{4!}\left\{h_2^4 + 6h_2^2(h_2^2 - 2h_1h_3 + 2h_4)\right.$$
$$+ 8h_2(h_2^3 - 3h_3h_2h_1 + 3h_3^2 + 3h_4h_1^2 - 3h_4h_2 - 3h_5h_1 + 3h_6)$$
$$+ 3(h_2^4 + 4h_3^2h_1^2 + 4h_4^2 - 4h_3h_2^2h_1 + 4h_4h_2^2 - 8h_4h_3h_1)$$
$$+ 6(h_2^4 - 4h_3h_2^2h_1 + 2h_3^2h_1^2 + 4h_3^2h_2 + 4h_4h_2h_1^2 - 4h_4h_2^2$$
$$- 8h_4h_3h_1 + 6h_4^2 - 4h_5h_1^3 + 8h_5h_2h_1 - 4h_5h_3$$
$$\left. + 4h_6h_1^2 - 4h_6h_2 - 4h_7h_1 + 4h_8)\right\}$$
$$= h_2^4 - 3h_3h_2^2h_1 - h_4h_2^2 + 2h_3^2h_2 + 2h_4h_2h_1^2 + 2h_5h_2h_1 + h_3^2h_1^2 + 2h_4^2$$
$$- 3h_4h_3h_1 - h_5h_1^3 - h_5h_3 + h_6h_1^2 - h_6h_2 - h_7h_1 + h_8$$
$$= h_8 - h_7h_1 - h_6h_2 + h_6h_1^2 - h_5h_3 + 2h_5h_2h_1 - h_5h_1^3 + 2h_4^2 - 3h_4h_3h_1 - h_4h_2^2$$
$$+ 2h_4h_2h_1^2 + 2h_3^2h_2 + h_3^2h_1^2 - 3h_3h_2^2h_1 + h_2^4,$$

$$h_{s^2} = \frac{1}{2!}\{h_4^2+h_4^2-2h_5h_3+2h_6h_2-2h_7h_1+2h_8\}$$

$$= h_8-h_7h_1+h_6h_2-h_5h_3+h_4^2,$$

and for present purposes we need calculate no further.

SECTION 10.

Groups of type (1^m).

Consider the expansion of

$$h_1^n = (\alpha+\beta+\gamma+\dots)^n.$$

It consists of products of the quantities $\alpha, \beta, \gamma, \dots$ of the n^{th} degree taken in all possible ways, repetitions and permutations being alike allowable. On this understanding the expansion consists of a number of terms each with coefficient unity. Suppose any such term to be

$$\alpha_1\beta_1\beta_2\alpha_2\alpha_3\gamma_1\beta_3\alpha_4\dots,$$

and place dots in any $m-1$ out of the $n-1$ intervals between the letters; this can be done in

$$\frac{(n-1)!}{(n-m)!\,(m-1)!}\text{ ways.}$$

A distribution (1^m) will correspond to each of these ways for every term of the expansion h_1^n.

Thus the distribution function of n objects into groups (1^m) is

$$Dg\,(n), (1^m) = \frac{(n-1)!}{(n-m)!\,(m-1)!}\,h_1^n,$$

and denoting by $\qquad B_{(p_1^{\pi_1}p_2^{\pi_2}\dots),\,(1^m)}$

the number of distributions of objects $(p_1^{\pi_1}p_2^{\pi_2}\dots)$ into groups (1^m), we have

$$B_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots),\,(1^m)} = \frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}(p_3!)^{\pi_3}\dots}\cdot\frac{(n-1)!}{(n-m)!\,(m-1)!}.$$

The distribution function is the coefficient of x^n in

$$(h_1x+h_1^2x^2+h_1^3x^3+h^4x^4+\dots)^m.$$

SECTION 11.

Groups of type (m).

Consider the symmetric function sum

$$\Sigma\frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots}(p_1^{\pi_1}p_2^{\pi_2}\dots) = h_1^n$$

arranged as a sum of products of letters $\alpha, \beta, \gamma, \ldots$, each permutation of every product occurring as a term, so that only coefficients equal to unity present themselves.

We require the homogeneous product sum of all these terms, of any desired order.

Putting $n = 2, 3, \ldots r$ successively, we may write as generating functions

$$\frac{1}{(1-\alpha^2 ax^2)(1-\beta^2 ax^2)\ldots(1-\alpha\beta ax^2)^2(1-\alpha\gamma ax^2)^2\ldots}$$
$$= 1 + H_2 ax^2 + H_{2^2} a^2x^4 + H_{2^3} a^3x^6 + \ldots$$

$$\frac{1}{(1-\alpha^3 ax^3)(1-\beta^3 ax^3)\ldots(1-\alpha^2\beta ax^3)^3\ldots(1-\alpha\beta\gamma ax^3)^6\ldots}$$
$$= 1 + H_3 ax^3 + H_{3^2} a^2x^6 + \ldots,$$

&c.,

and generally

$$\frac{1}{(1-\alpha^r ax^r)\ldots(1-\alpha^{r-1}\beta ax^r)^r\ldots(1-\alpha\beta\gamma\ldots ax^r)^{r!}\ldots}$$
$$= 1 + H_r ax^r + H_{r^2} a^2x^{2r} + H_{r^3} a^3x^{3r} + \ldots,$$

wherein H_{r^s} represents the s^{th} order homogeneous product sum of all the separate terms which arise when h_1^r is multiplied out *in extenso.*

By reasoning similar to that employed in the discussion of "Parcels," we see that

$$H_{r^s}$$

denotes the distribution function of sr objects in groups (s) in suchwise that each group shall consist of r objects.

Also that the distribution function of n objects into groups (m) is

$$\Sigma H_{t_1^{r_1}} H_{t_2^{r_2}} H_{t_3^{r_3}} \ldots,$$

the summation taking place over every partition of n

$$(t_1^{r_1}\, t_2^{r_2}\, t_3^{r_3} \ldots)$$

which contains exactly $m\ [= \Sigma r]$ parts.

Thus $$Dg(n), (m) = \Sigma H_{t_1^{r_1}} H_{t_2^{r_2}} H_{t_3^{r_3}} \ldots,$$

and it is the coefficient of $x^n a^m$ in the expansion of

$$\prod_{s=1}^{s=\infty} (1 + H_s x^s a + H_{s^2} x^{2s} a^2 + \ldots),$$

which is, therefore, the generating function.

Section 12.

Groups of type $(m_1 m_2 m_3 \ldots)$.

The law of derivation of the distribution functions of groups of many part partition types is precisely the same in the case of groups as in the case of parcels.

I, therefore, proceed at once to the examination of the new symmetric function

$$H_{rs}.$$

Section 13.

The symmetric function H_{rs}.

Let

$$x^w - k_1 x^{w-1} + k_2 x^{w-2} - \ldots = 0$$

be the equation, having for its roots the several quantities of which H_{rs} is the homogeneous product sum of order s.

Then

$$k_1 = h_1^r = H_{r1}.$$

Further, let σ_t denote the sum of the t^{th} powers of the roots of this equation.

If partitions in ( ) refer to the symmetric functions of the equation

$$(x-\alpha)(x-\beta)(x-\gamma)\ldots = 0,$$

we have

$$\sigma_t = (t)^r;$$

also

$$k_2 = \tfrac{1}{2}(\sigma_1^2 - \sigma_2) = \frac{(1)^{2r}}{2!} - \frac{(2)^r}{2};$$

hence

$$H_{r2} = k_1^2 - k_2 = \frac{\sigma_1^2}{2!} + \frac{\sigma_2}{2}$$

$$= \frac{(1)^{2r}}{2!} + \frac{(2)^r}{2}.$$

Also, since

$$k_3 = \frac{\sigma_1^3}{3!} - \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3},$$

we find

$$H_{r3} = k_1^3 - 2k_1k_2 + k_3$$

$$= \frac{\sigma_1^3}{3!} + \frac{\sigma_1\sigma_2}{2} + \frac{\sigma_3}{3}$$

$$= \frac{(1)^{3r}}{3!} + \frac{(1)^r(2)^r}{2} + \frac{(3)^r}{3},$$

and so forth.

Hence, finally, transforming as before to symmetric functions of the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0,$$

$$H_{r^1} = (1)^r = [1]^r,$$

$$H_{r^2} = \frac{[1]^{2r}}{2\,!} + (-)^r \frac{[2]^r}{2},$$

$$H_{r^3} = \frac{[1]^{3r}}{3\,!} + (-)^r \frac{[1]^r [2]^r}{2} + \frac{[3]^r}{3},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

The symmetric function H_{r^s} can be thus expressed in terms of h_1, h_2, h_3.

These results should be compared with those obtained in section 9 for the case of distribution into parcels.

It will be noticed that H_{r^s} is derived from h_{r^s} by simply writing $[k]^r$ in place of $[k^r]$.

SECTION 14.

Restricted distributions into Parcels.

The distributions considered in the foregoing sections were not subject to any restriction. There was no limit to the number of similar objects that it was permissible to distribute either into a single parcel or into a set of similar parcels. This freedom from restriction led naturally to the invariable appearance of the symmetric functions, which express the sums of the homogeneous products of the quantities, in the distribution functions.

In order to find the distribution function of n objects in n parcels, one object in each parcel, subject to the restriction that no two objects of the same kind are to appear in parcels of the same kind, we have merely to employ the elementary symmetric functions

$$a_1, a_2, a_3, \ldots$$

instead of the homogeneous product sums

$$h_1, h_2, h_3, \ldots.$$

The product $\qquad a_{p_1} a_{q_1} a_{r_1} \ldots$

is necessarily the distribution function of n objects into parcels $(p_1 q_1 r_1 \ldots)$, where $p_1 + q_1 + r_1 + \ldots = n$, subject to the restriction that no two similar objects are to appear in similar parcels. Thus, since

$$a_2 a_3 = (1^2)(1^3) = (2^2 1) + 3(21^3) + 10(1^5),$$

we discover that, subject to the restriction, objects (21^3) can be distributed into parcels (32) in three different ways. These three ways are apparent in the scheme :—

A	A	A	B	B
α	β	γ	δ	α
α	γ	δ	β	α
α	δ	β	γ	α

We wish now to impose the restriction that not more than t similar objects are to be distributed into similar parcels. For this purpose, form the symmetric functions

$$t_1,\ t_2,\ t_3,\ t_4,\ \ldots,$$

where t_s is defined to be that portion of the homogeneous product sum h_s in which no quantity occurs to a higher power than t.

In the product $$t_{p_1} t_{q_1} t_{r_1} \ldots,$$

we may suppose any term composing t_{p_1} to be written out with the letters in any order and a dot placed between each consecutive pair of letters. We consider the p_1 letters to denote p_1 objects distributed into p_1 similar parcels. Obviously, not more than t similar objects thus appear in similar parcels. By reasoning similar to that employed in section 1, it is established that, in the product

$$t_{p_1} t_{q_1} t_{r_1} \ldots,$$

when expanded, the symmetric function $(pqr \ldots)$ will appear with a coefficient which represents the number of ways that it is possible to distribute objects $(pqr \ldots)$ into parcels $(p_1 q_1 r_1 \ldots)$, one object in each parcel, subject to the restriction that not more than t similar objects are to appear in similar parcels. This restriction does not alter the reciprocal nature of the distribution. It is immaterial whether we regard the objects distributed into the parcels or the parcels distributed into the objects. We may say that not more than t similar objects are to be contained in similar parcels, or we may say that not more than t similar parcels are to contain similar objects. The restriction does not affect the reciprocity.

Theorem.—The number of ways of distributing objects $(pqr \ldots)$ into parcels $(p_1 q_1 r_1 \ldots)$ is equal to the number of ways of distributing objects $(p_1 q_1 r_1 \ldots)$ into parcels $(pqr \ldots)$; the distributions being subject to the restriction that not more than t similar objects are to present themselves in similar parcels.

This theorem points to a general algebraic law of symmetry.

Theorem.—The coefficient of symmetric function $(pqr\,...)$ in the development of
$$t_{p_1}\,t_{q_1}\,t_{r_1}\,...$$
is equal to the coefficient of symmetric function $(p_1\,q_1\,r_1\,...)$ in the development of
$$t_p\,t_q\,t_r\,....$$
This theorem includes all previous laws of symmetry.

The observation is made that, if any table of functions be found to possess symmetry of this nature, it follows, as a necessary and easily established result, that the " inverse table " also possesses the same symmetry.

The laws of symmetry, as apparent in ordinary tables of symmetric functions, are included in the above theorem. Still retaining the same restriction, it is easy to prove that the distribution function of n objects into parcels (1^m) is

$$\Sigma_t A_{(pqr\,...),\,(1^m)}\,(pqr\,...) = \Sigma\,\frac{m\,!}{\pi_1\,!\;\pi_2\,!\;\pi_3\,!\,...}\,t_{p_1}^{\pi_1}\,t_{p_2}^{\pi_2}\,t_{p_3}^{\pi_3}\,...,$$

wherein $\qquad \Sigma\pi = m\,;\quad \Sigma p\pi = n.$

SECTION 15.

General value of $_tA_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\,...),\,(1^m)}$.

We require the coefficient of x^n in the expansion of
$$(t_1x+t_2x^2+t_3x^3+\,...)^m,$$
and therein, the coefficient of the symmetric function
$$(p_1^{\pi_1}\,p_2^{\pi_2}\,p_3^{\pi_3}\,...).$$

Put $\qquad t_1x+t_2x^2+t_3x^3+\,...=u,$

so that $\qquad 1+u = \dfrac{1-\alpha^{t+1}x^{t+1}}{1-\alpha x}\cdot\dfrac{1-\beta^{t+1}x^{t+1}}{1-\beta x}\cdot\dfrac{1-\gamma^{t+1}x^{t+1}}{1-\gamma x}\,...,$

and $\qquad (1+u)^m = \Pi\left\{1-m\alpha^t x^t+\dfrac{m\,(m-1)}{2\,!}\,\alpha^{2t}x^{2t}-\,...\right\}$
$$\times\left\{1+m\alpha x+\frac{m\,(m+1)}{2\,!}\,\alpha^2x^2+\,...\right\}.$$

In this product, the coefficient of the symmetric function
$$(p_1^{\pi_1}\,p_2^{\pi_2}\,p_3^{\pi_3}\,...)$$

is
$$\left\{\frac{(m+p_1-1)!}{(m-1)!\,p_1!}-m\frac{(m+p_1-t-2)!}{(m-1)!\,(p_1-t-1)!}+\ldots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-1)!}{(m-1)!\,p_2!}-m\frac{(m+p_2-t-2)!}{(m-1)!\,(p_2-t-1)!}+\ldots\right\}^{\pi_2}\ldots,$$

and, since
$$u^m=(1+u-1)^m=(1+u)^m-m(1+u)^{m-1}+\frac{m(m-1)}{2!}(1+u)^{m-2}-\ldots,$$

we find
$${}_tA_{(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),\,(1^m)}$$
$$=\left\{\frac{(m+p_1-1)!}{(m-1)!\,p_1!}-m\frac{(m+p_1-t-2)!}{(m-1)!\,(p_1-t-1)!}+\ldots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-1)!}{(m-1)!\,p_2!}-m\frac{(m+p_2-t-2)!}{(m-1)!\,(p_2-t-1)!}+\ldots\right\}^{\pi_2}\ldots$$
$$-m\left\{\frac{(m+p_1-2)!}{(m-2)!\,p_1!}-(m-1)\frac{(m+p_1-t-3)!}{(m-2)!\,(p_1-t-1)!}+\ldots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-2)!}{(m-2)!\,p_2!}-(m-1)\frac{(m+p_2-t-3)!}{(m-2)!\,(p_2-t-1)!}+\ldots\right\}^{\pi_2}\ldots$$
$$+\frac{m(m-1)}{2!}\left\{\frac{(m+p_1-3)!}{(m-3)!\,p_1!}-(m-2)\frac{(m+p_1-t-4)!}{(m-3)!\,(p_1-t-1)!}+\ldots\right\}^{\pi_1}$$
$$\times\left\{\frac{(m+p_2-3)!}{(m-3)!\,p_2!}-(m-2)\frac{(m+p_2-t-4)!}{(m-3)!\,(p_2-t-1)!}+\ldots\right\}^{\pi_2}\ldots$$
$$-\ldots\ldots.$$

There is no difficulty in continuing the theory of this restriction. I have not thought it advantageous to proceed further with it in the case of distributions into parcels.

Section 16.

Restricted Distributions into Groups.

It is convenient to write
$$h_1^s=H_s.$$

The distribution function of the unrestricted distribution of n objects into groups
$$(1^m)$$
is then the coefficient of x^n in the expansion of
$$(H_1x+H_2x^2+H_3x^3+\ldots)^m,$$

where
$$\begin{aligned} H_1 &= h_1 = (1), \\ H_2 &= h_1^2 = (2)+2\,(1^2), \\ H_3 &= h^3 = (3)+3\,(21)+6\,(1^3), \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

I further denote by $\quad A_s,\ B_s,\ C_s\ \ldots\ T_s\ \ldots$

those portions of H_s which involve partitions containing no part greater than

$$1,\ 2,\ 3,\ \ldots\ t,\ \ldots \text{ respectively.}$$

It is easily seen that the distribution function of n objects into groups (1^m), subject to the restriction that not more than t objects of the same kind are to present themselves in groups of the same kind, is given by the coefficient of x^n in

$$(T_1x+T_2x^2+T_3x^3+\ldots)^m.$$

Section 17.

Algebraic Theorems derived from the Theory of Distributions.

Definition.

Of a number n, take any partition

$$(\lambda_1\lambda_2\lambda_3\ldots\lambda_s).$$

It becomes necessary to consider the separation of such a partition into component partitions. Such a separation may be represented by enclosing the component partitions in brackets; thus:

$$(\lambda_1\lambda_2)\,(\lambda_3\lambda_4\lambda_5)\,(\lambda_6)\ldots.$$

It is convenient to arrange the components in descending order as regards their weight or content, and, if these successive weights are in order

$$p,\ q,\ r,\ \ldots,$$

to speak of a separation of species $(pqr\ldots)$.

Just as we speak of the degree of a partition, meaning the magnitude of the largest part in such partition, so we may speak of the degree of a separation, meaning the sum of the largest parts in its components.

We have thus, primarily, three characteristics of a separation, viz.,

(i.) the separable partition,

(ii.) the species,

(iii.) the degree.

General Theorem of Algebraic Reciprocity.

In § 1, I considered the distribution of n objects into n parcels, and showed that the distribution function of objects into parcels

$$(p_1 q_1 r_1 \ldots)$$

is

$$h_{p_1} h_{q_1} h_{r_1} \ldots .$$

We may analyse this result in the following manner:—

Write

$$\begin{aligned}
X_1 &= (1)\, x_1, \\
X_2 &= (2)\, x_2 + (1^2)\, x_1^2, \\
X_3 &= (3)\, x_3 + (21)\, x_2 x_1 + (1^3)\, x_1^3, \\
X_4 &= (4)\, x_4 + (31)\, x_3 x_1 + (2^2)\, x_2^2 + (21^2)\, x_2 x_1^2 + (1^4)\, x_1^4, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

and generally

$$X_s = \Sigma\, (\lambda\mu\nu \ldots)\, x_\lambda x_\mu x_\nu \ldots,$$

the summation being in regard to every partition of s.

Consider the result of multiplication

$$X_{p_1} X_{q_1} X_{r_1} \ldots = \Sigma P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots .$$

P consists of an aggregate of terms, each of which, to a numerical factor *près*, is a separation of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots)$$

of species

$$(p_1 q_1 r_1 \ldots).$$

P, further, is the distribution function of objects into parcels

$$(p_1 q_1 r_1 \ldots),$$

subject to certain restrictions.

If in any distribution of n objects into n parcels (one object into each parcel) we write down a number

$$\xi$$

whenever we observe ξ similar objects in similar parcels, we write down a succession of numbers

$$\xi_1,\ \xi_2,\ \xi_3,\ \ldots,$$

where

$$(\xi_1 \xi_2 \xi_3 \ldots)$$

is some partition of n.

We may be given these numbers, and say that the distribution is subject to a restriction of partition

$$(\xi_1 \xi_2 \xi_3 \ldots).$$

Subject to this restriction, there are a certain number of distributions. In the present case, if we put

$$x_1 = x_2 = x_3 = \ldots = 1,$$

$$\Sigma P$$

is obviously the distribution function of n objects into n parcels without restriction.

P itself is manifestly the distribution function subject to the restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots).$$

Employing a more general notation, we may write

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \Sigma P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

and then P is the distribution function of objects into parcels

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

subject to the restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots).$$

Multiplying out P, we get the result

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \Sigma \theta \, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

indicating that, with a restriction of partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots),$$

there are precisely θ ways of distributing n objects

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots)$$

amongst n parcels $\qquad (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$

one object into each parcel.

Now, it is seen intuitively that, since there is one object in every parcel, it is immaterial whether we regard an object attached to a parcel or a parcel attached to an object, and that making this exchange does not alter the partition of restriction.

Hence the number of distributions must be the same, and if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots = \ldots + \theta \, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots,$$

then also $\quad X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} X_{\lambda_3}^{l_3} \ldots = \ldots + \theta \, (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots) \, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots.$

This extensive theorem of algebraic reciprocity includes all known theorems of symmetry in symmetric functions.

Limiting attention to the powers of

$$x_1,$$

we immediately obtain Cayley's law of symmetry.

Putting, further, $\qquad x_1 = 0,$

we obtain a theorem of wide application in the multiplication of covariants of binary quantics.

We may enunciate it as follows:—

Theorem.—Selecting at pleasure any three partitions of n

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots),$$

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \dots),$$

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots),$$

separate in any manner the numbers occurring in

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots)$$

into $\qquad \pi_1$ portions of content p_1,

π_2 " " p_2,

π_3 " " p_3,

$\vdots$ " " $\vdots$

Multiply the product of partitions thus formed by the number which expresses the number of ways of permuting the product, the only permutations allowable being those amongst partitions of the same content; take the sum of all such separations of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots),$$

each multiplied by the proper number determined as explained above. The coefficient of the symmetric function

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \dots),$$

in this sum of compound symmetric functions, will be precisely the same as if in the process we had interchanged the partitions

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots), \quad (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \dots).$$

Generalisation of Waring's Formula.

Waring's formula for the expression of the n^{th} power sum of the roots of an equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots = 0,$$

[redacted]

in terms of the coefficients, is usually written

$$S_m = \Sigma \frac{(-)^{m+\Sigma\lambda} (\Sigma\lambda - 1)!\, m}{\lambda_1!\, \lambda_2! \dots \lambda_n!}\, a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}.$$

Write this in the following form, viz.,

$$\frac{(-)^m (m-1)!}{m!}\, S(1^m) = \Sigma \frac{(-)^{\Sigma\lambda} (\Sigma\lambda-1)!}{\lambda_1!\, \lambda_2! \dots \lambda_n!}\, (1)^{\lambda_1} (1^2)^{\lambda_2} \dots (1^n)^{\lambda_n}.$$

Observe that this formula expresses the sum of the m^{th} powers of the roots in terms of separations of the partition

$$(1^m);$$

the typical separation $\quad (1)^{\lambda_1} (1^2)^{\lambda_2} \dots (1^n)^{\lambda_n}$

is of species $\quad (1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}),$

and of degree $\quad \Sigma\lambda.$

I proceed to demonstrate a formula for the expression of the m^{th} power sum of the roots as a linear function of separations of any partition whatever of m.

The general formula to be established is

$$(-)^{l+m+\dots} \frac{(l+m+\dots-1)!}{l!\, m! \dots}\, S(\lambda^l \mu^m \dots)$$

$$= \Sigma\, (-)^{j_1+j_2+\dots} \frac{(j_1+j_2+\dots-1)!}{j_1!\, j_2! \dots}\, (J_1)^{j_1} (J_2)^{j_2} \dots,$$

wherein $(\lambda^l \mu^m \dots)$ is the separable partition, $(J_1)^{j_1} (J_2)^{j_2} \dots$ is a separation of $(\lambda^l \mu^m \dots)$, and the summation is in regard to every such separation.

In this formula, $\quad S(\lambda^l \mu^m \dots)$

denotes the sum of the n^{th} powers of the roots $(l\lambda + m\mu + \dots = n)$ in terms of separations of $(\lambda^l \mu^m \dots)$.

Write down the series of relations

$$(a_1) = S_{a_1},$$

$$(a_1 a_2) = S_{a_1} S_{a_2} - S_{a_1+a_2},$$

$$(a_1 a_2 a_3) = S_{a_1} S_{a_2} S_{a_3} - S_{a_1} S_{a_2+a_3} - S_{a_2} S_{a_1+a_3} - S_{a_3} S_{a_1+a_2} + 2 S_{a_1+a_2+a_3},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

from the first two of these we find

$$S_{a_1+a_2} = (a_1)(a_2) - (a_1 a_2),$$

or, as this may be written,

$$S(a_1a_2) = (a_1)(a_2)-(a_1a_2),$$

and from the third we get, after reduction,

$$2S(a_1a_2a_3) = 2(a_1)(a_2)(a_3)-(a_1)(a_2a_3)-(a_2)(a_1a_3)-(a_3)(a_1a_2)+(a_1a_2a_3).$$

It is obvious that we can continue this series indefinitely, and express $S(a_1a_2a_3a_4)$, $S(a_1a_2a_3a_4a_5)$, ... in terms of separations of the partitions

$$(a_1a_2a_3a_4),\quad (a_1a_2a_3a_4a_5), \dots .$$

This holds also notwithstanding any equalities that may exist between the parts a_1, a_2, a_3, ... of the separable partition. The formulæ would, however, require modification in those cases.

First, suppose that no equalities exist between the parts of the separable partition; we require the expression of

$$S(a_1a_2\dots a_n)$$

in terms of separations of $(a_1a_2\dots a_n)$.

One such separation is, for example,

$$(a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{t\nu}),$$

where the successive component partitions have $p, q, \dots \nu$ parts, respectively, and there are t components.

Of this type there are in all

$$\frac{n!}{p!\,q!\dots\nu!} \text{ separations,}$$

and by symmetry we see that in the expression of

$$S(a_1a_2\dots a_n),$$

each such separation must be affected by the same coefficient.

Write, then,

$$S(a_1a_2\dots a_n) = \Sigma P \Sigma (a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{t\nu}).$$

To determine P, observe that if

$$a_1 = a_2 = \dots = a_n,$$

the formula should reduce to Waring's, viz.—

$$S(a_1^n) = \Sigma(-)^{n+t}(t-1)!\,n\,(a_1^p)(a_1^q)\dots(a_1^\nu),$$

where for simplicity it is supposed that no equalities exist between the ntegers $p, q, \dots \nu$.

On this supposition of the equality of the parts of the separable partition, the assumed formula becomes

$$S(a_1^n) = \Sigma P \frac{n!}{p!\,q!\dots\nu!}\,p!\,(a_1^p)\,q!\,(a_1^q)\dots\nu!\,(a_1^\nu),$$

and, equating these two expressions for $S(a_1^n)$, we find

$$P = (-)^{n+t}\frac{(t-1)!}{(n-1)!},$$

and we thus reach the formula

$$(-)^n(n-1)!\,S(a_1a_2\dots a_n)$$
$$= \Sigma(-)^t(t-1)!\,\Sigma(a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{tr}),$$

or, as this may be written,

$$(-)^n(n-1)!\,S(a_1a_2\dots a_n)$$
$$= \Sigma(-)^t(t-1)!\,(a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{tr}).$$

The supposition of any number of equalities between the integers $p, q, \dots \nu$ renders requisite an easy modification of the proof, and leads to the same final result.

I pass on to the general case

$$S(\lambda^l\mu^m\dots),$$

and put

$$S(\lambda^l\mu^m\dots) = \Sigma P\,(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}\,(\lambda^{l_2}\mu^{m_2}\dots)^{j_2}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu}.$$

Starting with the formula

$$(-)^n(n-1)!\,S(a_1a_2\dots a_n)$$
$$= \Sigma(-)^t(t-1)!\,(a_{11}a_{12}\dots a_{1p})(a_{21}a_{22}\dots a_{2q})\dots(a_{t1}a_{t2}\dots a_{tu}),$$

suppose that of the numbers

$$p, q, \dots u,$$

j_1 have the value $l_1+m_1+\dots$,

j_2 have the value $l_2+m_2+\dots$,

... ...

j_ν have the value $l_\nu+m_\nu+\dots$.

We may give such values to the quantities a, that certain of the separations under the summation sign shall become

$$(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}\,(\lambda^{l_2}\mu^{m_2}\dots)^{j_2}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu};$$

viz.,—we must put l of them equal to λ, m of them equal to μ, and so on. The number of separations which thus become of the required form is easily found to be

$$\frac{l\,!\,m\,!}{(l_1\,!\,m_1\,!\,\ldots)^{j_1}\,(l_2\,!\,m_2\,!\,\ldots)^{j_2}\ldots(l_\nu\,!\,m_\nu\,!\,\ldots)^{j_\nu}\,j_1\,!\,j_2\,!\ldots j_\nu\,!}.$$

Also, on replacing a component $(\alpha_{11}\,\alpha_{12}\ldots\alpha_{1p})$ by $(\lambda^{l_1}\mu^{m_1}\ldots)$, where

$$l_1+m_1+\ldots=p,$$

we must multiply by $\qquad l_1\,!\,m_1\,!\ldots;$

we thus get a multiplier

$$(l_1\,!\,m_1\,!\ldots)^{j_1}\,(l_2\,!\,m_2\,!\ldots)^{j_2}\ldots(l_\nu\,!\,m_\nu\,!\ldots)^{j_\nu},$$

and, further, t is equivalent to Σj.

Thus,

$$P=(-)^{l+m+\ldots+\Sigma j}\,\frac{(\Sigma j-1)\,!}{(l+m+\ldots-1)\,!}$$

$$\times\frac{l\,!\,m\,!\ldots}{(l_1\,!\,m_1\,!\ldots)^{j_1}\,(l_2\,!\,m_2\,!\ldots)^{j_2}\ldots(l_\nu\,!\,m_\nu\,!\ldots)^{j_\nu}\,j_1\,!\,j_2\,!\ldots j_\nu\,!}$$

$$\times(l_1\,!\,m_1\,!\ldots)^{j_1}\,(l_2\,!\,m_2\,!\ldots)^{j_2}\ldots(l_\nu\,!\,m_\nu\,!\ldots)^{j_\nu}$$

$$=(-)^{l+m+\ldots+\Sigma j}\,\frac{l\,!\,m\,!\ldots}{(l+m+\ldots-1)\,!}\cdot\frac{(\Sigma j-1)\,!}{j_1\,!\,j_2\,!\ldots j_\nu\,!},$$

leading to the formula

$$\frac{(-)^{l+m+\ldots}\,(l+m+\ldots-1)\,!}{l\,!\,m\,!\ldots}\,S(\lambda^l\mu^m\ldots)$$

$$=\Sigma\,(-)^{\Sigma j}\,\frac{(\Sigma j-1)}{j_1\,!\,j_2\,!\ldots j_\nu\,!}\,(\lambda^{l_1}\mu^{m_1}\ldots)^{j_1}\,(\lambda^{l_2}\mu^{m_2}\ldots)^{j_2}\ldots(\lambda^{l_\nu}\mu^{m_\nu}\ldots)^{j_\nu}.$$

Assuming the form

$$S(\lambda^l\mu^m\ldots)=\Sigma P\Sigma\,(\lambda^{l_1}\mu^{m_1}\ldots)^{j_1}\,(\lambda^{l_2}\mu^{m_2}\ldots)^{j_2}\ldots(\lambda^{l_\nu}\mu^{m_\nu}\ldots)^{j_\nu},$$

another proof may be given.

In this form the sums—

$$\begin{gathered}l_1+m_1+\ldots,\\ l_2+m_2+\ldots,\\ \ldots\quad\ldots\quad\ldots\\ l_\nu+m_\nu+\ldots,\end{gathered}$$

are each considered constant.

Putting each part equal to λ, we must multiply every resulting component $(\lambda^{l_1+m_1+\dots})$ by

$$\frac{(l_1+m_1+\dots)\,!}{l_1\,!\;m_1\,!\;\dots}.$$

Thus $$S\,(\lambda^{l+m+\dots})$$

$$= \Sigma P \left\{ \Sigma \left(\frac{(l_1+m_1+\dots)\,!}{l_1\,!\;m_1\,!\;\dots}\right)^{j_1} \left(\frac{(l_2+m_2+\dots)\,!}{l_2\,!\;m_2\,!\;\dots}\right)^{j_2} \dots \left(\frac{(l_\nu+m_\nu+\dots)\,!}{l_\nu\,!\;m_\nu\,!\;\dots}\right)^{j_\nu} \right\}$$

$$\times\,(\lambda^{l_1}\mu^{m_1}\dots)^{j_1}\dots(\lambda^{l_\nu}\mu^{m_\nu}\dots)^{j_\nu}.$$

Now, $$\Sigma\,\frac{\{(l_1+m_1+\dots)\,!\}^{j_1}}{(l_1\,!\;m_1\,!\;\dots)^{j_1}}\,\frac{\{(l_2+m_2+\dots)\,!\}^{j_2}}{(l_2\,!\;m_2\,!\;\dots)^{j_2}}\dots\frac{\{(l_\nu+m_\nu+\dots)\,!\}^{j_\nu}}{(l_\nu\,!\;m_\nu\,!\;\dots)^{j_\nu}}$$

$$= \frac{(l+m+\dots)\,!}{l\,!\;m\,!\;\dots},$$

for each represents the total number of permutations of $l+m+\dots$ things, of which l are of one sort, m of a second, &c.

Hence

$$S\,(\lambda^{l+m+\dots}) = \Sigma\,\frac{(l+m+\dots)\,!}{l\,!\;m\,!\;\dots}\,P\,(\lambda^{l_1+m_1+\dots})^{j_1}(\lambda^{l_2+m_2+\dots})^{j_2}\dots(\lambda^{l_\nu+m_\nu+\dots})^{j_\nu}.$$

Comparing this with the known formula

$$\frac{(-)^{l+m+\dots}}{l+m+\dots}\,S\,(\lambda^{l+m+\dots})$$

$$= \Sigma\,(-)^{\Sigma j}\,\frac{(\Sigma j-1)\,!}{j_1\,!\;j_2\,!\;\dots j_\nu\,!}\,(\lambda^{l_1+m_1+\dots})^{j_1}(\lambda^{l_2+m_2+\dots})^{j_2}\dots(\lambda^{l_\nu+m_\nu+\dots})^{j_\nu},$$

we find, as before,

$$P = (-)^{l+m+\dots+\Sigma j}\,\frac{l\,!\;m\,!\;\dots}{(l+m+\dots-1)\,!}\cdot\frac{(\Sigma j-1)\,!}{j_1\,!\;j_2\,!\;\dots}.$$

It will be noticed that the general result involves only the numbers $l, m, \dots j_1, j_2, \dots$; so that, merely attending to these multiplicities, we may write the result in the hypersymbolic and compact form—

$$(-)^{l+m+\dots}\,\frac{(l+m+\dots-1)\,!}{l\,!\;m\,!\;\dots}\,S\,|\,lm\,\dots| = \Sigma\,(-)^{\Sigma j}\,\frac{(\Sigma j-1)\,!}{j_1\,!\;j_2\,!\;\dots}\,|\,j_1 j_2 j_3\,\dots\,|,$$

where $|\,j_1 j_2 j_3\,\dots\,|$ denotes the sum of all the corresponding separations.

This theorem enables us at once to write down an expression for

the s^{th} power of the roots corresponding to every partition of s. Thus for $s = 6$, the series is

$$S(6) = (6),$$

$$S(51) = (5)(1)-(51),$$

$$S(42) = (4)(2)-(42),$$

$$S(41^2) = (4)(1)^2-(41)(1)-(4)(1^2)+(41^2),$$

$$\tfrac{1}{2}S(3^2) = \tfrac{1}{2}(3)^2-(3^2),$$

$$2S(321) = 2(3)(2)(1)-(32)(1)-(31)(2)-(21)(3)+(321),$$

$$S(31^3) = (3)(1)^3-(31)(1)^2-2(3)(1^2)(1)$$
$$+(31)(1^2)+(31^2)(1)+(3)(1^3)-(31^3),$$

$$\tfrac{1}{3}S(2^3) = \tfrac{1}{3}(2)^3-(2^2)(2)+(2^3),$$

$$\tfrac{3}{2}S(2^21^2) = \tfrac{3}{2}(2)^2(1)^2-2(21)(2)(1)-(2^2)(1)^2-(1^2)(2)^2$$
$$+(2^2)(1^2)+\tfrac{1}{2}(21)^2+(21^2)(2)+(2^21)(1)-(2^21^2),$$

$$S(21^4) = (2)(1)^4-(21)(1)^3-3(2)(1^2)(1)^2+(21)(1)^2+2(1^3)(1)(2)$$
$$+2(21)(1^2)(1)+(2)(1^2)^2-(2)(1^4)$$
$$-(21)(1^3)-(21^2)(1^2)-(21^3)(1)+(21^4),$$

$$\tfrac{1}{6}S(1^6) = \tfrac{1}{6}(1)^6-(1^2)(1)^4+\tfrac{3}{2}(1^2)^2(1)^2+(1^3)(1)^3-\tfrac{1}{3}(1^2)^3$$
$$-2(1^3)(1^2)(1)-(1^4)(1)^2+\tfrac{1}{2}(1^3)^2+(1^4)(1^2)+(1^5)(1)-(1^6).$$

New Tables of Symmetric Functions.

It may be gathered from the foregoing section that it is possible to form tables of symmetric functions, of a symmetrical character, corresponding to every partition of every number. We may select at pleasure any partition as the partition of restriction, and write down partitions representing every possible species of its separations; by the side of these partitions we may write down the compound symmetric functions represented by the corresponding separations. In the expansion of these compounds in a series of monomial symmetric functions, only those monomials will occur which have partitions identical with those representing the species of the separations; this follows naturally from the law of algebraic reciprocity. Thus a symmetrical table necessarily results. To make the method clear, I instance the partition

$$(21^3),$$

and exemplify, in full, the corresponding symmetric function table.

Form two columns—

$$\begin{array}{ll} (5) & (21^3) \\ (41) & (21^2)(1) \\ (32) & (21)(1^2)+(1^3)(2) \\ (31^2) & (21)(1)^2 \\ (2^21) & 2\,(2)(1^2)(1) \\ (21^3) & (2)(1)^3. \end{array}$$

The left-hand column gives the species of possible separations of

$$(21^3).$$

The right-hand column gives the corresponding separations as derived from the X products (*vide* previous section).

Thus $(21)(1^2)+(1^3)(2)$ is coefficient of $x_2x_1^3$ in X_3X_2,

and $2\,(2)(1^2)(1)$ „ $x_2x_1^3$ in $X_2^2X_1$.

We may then set out in any convenient way the following table :—

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)
(21^3)						1
$(21^2)(1)$				1	2	3
$(21)(1^2)+(1^3)(2)$			1	3	2	4
$(21)(1)^2$		1	3	4	6	6
$2\,(2)(1^2)(1)$		2	2	6	4	6
$(2)(1)^3$	1	3	4	6	6	6

which reads the same by rows as by columns.

In this way we may treat every partition of every number.

We may invert these tables so as to exhibit the single partition symmetric functions in terms of compound symmetric functions symbolised by separations.

We reach then the cardinal and very important theorem of expressibility, which I now enunciate.

Theorem.—" Being given any symmetric function, of partition

$$(\lambda\mu\nu\,...),$$

let $(\lambda_1\lambda_2\lambda_3 \ldots)$ be any partition of λ,

$(\mu_1\mu_2\mu_3 \ldots)$ " " μ,

$(\nu_1\nu_2\nu_3 \ldots)$ " " ν,

... ...

Then the symmetric function

$$(\lambda\mu\nu \ldots)$$

is expressible by means of compound symmetric functions which are symbolised by separations of the partition

$$(\lambda_1\lambda_2\lambda_3 \ldots \mu_1\mu_2\mu_3 \ldots \nu_1\nu_2\nu_3 \ldots)."$$

In the example above of the partition

$$(21^3),$$

it will be noticed that there are 7 separations and 6 species of separations; there is thus

$$7-6=1,$$

syzygy between the separations.

The syzygy in question is, in fact, derivable from the separation

$$(21)(2),$$

for we may either express (21) in terms of separations of (1^3), leaving (2) unchanged, or we may leave (21) unchanged and express (2) in terms of separations of (1^2); thus the syzygy is

$$(2)\left\{(1^2)(1)-3(1^3)\right\}-(21)\left\{(1)^2-2(1^2)\right\}=0,$$

or
$$(2)(1^2)(1)-3(2)(1^3)-(21)(1)^2+2(21)(1^2)=0.$$

In general, if there are θ separations of any partition and ϕ species of separation, there must be

$$\theta-\phi$$

syzygies between the θ separations.

The h Tables direct.

	(1)
h_1	1

	(2)	(1^2)
h_2	1	1
h_1^2	1	2

	(3)	(21)	(1^3)
h_3	1	1	1
h_2h_1	1	2	3
h_1^3	1	3	6

	(4)	(31)	(2^2)	(21^2)	(1^4)
h_4	1	1	1	1	1
h_3h_1	1	2	2	3	4
h_2^2	1	2	3	4	6
$h_2h_1^2$	1	3	4	7	12
h_1^4	1	4	6	12	24

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
h_5	1	1	1	1	1	1	1
h_4h_1	1	2	2	3	3	4	5
h_3h_2	1	2	3	4	5	7	10
$h_3h_1^2$	1	3	4	7	8	13	20
$h_2^2h_1$	1	3	5	8	11	18	30
$h_2h_1^3$	1	4	7	13	18	33	60
h_1^5	1	5	10	20	30	60	120

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
h_6	1	1	1	1	1	1	1	1	1	1	1
h_5h_1	1	2	2	3	2	3	4	3	4	5	6
h_4h_2	1	2	3	4	3	5	7	6	8	11	15
$h_4h_1^2$	1	3	4	7	4	8	13	9	14	21	30
h_3^2	1	2	3	4	4	6	8	7	10	14	20
$h_3h_2h_1$	1	3	5	8	6	12	19	15	24	38	60
$h_3h_1^3$	1	4	7	13	8	19	34	24	42	72	120
h_2^3	1	3	6	9	7	15	24	21	33	54	90
$h_2^2h_1^2$	1	4	8	14	10	24	42	33	58	102	180
$h_2h_1^4$	1	5	11	21	14	38	72	54	102	192	360
h_1^6	1	6	15	30	20	60	120	90	180	360	720

The h Tables—inverse.

	h_1
(1)	1

	h_2	h_1^2
(2)	2	−1
(1^2)	−1	1

	h_3	h_2h_1	h^3
(3)	3	−3	1
(21)	−3	4	−2
(1^3)	1	−2	1

	h_4	h_3h_1	h_2^2	$h_2h_1^2$	h_1^4
(4)	4	−4	−2	4	−1
(31)	−4	7	2	−7	2
(2^2)	−2	2	3	−4	1
(21^2)	4	−7	−4	10	−3
(1^4)	−1	2	1	−3	1

	h_5	h_4h_1	h_3h_2	$h_3h_1^2$	$h_2^2h_1$	$h_2h_1^3$	h_1^5
(5)	5	−5	−5	5	5	−5	1
(41)	−5	9	5	−9	−7	9	−2
(32)	−5	5	11	−8	−11	10	−2
(31^2)	5	−9	−8	12	10	−13	3
(2^21)	5	−7	−11	10	14	−14	3
(21^3)	−5	9	10	−13	−14	17	−4
(1^5)	1	−2	−2	3	3	−4	1

	h_6	h_5h_1	h_4h_2	$h_4h_1^2$	h_3^2	$h_3h_2h_1$	h_3h_1	h_2^3	$h_2^2h_1^2$	$h_2h_1^4$	h_1^6
(6)	6	−6	−6	6	−3	12	−6	2	−9	6	−1
(51)	−6	11	6	−11	3	−17	11	−2	14	−11	2
(42)	−6	6	14	−10	3	−20	10	−6	19	−12	2
(41^2)	6	−11	−10	15	−3	21	−15	4	−20	16	−3
(3^2)	−3	3	3	−3	6	−15	6	−1	9	−6	1
(321)	12	−17	−20	21	−15	61	−30	8	−48	34	−6
(31^3)	−6	11	10	−15	6	−30	19	−4	26	−21	4
(2^3)	2	−2	−6	4	−1	8	−4	4	−10	6	−1
(2^21^2)	−9	14	19	−20	9	−48	26	−10	46	−33	6
(21^4)	6	−11	−12	16	−6	34	−21	6	−33	26	−5
(1^6)	−1	2	2	−3	1	−6	4	−1	6	−5	1

On the General Linear Differential Equation of the Second Order.

By Sir JAMES COCKLE, F.R.S.

[*Read Nov.* 10*th*, 1887.]

The Sections and Articles of this paper are numbered consecutively with those of my paper "On the Equation of Riccati" (Vol. XVIII., pp. 180—202), to which it is a complement.

§ VIII. *On Decomposable Forms and their Notation.*

84. I call a form which can be expressed in two ways, viz., either by means of synthemes alone, or by means of synthemes and of u, u', u'', $\dot{u}$, $\dot{u}'$ and $\dot{u}''$ (to the exclusion of v, $\dot{v}$ and their derivatives), a pure decomposable. Of such forms (1, 2, 3) of Art. 9 and (5) of Art. 73 afford examples.

85. The simplest of pure decomposable forms, which I therefore call primary, are herein represented by the following expressions:—

$$\begin{aligned}
(0, 0)(1, 1)-(1, 0)(0, 1) &= AD-BC = \quad i = a_1,\\
(0, 0)(2, 2)-(2, 0)(0, 2) &= AI-EF = \quad d = a_2,\\
(1, 1)(2, 2)-(2, 1)(1, 2) &= DI-GH = \quad a = a_3,\\
(0, 0)(2, 1)-(2, 0)(0, 1) &= AG-CE = -h = a_4,\\
(0, 0)(1, 2)-(1, 0)(0, 2) &= AH-BF = -g = a_5,\\
(1, 0)(2, 1)-(2, 0)(1, 1) &= BG-DE = \quad f = a_6,\\
(0, 1)(1, 2)-(1, 1)(0, 2) &= CH-DF = \quad e = a_7,\\
(1, 0)(2, 2)-(2, 0)(1, 2) &= BI-EH = -c = a_8,\\
(0, 1)(2, 2)-(2, 1)(0, 2) &= CI-FG = -b = a_9.
\end{aligned}$$

86. These expressions, which exhaust the primaries of the second degree in the synthemes and which are in fact the minors of the determinant

$$\begin{vmatrix} A, & C, & F \\ B, & D, & H \\ E, & G, & I \end{vmatrix},$$

may be easily deduced. Thus

$$\dot{v}''v'.\dot{v}v = (G-\dot{u}''u')(A-\dot{u}u) = AG-A\dot{u}''u'-G\dot{u}u+\dot{u}''u'\dot{u}u$$

$$= \dot{v}''v.\dot{v}v' = (E-\dot{u}''u)(C-\dot{u}u') = CE-C\dot{u}''u-E\dot{u}u'+\dot{u}''u\dot{u}u',$$

whence $\quad AG-CE = A\dot{u}''u'-C\dot{u}''u-E\dot{u}u'+G\dot{u}u,$

and h or a_4 is decomposable. This result I shall express by

$$-h = -\delta h, \text{ or } a_4 = \delta a_4;$$

and so in other cases.

87. More generally, put

$$\dot{v}^{(r)}v^{(s)} = (r,s)-\dot{u}^{(r)}u^{(s)}, \quad \dot{v}^{(p)}v^{(q)} = (p,q)-\dot{u}^{(p)}u^{(q)},$$

then

$$\dot{v}^{(r)}v^{(s)}.\dot{v}^{(p)}v^{(q)} = (p,q)(r,s)-(p,q)\,\dot{u}^{(r)}u^{(s)}-(r,s)\,\dot{u}^{(p)}u^{(q)}+\dot{u}^{(r)}u^{(s)}\dot{u}^{(p)}u^{(q)};$$

again, put $\dot{v}^{(p)}v^{(s)} = (p,s)-\dot{u}^{(p)}u^{(s)}, \quad \dot{v}^{(r)}v^{(q)} = (r,q)-\dot{u}^{(r)}u^{(q)},$

then

$$\dot{v}^{(p)}v^{(s)}.\dot{v}^{(r)}v^{(q)} = (p,s)(r,q)-(p,s)\,\dot{u}^{(r)}u^{(q)}-(r,q)\,\dot{u}^{(p)}u^{(s)}+\dot{u}^{(p)}u^{(s)}\dot{u}^{(r)}u^{(q)},$$

and, subtracting and transposing, we get

$$(p,q)(r,s)-(p,s)(r,q)$$

$$= (p,q)\,\dot{u}^{(r)}u^{(s)}-(p,s)\,\dot{u}^{(r)}u^{(q)}-(r,q)\,\dot{u}^{(p)}u^{(s)}+(r,s)\,\dot{u}^{(p)}u^{(q)},$$

whence the nine primaries of Art. 85, and no others, can be deduced.

88. If we frame the two schemes*

$$\begin{array}{c|ccc} & u,v; & u',v'; & u'',v'' \\ \hline \dot{u},\dot{v} & A & C & F \\ \dot{u}',\dot{v}' & B & D & H \\ \dot{u}'',\dot{v}'' & E & G & I \end{array} \qquad \text{and} \qquad \begin{array}{c|ccc} & u & u' & u'' \\ \hline \dot{u} & a & c & f \\ \dot{u}' & b & d & h \\ \dot{u}'' & e & g & i \end{array}$$

the first scheme may be regarded as giving the equations of syn-

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* It has been suggested that a change of my original notation would considerably facilitate the reading of this paper and the detection of identities; and that when the synthemes of the former paper are denoted as in the first of the above schemes (viz., $A = \dot{u}u+\dot{v}v$, $B = \dot{u}'u+\dot{v}'v$, &c.), then the decomposables of the present paper are the first minors of this array. This criticism has enabled me to simplify, and other criticisms to amend, the paper. In giving an additional example, I follow a suggestion that an example or two would be useful.

themes (Art. 8), and the other as representing the six equations

$$i\dot{u}''+h\dot{u}'+f\dot{u}=0,\qquad iu''+gu'+eu=0,$$

$$g\dot{u}''+d\dot{u}'+c\dot{u}=0,\qquad hu''+du'+bu=0,$$

$$e\dot{u}''+b\dot{u}'+a\dot{u}=0,\qquad fu''+cu'+au=0.$$

89. Now, taking the two determinants

$$\begin{vmatrix} A, & C, & F \\ B, & D, & H \\ E, & G, & I \end{vmatrix} \quad\text{and}\quad \begin{vmatrix} a, & c, & f \\ b, & d, & h \\ e, & g, & i \end{vmatrix},$$

or J and j, wherein a, b, and so on, are the several minors corresponding to A, B and so on, respectively, we get (six, or, changing rows into columns) twelve expressions of the type $Ab+Cd+Fh$, or, say, ΣAb, each of which vanishes. We also get (three, or, changing as before) six of the type $Aa+Cc+Ff$, or, say ΣAa, each of which represents J. And J vanishes (not identically but) in virtue of the nine equations of synthemes given in Art. 8. This will be seen on actually substituting for A, B, and so on, their respective values $\dot{u}u+\dot{v}v$, $\dot{u}'u+\dot{v}'v$, and so on, or may be shown as in Arts. 98, 99.

90. If δ be a distributive operator affecting only the letters $A, B, \ldots I$ (and not $\dot{u}$ or u), and such that its effect on a single syntheme (say A) is to destroy the term containing $\dot{v}$ and v (so that, for instance, $\delta A = \dot{u}u$); and, if moreover the effect of δ on a product be made to resemble that of the d of differentiation (so that, for example, $\delta\,.\,AD = A\delta D + D\delta A$), then we may represent any one of the decompositions of Art. 85 by $a = \delta a$.

§ IX. *On certain Identities.*

91. The above formulæ involve identities, which I write here. Thus

$$\begin{aligned}
Aa-Gg &= Ii-Bb &&= Dd-Ff\,(=ADI-BFG),\\
Aa-Hh &= Ii-Cc &&= Dd-Ee\,(=ADI-CEH),\\
Gg-Hh &= Bb-Cc &&= Ff-Ee\,(=BFG-CEH),\\
Gg-Ff &= Aa-Dd &&= Hh-Ee\,(=DEF-AGH),\\
Aa-Ii &= Gg-Bb &&= Hh-Cc\,(=IBC-AGH),\\
Dd-Ii &= Ee-Cc &&= Ff-Bb\,(=IBC-DEF),
\end{aligned}$$

whence we can form systems of four equations homogeneous and linear in A, B ... and also in a, b ..., and independent: for example,

$$Ee-Ff+Gg-Hh=0=Ee-Ff+Bb-Cc,$$

$$Dd-Ee-Ii+Cc=0=Dd-Ee-Aa+Hh.$$

92. There is another class of identities, which also I write down here, premising a fully worked out example. We have

$$Ab+Cd+Fh=0=A\delta b+C\delta d+F\delta h$$
$$=A\delta\,(FG-CI)+C\delta\,(AI-EF)+F\,(CE-AG).$$

But
$$A\delta\,(FG-CI)=A\,(F\dot{u}''u'+G\dot{u}u''-C\dot{u}''u''-I\dot{u}u'),$$
$$C\delta\,(AI-EF)=C\,(A\dot{u}''u''+I\dot{u}u-E\dot{u}u''-F\dot{u}''u),$$
$$F\delta\,(CE-AG)=F\,(C\dot{u}''u+E\dot{u}u'-A\dot{u}''u'-G\dot{u}u);$$

therefore

$$Ab+Cd+Fh=(AG-CE)\,\dot{u}u''+(EF-AI)\,\dot{u}u'+(CI-FG)\,\dot{u}u$$
$$=-\dot{u}\,(hu''+du'+bu).$$

Proceeding in this manner, I get the following twelve equations, whereof the sinisters, being all of the type ΣAb, vanish; and if we reject the monomial solutions $\dot{u}, \dot{u}' \ldots u', u''=0$, we are led to the system of Art. 88.

Now, unless j (and therefore J) vanishes the system of Art. 88 will lead to the rejected monomial solutions. But J (and therefore j) does in fact vanish when we postulate the nine equations of synthemes (Art. 8). This will be seen on substituting for A, B, ... I their values, or may be shown as in Arts. 98, 99.

$$Ba+Dc+Hf=-\dot{u}'\,(fu''+cu'+au),$$
$$Ea+Gc+If=-\dot{u}''\,(fu''+cu'+au),$$
$$Ab+Cd+Fh=-\dot{u}\,(hu''+du'+bu),$$
$$Eb+Gd+Ih=-\dot{u}''\,(hu''+du'+bu),$$
$$Ae+Cg+Fi=-\dot{u}\,(iu''+gu'+eu),$$
$$Be+Dg+Hi=-\dot{u}'\,(iu''+gu'+eu),$$

$$Ca+Db+Ge=-u'\,(e\dot{u}''+b\dot{u}'+a\dot{u}),$$

$$Fa+Hb+Ie=-u''\,(e\dot{u}''+b\dot{u}'+a\dot{u}),$$

$$Ac+Bd+Eg=-u\,(g\dot{u}''+d\dot{u}'+c\dot{u}),$$

$$Fc+Hd+Ig=-u''\,(g\dot{u}''+d\dot{u}'+c\dot{u}),$$

$$Af+Bh+Ei=-u\,(i\dot{u}''+h\dot{u}'+f\dot{u}),$$

$$Cf+Dh+Gi=-u'\,(i\dot{u}''+h\dot{u}'+f\dot{u}).$$

93. The simultaneous interchanges (B, C), (E, F), and (G, H) have no effect upon a_1, a_2, a_3. In other cases they change a_{2m} into a_{2m+1} or a_{2m+1} into a_{2m}. These interchanges would be made by a shifting of accents in Art. 8 (say, for instance, by changing $\dot{u}'u$ into $u\dot{u}'$). But such shifting would have no effect upon A, D, or I.

94. I remark that

$$ADI-BFG=(2,2)(1,1)(0,0)-(2,1)(1,0)(0,2),$$

$$ADI-CEH=(2,2)(1,1)(0,0)-(2,0)(1,2)(0,1),$$

and that a similar form for $BFG-CEH$ may be found by subtraction.

§ X. *Differentiations and Verifications; Simplifications.*

95. The foregoing results are general and not confined to the cases in which the accents denote differentiations. They are true when the accents are regarded merely as marks to distinguish different quantities. But, treating the accents as differentiations and recurring to § II., we get

$$i'=-(g+h),$$

$$d'=-(\dot{p}+p)\,d+\dot{q}g+qh-(b+c),$$

$$a'=-(\dot{p}+p)\,a+\dot{r}e+rf,$$

$$-h'=f+d+\dot{p}h-\dot{q}i,$$

$$-g'=e+d+pg-qi,$$

$$f'=-c-\dot{p}f+\dot{r}i,$$

$$-e'=-b-pe+ri,$$

$$-c'=(\dot{p}+p)\,c-qf-\dot{r}g+a,$$

$$-b'=(\dot{p}+p)\,b-\dot{q}e-rh+a.$$

96. If we differentiate the identity

$$Ff-Ee-Gg+Hh=0,$$

which (with signs changed) is one of the identities of Art. 91, we get, after substitutions and reductions,

$$\begin{aligned}&(Ff-Ee-Gg+Hh)'\\ &=-(\dot{p}+p)(Ff-Ee-Gg+Hh)-Fc+Eb+(G-H)\,d-I\,(g-h)\\ &\quad+\dot{q}\,(Hi+Dg+Be)-q\,(Gi+Cf+Dh)\\ &\quad+\dot{r}\,(Fi+Cg+Ae)-r\,(Ei+Af+Bh)\,;\end{aligned}$$

but $-Fc+Eb+(G-H)\,d-I\,(g-h)$ vanishes identically, as will be found on substitution, and the rest vanishes in virtue of the identities of Arts. 91, 92. All this is right.

97. Again, take the identity

$$Ei+Bh+Af=0\,;$$

we get

$$(Ei+Bh+Af)'=Gi+Dh+Cf-(Ei+Bh+Af)\,\dot{p}=0+0\dot{p}=0,$$

as is seen on turning to Art. 92. This too is right.

98. And I here add that

$$\begin{aligned}Aa+Bb+Ee&=u''\,(i\dot{u}''+h\dot{u}'+f\dot{u})+u'\,(g\dot{u}''+d\dot{u}'+c\dot{u}),\\ Aa+Cc+Ff&=\dot{u}''\,(iu''+gu'+eu)+\dot{u}'\,(hu''+du'+bu),\\ Bb+Dd+Hh&=\dot{u}\,(fu''+cu'+au)+\dot{u}''\,(iu''+gu'+eu),\\ Cc+Dd+Gg&=u\,(e\dot{u}''+b\dot{u}'+a\dot{u})+u''\,(i\dot{u}''+h\dot{u}'+f\dot{u}),\\ Ee+Gg+Ii&=\dot{u}'\,(hu''+du'+bu)+\dot{u}\,(fu''+cu'+au),\\ Ff+Hh+Ii&=u'\,(g\dot{u}''+d\dot{u}'+c\dot{u})+u\,(e\dot{u}''+b\dot{u}'+a\dot{u}).\end{aligned}$$

The first of these formulæ is obtained from

$$Aa+Bb+Ee=A\delta a+B\delta b+C\delta c,$$

by processes corresponding with those of Art. 92; the rest in a similar manner..

99. Rejecting the results $\dot{u},\ \dot{u}',\ \dot{u}'',\ \dots\ u''=0$, there remain in Arts. 88 and 92, six equations which show that the six forms of J given in Art. 98 vanish. Again, subtract the second form from the first. We get

$$Ee-Ff+Bb-Cc=\dot{u}\,(fu''+cu')-u\,(e\dot{u}''+b\dot{u}'),$$

whereof the sinister vanishes (see Art. 91), and the dexter reduces to $\dot{u}\,(-au)-u\,(-a\dot{u})$ or zero.

Formulæ such as those given in Arts. 88 and 92 may aid in keeping down elevation of degree arising from elimination.

100. In this paper, however, my object is not so much to consider the actual calculations incident to the eliminations as to conform, as nearly as may be, with the prior memoir and to ingraft upon it the proof that the general linear differential equation of the second order is soluble by means of an algebraical equation, the coefficients of which, however, will not in general be algebraical.

101. It is not meant to be asserted that such equation will itself be algebraically soluble. For, although one of its roots will be a rational function of two other of them and of the coefficients, still, unless its degree be prime, its solubility cannot be affirmed.

102. The sinisters of $a = \delta a$ will consist of terms of the form $\beta\gamma$ (using β, γ, ϵ, &c., to represent letters of the set $A, B, \dots I$), and the dexters will consist of terms of the form $\epsilon \dot{u}^{(r)} u^{(s)}$.

103. The expressions $a_m \delta a_n - a_n \delta a_m$ will be linear and homogeneous in $\dot{u}$ and u; so that on dividing by $\dot{u}u$ we shall get equations which the elimination of $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2 will enable us to express in terms of $A, B, \dots I$. But (anticipating Art. 133) the expressions

$$(\Theta_0\Theta_3-\Theta_1\Theta_2)\,a_m - c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx}\,\delta a_m,$$

similar in the linearity and homogeneity to the former, will be more advantageous, being of lower dimensions in $A, B, \dots I$. By means of Art. 88, the $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2 of Art. 79 can be expressed rationally in terms of the minors. But the forms $(0:0)$ are useless.

104. As the δ of Arts. 86, 90 does not affect $\dot{u}$ or u, so neither does it affect $p, q \dots \dot{q}, \dot{r}$, nor indeed any function X of x; so that, for example, $\delta\,.\,Xa = X\delta a$.

§ XI. *Introduction of the General Biordinal.*

105. Let p and r be independent and arbitrary functions of x, and let m be any constant (other than 0 or ∞). Then the general linear and homogeneous biordinal may be represented by

$$z'' + \left(p + \frac{m}{x}\right) z' + \left\{ \frac{1}{x}\left(p + \frac{m}{x}\right) - \frac{xr}{m} - \frac{2}{x^2} \right\} z = 0.$$

106. Effecting the transformation indicated by

$$\left\{\frac{d^2}{dx^2}+\left(p+\frac{m}{x}\right)\frac{d}{dx}+\frac{1}{x}\left(p+\frac{m}{x}\right)-\frac{xr}{m}-\frac{2}{x^2}\right\}\left(\frac{d}{dx}-\frac{m}{x}\right)y=0,$$

we get $$y'''+py''+qy'+ry=0,$$

wherein $$q=-\left\{\frac{m-1}{x}\left(p+\frac{m-2}{x}\right)+\frac{xr}{m}\right\}.$$

107. The complete solution of this terordinal involves that of the biordinal. Let $y=u,\ v,\ w$ denote three independent solutions of the terordinal. Then $(y=)\,w=x^m$ may be taken as one of these solutions.

108. It follows (Arts. 4 and 15; also Art. 55) that $\dfrac{x^{m-1}}{r}$ is an integrating factor of the deformation.

109. The value $m=2$ is that best adapted to the case of the particular Riccatian. I retain it here, and the results of the present are easily compared with those of my prior paper. Put, then, $w=x^2$.

110. The terordinal will be

$$y'''+py''-\left(\frac{p}{x}+\tfrac{1}{2}xr\right)y'+ry=0,$$

and, multiplied into x, will take the form

$$\Theta''+p\Theta'-\tfrac{1}{2}xr\Theta=0,$$

wherein $$\Theta=xy'-2y.$$

111. Its deformation will be

$$Y'''-\left(p+2\frac{r'}{r}\right)Y''$$
$$+\left\{\left(p+\frac{r'}{r}\right)\frac{r'}{r}-\left(p+\frac{r'}{r}\right)'-\frac{p}{x}-\tfrac{1}{2}xr\right\}Y'-rY=0,$$

which, multiplied into $\dfrac{x}{r}$ and integrated, yields

$$\frac{x}{r}Y''-\left(\frac{x}{r}p+x\frac{r'}{r^2}+\frac{1}{r}\right)Y'-\tfrac{1}{2}x^2Y=\text{constant}$$

whence, if the constant be supposed to vanish,

$$Y''-\left(p+\frac{r'}{r}+\frac{1}{x}\right)Y'-\tfrac{1}{2}xrY=0.$$

112. The mixed integral of Art. 16 written in the form

$$y'\left\{Y''-\left(p+\frac{r'}{r}\right)Y'\right\}-y''Y'-ryY=cr,$$

therefore, becomes

$$y'\left(\frac{1}{x}Y'+\tfrac{1}{2}xrY\right)-y''Y'-ryY=cr,$$

or $$-\frac{1}{x}(xy''-y')\,Y'+\tfrac{1}{2}r(xy'-2y)\,Y=cr,$$

which is equivalent to

$$-\Theta'Y'+\tfrac{1}{2}xr\Theta Y=cxr.$$

113. The results are the same in form for the dotted as for the undotted letters. And by precisely similar steps we are led, in the correlate system, to the mixed integral

$$-\dot{\Theta}'\dot{Y}'+\tfrac{1}{2}x\dot{r}\dot{\Theta}\dot{Y}=\dot{c}x\dot{r}.$$

114. The unsuffixed Θ of this paper is essentially different from the suffixed Θ's, say the Θ_m's. Thus, Θ_m means a function of x and of synthemes; while Θ is the dependent variable in the biordinal of Art. 110, whereof $\Theta=\theta$, and $\Theta=\vartheta$, are supposed to be independent particular integrals. And $\dot{\Theta}\ (=x\dot{y}'-2\dot{y})$ is the dependent variable in

$$\dot{\Theta}''+\dot{p}\dot{\Theta}'-\tfrac{1}{2}x\dot{r}\dot{\Theta}=0,$$

whereof $\dot{\Theta}=\dot{\theta}$, and $\dot{\Theta}=\dot{\vartheta}$, are to be taken as independent particular solutions.

115. Except in so far as it is necessary in particular cases to substitute appropriate particular values for p and r the process is, up to a certain point and so long as we keep to the value 2 of the exponent m of Art. 107, the same for all biordinals. The equations of Arts. 8—12 are the same for all. The meaning of θ, ϑ, $\dot{\theta}$, and $\dot{\vartheta}$ is the same for all. The quantities Θ_0, Θ_1, Θ_2, and Θ_3 are, when expressed in terms of x and synthemes, the same for all. But the expressions for Θ_0, Θ_1, Θ_2, and Θ_3 in terms of M and x are peculiar to the Riccatian discussed. So that while the relations of Art. 58 are always true those of Art. 68 are, so far at least as M and all which follows the "=" that precedes it are concerned, true only for the particular Riccatian.

116. The equation of Art. 16 holds for all values of Y and y.

Keeping to the same Y, let y_1, y_2, and y_3 be any three distinct values of y. Then the three equations

$$y_1'Y''-y_1''Y'-\left(p+\frac{r'}{r}\right)y_1'Y'-ry_1Y=c_1r,$$

$$y_2'Y''-y_2''Y'-\left(p+\frac{r'}{r}\right)y_2'Y'-ry_2Y=c_2r,$$

$$y_3'Y''-y_3''Y'-\left(p+\frac{r'}{r}\right)y_3'Y'-ry_3Y=c_3r,$$

always hold when proper values or ratios are given or assigned to or among the arbitrary constants c_1, c_2, and c_3.

117. Writing this system thus—

$$\begin{array}{lll} Y'', & Y', & Y, \\ y_1', & -y_1''-\left(p+\frac{r'}{r}\right)y_1', & -ry_1=c_1r, \\ y_2', & -y_2''-\left(p+\frac{r'}{r}\right)y_2', & -ry_2=c_2r, \\ y_3', & -y_3''-\left(p+\frac{r'}{r}\right)y_3', & -ry_3=c_3r, \end{array}$$

and dealing with the determinant on the sinister in the same way as the determinant δ was dealt with in Art. 21, we get

$$Y=e^{\int pdx}\left(-c_1\begin{vmatrix} y_2'', & y_2' \\ y_3'', & y_3' \end{vmatrix}-c_2\begin{vmatrix} y_3'', & y_3' \\ y_1'', & y_1' \end{vmatrix}-c_3\begin{vmatrix} y_1'', & y_1' \\ y_2'', & y_2' \end{vmatrix}\right).$$

118. Putting $c_1=0=c_2$, replacing Y, y_1, and y_2 by W, u, and v respectively, and merging $-c_3$ in the constant of integration, we get

$$W=e^{\int pdx}\begin{vmatrix} u'', & u' \\ v'', & v' \end{vmatrix},$$

and, by corresponding operations or by cyclical changes, we get the systems of Arts. 18 and 19.

119. If in Arts. 96, 97, we put $y_1, y_2, y_3=u, v, x^2$; then

$$u'Y''-u''Y'-\left(p+\frac{r'}{r}\right)u'Y'-ruY=c_1r,$$

$$v'Y''-v''Y'-\left(p+\frac{r'}{r}\right)v'Y'-rvY=c_2r,$$

$$Y''-\left(p+\frac{r'}{r}+\frac{1}{x}\right)Y'-\tfrac{1}{2}xrY=\tfrac{1}{2}c_3\frac{r}{x},$$

and if we suppose, in conformity with Art. 111, that the arbitrary constant in the integrated deformation vanishes, then $c_2 = 0$.

120. On these suppositions the formulæ of Art. 112 lead to

$$-\theta' Y' + \tfrac{1}{2}xr\theta Y = c_1 xr,$$

$$-\vartheta' Y' + \tfrac{1}{2}xr\vartheta Y = c_2 xr,$$

whence, eliminating Y', reducing and recalling Art. 56,

$$(\theta\vartheta' - \vartheta\theta')\, Y = 2\,(c_1\vartheta' - c_2\theta') = e^{-\int p\,dx}\,(c_1 U + c_2 V),$$

and $$e^{\int p\,dx}\,(\theta\vartheta' - \vartheta\theta') = \frac{c_1 U + c_2 V}{Y}.$$

121. The sinister is a constant, for

$$\left\{e^{\int p\,dx}\,(\theta\vartheta' - \vartheta\theta')\right\}'$$

vanishes identically when θ'' and ϑ'' are eliminated (for θ and ϑ are solutions of the biordinal of Art. 110); and $C_1 U + C_2 V$ being substituted for Y on the dexter, we have

$$c_1 : C_1 = c_2 : C_2 = k,$$

where $\log k$ is the constant of integration in $\int p\,dx$. Hence c_1 and c_2 cannot be supposed to vanish simultaneously without leading to a useless result. But either c_1 or c_2 may vanish separately, and in fact one (only) of them is supposed to vanish in deducing the systems of Arts. 18 and 19. This evanescence enabled us to give to the systems a shape which, though not the most general in form, is in substance general, and the simplest which can be constructed.

122. Let $Z = U$ or V; then by $\{3\}$ and $\{4\}$ of Art. 59, we have

$$Z = \pm 2e^{\int p\,dx}\,\Theta',$$

the positive sign $(+)$ and the value $\Theta = \vartheta$ being taken when $Z = U$, and the negative sign $(-)$ and the value $\Theta = \theta$ being taken when $Z = V$. Hence

$$Z' = pZ \pm 2e^{\int p\,dx}\,\Theta'' = pZ \pm 2e^{\int p\,dx}\,(-p\Theta' + \tfrac{1}{2}xr\Theta) = \pm\, e^{\int p\,dx}\,xr\Theta,$$

and $$\Theta' Z' = \pm\,\tfrac{1}{2}xr\,.\,2e^{\int p\,dx}\,\Theta'\Theta = \tfrac{1}{2}xr\Theta Z.$$

123. It follows that

$$-\vartheta' U' + \tfrac{1}{2}xr\vartheta U \quad\text{and}\quad -\theta' V' + \tfrac{1}{2}xr\theta V$$

both vanish identically. Hence, dealing with the first two equations of Art. 120, when we put $Y = U$, then we must put $c_2 = 0$, and

$$-\theta' U' + \tfrac{1}{2} xr\theta U = c_1 rx;$$

and when we put $Y = V$ then we must put $c_1 = 0$, and

$$-\vartheta' V' + \tfrac{1}{2} xr\vartheta V = c_2 rx.$$

124. When $c_2 = -c_1$ these last results coincide, for, multiplying both into $e^{\int p\,dx}$ and remembering Art. 122, each becomes

$$VU' - UV' = c_1 rx e^{\int p\,dx},$$

whence, since $R = -r$,

$$Re^{\int P\,dx}(VU' - UV') = -c_1 r^2 x e^{\int (p+P)\,dx} = -c_1 kKx,$$

a result which, when $-c_1 kK = 2$, coincides with the sixth relation of Art. 19. Here $\log K$ is supposed to be the constant in the integration $\int P\,dx$.

125. For the correlate system we get (since $\dot{R} = -\dot{r}$) a similar result

$$\dot{R}e^{\int \dot{P}\,dx}(\dot{V}\dot{U}' - \dot{U}\dot{V}') = -\dot{c}_1 \dot{r}^2 x e^{\int (\dot{p}+\dot{P})\,dx} = -\dot{c}_1 \dot{k}\dot{K}x,$$

and a similar condition $-\dot{c}_1 \dot{k}\dot{K} = 2$; $\log \dot{k}$ and $\log \dot{K}$ being the constants of integration in $\int \dot{p}\,dx$ and $\int \dot{P}\,dx$ respectively.

126. Thus, when the arbitrary constants are properly adjusted, the mixed integrals introduce no new conditions.

§ XII. *On Certain Special Cases.*

127. Putting $rx = 2$, multiplying the first mixed integral of Art. 123 into λ, and substituting for λU a value given by $\{3\}$ of Art. 59, we get

$$\theta' \{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\}' - \theta\{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\} = 2c_1\lambda,$$

or

$$\theta' \{x(1-x^2)\dot{\theta}'' + (1-2x^2)\dot{\theta}' + 2x\dot{\theta}\} - \theta\{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\} = 2c_1\lambda.$$

But, in the case of the Riccatian, the biordinal of Art. 114 gives

$$\dot{\theta}'' + \left(\frac{1}{x} - x\right)\dot{\theta}' + \dot{\theta} = 0, \text{ and } x(1-x^2)\dot{\theta}'' = -(1-x^2)^2\dot{\theta}' - x(1-x^2)\dot{\theta},$$

and the mixed integral becomes

$$\theta'\left[\{1-2x^2-(1-x^2)^2\}\dot{\theta}'+x(1+x^2)\dot{\theta}\right]-\theta\{x(1-x^2)\dot{\theta}'+x^2\dot{\theta}\}=2c_1\lambda,$$

or
$$-x^4\dot{\theta}'\theta'+x(1+x^2)\dot{\theta}\theta'-x(1-x^2)\dot{\theta}'\theta-x^2\dot{\theta}\theta=2c_1\lambda.$$

128. Again, the corresponding mixed integral of the correlate system becomes
$$-\dot{\theta}'\dot{U}'-\dot{\theta}\dot{U}=-2\dot{c}_1,$$
since $\dot{x}r=-2$. Multiplying into $\dot{\lambda}$ and substituting for $\dot{\lambda}\dot{U}$ a value given by $\{1\}$ of Art. 59, we get in the same way

$$\dot{\theta}'\{x(1+x^2)\theta''+(1+2x^2)\theta'-2x\theta\}+\dot{\theta}\{x(1+x^2)\theta'-x^2\theta\}=-2\dot{c}_1\dot{\lambda},$$

whence, eliminating θ'' by means of

$$\theta''+\left(\frac{1}{x}+x\right)\theta'-\theta=0,$$

obtained from the biordinal of Art. 90, we get

$$\dot{\theta}'\{-x^4\theta'-x(1-x^2)\theta\}+\dot{\theta}\{x(1+x^2)\theta'-x^2\theta\}=-2\dot{c}_1\dot{\lambda}.$$

129. The two mixed integrals coincide if $2c_1\lambda=-2\dot{c}_1\dot{\lambda}$. But (Art. 54) $\dot{\lambda}=-\lambda$. Hence $\dot{c}_1=c_1$, and recurring to Arts. 76 and 77, we have $2c_1\lambda=-2\dot{c}_1\dot{\lambda}=-c=\dot{c}$.

130. Unless we introduce transcendents these hybrid integrals, involving both $\dot{u}$ and u (and also their differential coefficients), can only be obtained in rare cases, of which the particular Riccatian is the most conspicuous example.

131. Suppose that the biordinal of Art. 105, put under Boole's form, is
$$D(D-b)z+x^2(D-a)z=0,$$
then, proceeding as indicated in Art. 106, we have the terordinal

$$D(D-2)(D-b)y+x^2(D-2)(D-a)y=0,$$

its deformation

$$D^2(D+b-2)Y-x^2(D+2)(D+a)Y=0,$$

a correlate

$$D(D-2)(D-b)\dot{y}-x^2(D-2)(D-a)\dot{y}=0,$$

and its deformation

$$D^2(D+b-2)\dot{Y}+x^2(D+2)(D+a)\dot{Y}=0.$$

132. Following Boole and transforming the deformation into the correlate by the substitution

$$Y = P_2 \frac{D\,.\,D+a-2\,.\,D\,.\,D-2\,.\,D-b}{D^2\,.\,D+b-2\,.\,D-4\,.\,D-a-2}\,\dot{y} = P_2 \frac{D-2\,.\,D+a-2\,.\,D-b}{D-4\,.\,D-a\,.\,D+b-2}\,\dot{y},$$

we see that when, and only when, a is (an) even (integer) or when $b-a$ is even, or when $2a$ and $2b$ are both even, then a hybrid integral can be found without introducing transcendents. The like would hold if we compared the given equation with the deformation of its correlate. And the process of the prior paper admits of extension to all cases in which a is odd and b even. But the better course would be to transform at once to the case already discussed.

§ XIII. *On the Functions* Θ_0, Θ_1, Θ_2, *and* Θ_3.

133. Multiplying together the two identities

$$\theta\vartheta' - \vartheta\theta' = c_1 e^{-\int p\,dx} \text{ and } \dot{\theta}\dot{\vartheta}' - \dot{\vartheta}\dot{\theta}' = \dot{c}_1 e^{-\int \dot{p}\,dx},$$

we get

$$\begin{vmatrix} \dot{\theta}, & \dot{\vartheta} \\ \dot{\theta}', & \dot{\vartheta}' \end{vmatrix} \cdot \begin{vmatrix} \theta, & \vartheta \\ \theta', & \vartheta' \end{vmatrix} = \begin{vmatrix} \dot{\theta}\theta + \dot{\vartheta}\vartheta, & \dot{\theta}\theta' + \dot{\vartheta}\vartheta' \\ \dot{\theta}'\theta + \dot{\vartheta}'\vartheta, & \dot{\theta}'\theta' + \dot{\vartheta}'\vartheta' \end{vmatrix} = c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx},$$

or

$$\Theta_0\Theta_3 - \Theta_1\Theta_2 = \dot{c}_1 c_1 e^{-\int(\dot{p}+p)\,dx} \quad\text{......................(B).}$$

134. In verification make the substitutions appropriate to the Riccatian; then (Arts. 41, 43, and 68) this (B) becomes

$$M^2x^2 - M^2\left(x^2 - \frac{1}{x^2}\right) = c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx} = \frac{c_1\dot{c}_1}{k\dot{k}x^2},$$

whence

$$M^2 = \frac{c_1\dot{c}_1}{k\dot{k}} = \frac{4c_1\dot{c}_1\lambda\dot{\lambda}}{4k\dot{k}\lambda\dot{\lambda}} = -(2c_1\lambda)(2\dot{c}\dot{\lambda}) = (2c_1\lambda)^2 = (-c)^2 = c^2,$$

as we see on turning to Arts. 67 and 109.

135. In further verification, differentiate. Then

$$(\Theta_0\Theta_3 - \Theta_1\Theta_2)' = \Theta_0\Theta_3' + \Theta_3\Theta_0' - \Theta_1\Theta_2' - \Theta_2\Theta_1' = -(\dot{p}+p)(\Theta_0\Theta_3 - \Theta_1\Theta_2),$$

as will be seen on substituting for the accented letters their respective values, viz.:

$$\begin{aligned} \Theta_3' &= -(\dot{p}+p)\,\Theta_3 + \tfrac{1}{2}x\dot{r}\,\Theta_2 + \tfrac{1}{2}xr\,\Theta_1, \\ \Theta_2' &= \Theta_3 - p\Theta_2 + \tfrac{1}{2}rx\,\Theta_0, \\ \Theta_1' &= \Theta_3 - \dot{p}\,\Theta_1 + \tfrac{1}{2}x\dot{r}\,\Theta_0, \\ \Theta_0' &= \Theta_2 + \Theta_1. \end{aligned}$$

136. By development

$$\Theta_0\Theta_3-\Theta_1\Theta_2 = x^4a+2x^3(b+c)+2x^2(2d+f+e)+4x(g+h)+4i,$$

and we have a decomposable form. Decomposing it, we get

$$\dot{\Theta}_0\delta\Theta_3+\Theta_3\delta\Theta_0-\Theta_1\delta\Theta_2-\Theta_2\delta\Theta_1 = x^3\Theta_0\dot{u}''u''-(x^2\Theta_1+x\Theta_0)\,\dot{u}'u''$$

$$-(x^2\Theta_2+x\,\Theta_0)\,\dot{u}''u'+2x\,\Theta_1\dot{u}u''+2x\,\Theta_2\,\dot{u}''u+\left\{x^2\Theta_3+x(\Theta_1+\Theta_2)+\dot{\Theta}_0\right\}\dot{u}'u'$$

$$-2\,(x\Theta_3+\Theta_1)\,\dot{u}u'-2\,(x\Theta_3+\Theta_2)\,\dot{u}'u+4\Theta_3\dot{u}u = c_1\dot{c}_1e^{-\int(\dot{p}+p)\,dx}.$$

137. In verification make the appropriate substitutions (Art. 68) and we get the mixed integral of Arts. 76 and 77; and, when this hybrid (Art. 130) integral is known, all the four Θ_m's are known.

138. Differentiation will elicit no new result from (B) nor, as Art. 95 shows, from the formulæ of Art. 85. And the last statement is confirmed when we follow the course of my prior paper. Six additional results are got by means of (2)′, (2)″, (3)′, and (3)″, or, in other words, by twice differentiating (2) and (3) of Art. 9, and then eliminating $\dot{u}'''$, u''', and the accented letters A', B', ... I' by means of the formulæ of Arts. 9, 11, and 12.

139. The relations so obtained will not, however, be in their simplest form. Remarking that (4) is (1)′, and that (5) of Art. 74 is in fact $(4)'-2\,(2)+(\dot{q}+q)\,(1)$, I form the several expressions

$$(2)'+(\dot{p}+p)\,(2), \text{ or say } (7);$$

$$(7)+(\dot{p}+p)\,(7)-2\{(3)+(\dot{q}+q)(2)+\dot{q}q\,(1)\}, \text{ or say } (8);$$

$$(3)'+(\dot{p}+p)(3), \text{ or say } (9);$$

and $$(9)'+(\dot{p}+p)(9)-2\dot{r}r\,(1), \text{ or say } (10);$$

and thus from that prior paper I get, putting

$$\dot{r}-\dot{p}\dot{q}-\dot{q}' = \dot{\rho} \text{ and } r-pq-q' = \rho,$$

$$i = \delta i,\quad d = \delta d,\quad a = \delta a \text{..................}(1, 2, 3),$$

$$g+h = \delta g+\delta h \text{ ....}(4),$$

$$f+e+\dot{p}h+pg = \delta f+\delta e+\dot{p}\delta h+p\delta g \text{.....................}(5),$$

$$-b-c+qh+\dot{q}g = -\delta b-\delta c+q\delta h+\dot{q}\delta g \text{..................}(7),$$

$$-2\dot{q}e-2qf-\rho h-\dot{\rho}g = -2\dot{q}\delta e-2q\delta f-\rho\delta h-\dot{\rho}\delta g \text{ ....}(8),$$

$$rf+\dot{r}e = r\delta f+\dot{r}\delta e \dots\dots\dots\dots (9),$$

$$-rc-\dot{r}b+\pi rf+\dot{\pi}re = -r\delta c-\dot{r}\delta b+\pi r\delta f+\dot{\pi}r\delta e \dots\dots\dots (10),$$

π and $\dot{\pi}$ being already (Arts. 18, 41, 43) defined. If we proceed to another differentiation we get

$$-2pf-2\dot{p}e-(\dot{p}^2-\dot{p}')\,h-(p^2-p')\,g$$
$$= -2p\delta f-2\dot{p}\delta e-(\dot{p}^2-\dot{p}')\,\delta h-(p^2-p')\,\delta g,$$

and inasmuch as this can be put under the form

$$\lambda\,(4)+\mu\,(5)+\nu\,(8)+\sigma\,(9),$$

where λ, μ, ν, and σ are all free from any or either of the quantities f, e, h, and g, no new result is gained.

140. Whichever course we adopt, viz., whether we use the system of Art. 85 or that of Art. 139, we get nine equations, equivalent however to five independent equations only. Denote the six forms of J, taken in the order in which they occur in (the sinisters of) Art. 98, by (1, 2, 3, 4, 5, 6) respectively. Then, whether the small letters represent minors or decomposables, or whether they are regarded as arbitrary and independent, the identity

$$(1)+(4)+(6)-(2)-(3)-(5) = 0$$

subsists; and the relations

$$(1) = (2) = (3) = (4) = (5) = (6)$$

imply four conditions only. But they do imply four distinct conditions capable of being exhibited in various ways. For instance, strike out (6); then $(1) = (2)$, $(1) = (3)$, $(1) = (4)$, $(1) = (5)$, will give four distinct conditions, for Ff occurs in $(1) = (2)$ only; Hh in $(1) = (3)$ only; Aa in $(1) = (4)$ only, and Ii in $(1) = (5)$ only. Again, recurring to Art. 91, Gg, Bb, Ii, and Hh severally occur only in the first, second, third, and fourth respectively of the relations given at its close. But the system of Art. 91 is not really different from the one just considered.

Thus the nine equations which enter (explicitly) reduce themselves (implicitly) to $(9-4$, or) five conditions. And these conditions may be expressed in terms of A, B, ... I without imposing any additional restriction. For when (and if) the four Θ_m's are properly determined (say each in the form $\Theta_m = X_m$), then the equation (B) of Art. 133 is necessarily satisfied, and the process of Art. 103 introduces no new

relation. And, since each of the Θ_m's is a linear function of some four of the nine quantities $A, B, \ldots I$, we have $(5+4$, or) nine relations, not all homogeneous, for determining the nine quantities $A, B, \ldots I$.

141. All this supposes that the four Θ_m's can be finitely determined.

§ XIV. *On the Solution of the General Biordinal.*

142. Assume

$$Y = L\dot{y}'' + M\dot{y}' + N\dot{y},$$

then

$$Y' = L_1\dot{y}'' + M_1\dot{y}' + N_1\dot{y},$$

$$Y'' = L_2\dot{y}'' + M_2\dot{y}' + N_2\dot{y},$$

where

$$L_1 = -\dot{p}L + L' + M, \quad L_2 = -\dot{p}L_1 + L_1' + M_1,$$

$$M_1 = -\dot{q}L + M' + N, \quad M_2 = -\dot{q}L_1 + M_1' + N_1,$$

$$N_1 = -\dot{r}L + N', \quad N_2 = -\dot{r}L_1 + N_1'.$$

143. Substituting these values of Y, Y', and Y'' in the biordinal of Art. 111, and putting for the moment $K = p + \dfrac{r'}{r} + \dfrac{1}{x}$, we get

$$(L_2 - KL_1 - \tfrac{1}{2}xrL)\,\dot{y}'' + (M_2 - KM_1 - \tfrac{1}{2}xrM)\,\dot{y}' + (N_2 - KN_1 - \tfrac{1}{2}xrN)\,\dot{y} = 0.$$

144. This will be satisfied if

$$L_2 - KL_1 - \tfrac{1}{2}xrL = 0, \quad M_2 - KM_1 - \tfrac{1}{2}xrM = 0, \quad N_2 - KN_1 - \tfrac{1}{2}xrN = 0.$$

145. An absolutely general solution of this system is not essential; but $L = 0$ will afford one of sufficient generality. I shall, however, for the present, defer the introduction of this condition ($L = 0$).

146. We see at once that

$$L_1 + xM_1 + \tfrac{1}{2}x^2N_1 = -(\dot{p} + x\dot{q} + \tfrac{1}{2}x^2\dot{r})\,L + (L + xM + \tfrac{1}{2}x^2N)';$$

but inasmuch as $y = x^2$ is a solution of the terordinal of Art. 90, and $\dot{y} = x^2$ of its correlate, therefore

$$p + xq + \tfrac{1}{2}x^2r = 0 = \dot{p} + x\dot{q} + \tfrac{1}{2}x^2\dot{r},$$

and, in virtue of the last relation,

$$L_1 + xM_1 + \tfrac{1}{2}x^2N_1 = (L + xM + \tfrac{1}{2}x^2N)'.$$

147. So

$$L_2 + xM_2 + \tfrac{1}{2}x^2N_2 = (L_1 + xM_1 + \tfrac{1}{2}x^2N_1)' = (L + xM + \tfrac{1}{2}x^2N)'';$$

148. In the simplified mixed integral of Art. 112 substitute for Θ and Θ' (see Art. 110) and for Y and Y' (see Art. 142). We get

$$-xL_1\dot{y}''y''+(\tfrac{1}{2}x^2rL+L_1)\,\dot{y}''y'-xM_1\dot{y}'y''-xrL\dot{y}''y-xN_1\dot{y}y''$$
$$+(\tfrac{1}{2}x^2rM+M_1)\dot{y}'y'-xrM\dot{y}'y+(\tfrac{1}{2}x^2rN+N_1)\,\dot{y}y'-xrN\dot{y}y=cxr.$$

149. In the last relation of Art. 136 replace $\dot{u}$ and u by $\dot{y}$ and y respectively; and for shortness represent $e^{-\int(\dot{p}+p)\,dx}$ by ϕ. The result will be identical with that of Art. 148 if

$$-\frac{L_1}{cr}=x^2\frac{\Theta_0}{\phi};\quad -\frac{L}{c}=2x\frac{\Theta_2}{\phi};\quad -\frac{N_1}{cr}=2x\frac{\Theta_1}{\phi};\quad -\frac{N}{c}=4\frac{\Theta_3}{\phi};$$

$$\tfrac{1}{2}x\,\frac{L}{c}+\frac{L_1}{cxr}=-x^2\frac{\Theta_2}{\phi}-x\frac{\Theta_0}{\phi};\quad -\frac{M_1}{cr}=-x^2\frac{\Theta_1}{\phi}-x\frac{\Theta_0}{\phi};$$

$$-\frac{M}{c}=-2x\frac{\Theta_3}{\phi}-2\frac{\Theta_2}{\phi};$$

$$\tfrac{1}{2}x\frac{M}{c}+\frac{M_1}{cxr}=x^2\frac{\Theta_3}{\phi}+x\frac{\Theta_1+\Theta_2}{\phi}+\frac{\Theta_0}{\phi}\quad \tfrac{1}{2}x\frac{N}{c}+\frac{N_1}{cxr}=-2x\frac{\Theta_3}{\phi}-2\frac{\Theta_1}{\phi}.$$

150. All the conditions are satisfied if

$$L=-2cxe^{\int(\dot{p}+p)\,dx}\,\Theta_2;\quad M=2ce^{\int(\dot{p}+p)\,dx}\,(x\Theta_3+\Theta_2);\quad N=-4ce^{\int(\dot{p}+p)\,dx}\,\Theta_3,$$

$$L_1=-cx^2re^{\int(\dot{p}+p)\,dx}\,\Theta_0;\; M_1=cxre^{\int(\dot{p}+p)\,dx}\,(x\Theta_1+\Theta_0);$$

$$N_1=-2cxre^{\int(\dot{p}+p)\,dx}\,\Theta_1.$$

I have often, perhaps unnecessarily, retained arbitrary constants which might in strictness have been suppressed or merged. But the retention is, I think, convenient for purposes of reference.

151. We now get

$$L+xM+\tfrac{1}{2}x^2N=0;\quad L_1+xM_1+\tfrac{1}{2}x^2N_1=0,$$

and (see Art. 147) consequently,

$$L_2+xM_2+\tfrac{1}{2}x^2N_2=0.$$

152. Eliminating L_2, L_1, and L, we get

$$M''+m_1M'+m_2M+n_1N'+n_2N=0,$$
$$N''+\nu_1N'+\nu_2N+\mu_1M'+\mu_2M=0;$$

and here it will be convenient to introduce a symbol k defined by

$$k = \dot{p} + K = \dot{p} + p + \frac{r'}{r} + \frac{1}{x},$$

K having the same signification as in Art. 143.

153. This being so, we have

$$m_1 = 2x\dot{q} - K, \qquad \mu_1 = 2x\dot{r},$$

$$m_2 = (x\dot{q})' - k\dot{q} + x(\dot{r} - \tfrac{1}{2}r), \qquad \mu_2 = (x\dot{r})' - 2x\dot{r},$$

$$n_1 = x^2\dot{q} + 2, \qquad \nu_1 = x^2\dot{r} - K,$$

$$n_2 = (\tfrac{1}{2}x^2\dot{q} + 1)' - k(\tfrac{1}{2}x^2\dot{q} + 1), \quad \nu_2 = (\tfrac{1}{2}x^2\dot{r})' - k(\tfrac{1}{2}x^2\dot{r}) + x(\dot{r} - \tfrac{1}{2}r);$$

wherein it may be noticed that we have

$$m_1 + \nu_1 = -2k, \qquad \nu_1 - \tfrac{1}{2}x\mu_1 = -K,$$

$$m_1 - \nu_1 = 2x\dot{q} - x^2\dot{r}, \qquad \nu_2 - \nu_1' - \frac{x}{2}(\mu_2 - \mu_1') = K' + \tfrac{1}{2}x(\dot{r} - r),$$

$$m_2 - \nu_2 = \tfrac{1}{2}(m_1 - \nu_1)' - \tfrac{1}{2}k(m_1 - \nu_1), \; \tfrac{1}{2}x(m_1 - \nu_1) - n_1 = -x^2\dot{r} - 2.$$

154. Multiply the first equation of Art. 152 into M, and the second into N, and subtract the last from the first result. Then $e^{-\int k\,dx}$ will be an integrating factor of the difference; and the integral may be written

$$MN' - NM' + \tfrac{1}{2}\mu_1 M^2 - \tfrac{1}{2}(m_1 - \nu_1)MN - \tfrac{1}{2}n_1 N^2 = e^{\int k\,dx}.$$

155. To verify this, substitute for N' and M' their values ($N' = N_1 + \dot{r}L$, $M' = M_1 + \dot{q}L - \dot{N}$) obtained from Art. 142. The integral becomes, in virtue of Art. 153,

$$MN_1 - NM_1 + (\dot{r}M - \dot{q}N)L + x\dot{r}M^2 - (x\dot{q} - \tfrac{1}{2}x^2\dot{r})MN - \tfrac{1}{2}x^2\dot{q}N^2$$

$$= MN_1 - NM_1 + (\dot{r}M - \dot{q}N)(L + xM + \tfrac{1}{2}x^2N) = MN_1 - NM_1 = e^{\int k\,dx},$$

as appears from Art. 151. But from the formulæ of Art. 150, we get

$$MN_1 - NM_1 = 4c^2xr(\Theta_0\Theta_3 - \Theta_1\Theta_2)\,e^{2\int(\dot{p}+p)\,dx} = 4c^2\dot{c}_1c_1xre^{\int(\dot{p}+p)\,dx}$$

$$= 4c^2\dot{c}_1c_1e^{\int k\,dx},$$

as follows from (B) of Art. 133, and from bringing xr under the exponential integral. Merging the arbitrary constant $4c^2\dot{c}_1c_1$, we get

for $MN_1 - NM_1$ the same value as before. This verifies not only the particular calculations but also the relations, assigned in Art 150, between the Θ_m's and L, M, and N.

156. In further verification, we have

$$L_1 + \dot{p}L - L' - M = ce^{\int(\dot{p}+p)\,dx}\left\{-x^2 r\Theta_0 - 2x\dot{p}\Theta_2 + 2\left[1 + x(\dot{p}+p)\right]\Theta_2 + 2x\Theta_2' - 2(x\Theta_3 + \Theta_2)\right\}$$
$$= ce^{\int(\dot{p}+p)\,dx}\left\{-x^2 r\Theta_0 + 2xp\Theta_2 - 2x\Theta_3 + 2x\Theta_2'\right\},$$

which vanishes, as it ought, in virtue of Art. 135. Another easy test is obtained from $N_1 = -\dot{r}L + N'$.

157. A strong verification is had by applying the foregoing formulæ to the particular Riccatian discussed in my prior paper, for which we have

$$L = x^4 - x^2;\ \ M = x - 2x^3;\ \ N = 2x^2;\ \ L_1 = x^5;$$
$$M_1 = -2x^4 - 2x^2;\ \ N_1 = 2x^3 + 2x.$$

These values, being substituted in each of the three equations

$$-pL_1 + L_1' + M_1 - KL_1 - \tfrac{1}{2}xrL = 0 = -\dot{q}L_1 + M_1' + N_1 - KM_1 - \tfrac{1}{2}xrM$$
$$= -\dot{r}L_1 + N_1' - pN - \tfrac{1}{2}xrN$$

(which are obtained by a combination of results given in Arts. 143—7, 151), satisfy all three of them.

158. I now introduce a symbol λ defined by

$$e^{\int K\,dx}\int e^{-\int K\,dx}\,dx = \lambda, \text{ or by } \lambda' = K\lambda + 1,$$

and which will therefore be in general a transcendent. And I give to the coefficients of the correlate the following values, viz.,

$$\dot{p} = K + \frac{2}{\lambda};\ \ \dot{q} = K' - \frac{1}{x}K - \frac{2}{x\lambda} - \tfrac{1}{2}xr;\ \ \dot{r} = r - \frac{2K'}{x};$$

values which satisfy the condition of Art. 146. This being done, I say that

$$L = 0,\ \ M = -\tfrac{1}{2}x\lambda,\ \ N = \lambda$$

will fulfil all the conditions.

159. Put then

$$Y = L\dot{y}'' + M\dot{y}' + N\dot{y} = \lambda(\dot{y} - \tfrac{1}{2}x\dot{y}') = -\tfrac{1}{2}\lambda\dot{\Theta};$$

it follows that

$$Y' = -\tfrac{1}{2}(K\lambda+1)\,\dot{\theta} - \tfrac{1}{2}\lambda\dot{\theta}',$$

$$Y'' = -\tfrac{1}{2}\left\{K'\lambda + K(K\lambda+1)\right\}\dot{\theta} - (K\lambda+1)\,\dot{\theta}' - \tfrac{1}{2}\lambda\dot{\theta}''.$$

160. Hence

$$\begin{aligned} Y'' - KY' &= -\tfrac{1}{2}K'\lambda\dot{\theta} - (\tfrac{1}{2}K\lambda+1)\,\dot{\theta}' - \tfrac{1}{2}\lambda\dot{\theta}'' \\ &= -\tfrac{1}{2}K'\lambda\dot{\theta} - (\tfrac{1}{2}K\lambda+1)\,\dot{\theta}' - \tfrac{1}{2}\lambda\,(\tfrac{1}{2}x\dot{r}\dot{\theta} - \dot{p}\dot{\theta}'), \quad \text{[Art. 11} \\ &= -\tfrac{1}{2}\lambda\left\{(K'+\tfrac{1}{2}x\dot{r})\,\dot{\theta} + (K-\dot{p})\,\dot{\theta}'\right\} - \dot{\theta}' \\ &= -\tfrac{1}{2}\lambda\left\{\tfrac{1}{2}xr\dot{\theta} - \frac{2}{\lambda}\,\dot{\theta}'\right\} - \dot{\theta}', \\ &= -\tfrac{1}{4}xr\lambda\dot{\theta} = \tfrac{1}{2}xrY; \end{aligned}$$

therefore $Y'' - KY' - \tfrac{1}{2}xrY = 0 = Y'' - \left(p + \dfrac{r'}{r} + \dfrac{1}{x}\right)Y' - \tfrac{1}{2}xrY,$

and the equation of Art. 111 is satisfied.

161. By way of example: in the biordinal of Art. 105 put $p =$ $m = 2$, $r = x^n$, and it becomes $z'' + \dfrac{2}{x}\,z' - \tfrac{1}{2}x^{n+1}z = 0$, which is tra formed into $\zeta'' - \tfrac{1}{2}x^{n+1}\zeta = 0$, by the substitution $xz = \zeta$. T terordinal (Art. 110) is

$$y''' + 0\,.\,y'' - \tfrac{1}{2}x^{n+1}y' + x^n y = 0,$$

the deformation is

$$Y''' - \frac{2n}{x}\,Y'' + \left\{\frac{n(n+1)}{x^2} - \tfrac{1}{2}x^{n+1}\right\}Y' - x^n Y = 0, \quad \text{[Art. 111.]}$$

and the correlate is

$$\dot{y}''' + \frac{1-n}{x}\,\dot{y}'' - \left(\frac{2}{x^2} + \tfrac{1}{2}x^{n+1}\right)\dot{y}' + \left(2\,\frac{n+1}{x^3} + x^n\right)\dot{y} = 0,$$

for $$K = \frac{1+n}{x}, \quad \lambda = -\frac{x}{n},$$

and, by Art. 158,

$$\dot{p} = \frac{1-n}{x}, \quad \dot{q} = -\frac{2}{x^2} - \tfrac{1}{2}x^{n+1}, \quad \dot{r} = x^n + 2\,\frac{n+1}{x^3}.$$

162. Hence $$Y = -\frac{x}{n}\left(\dot{y} - \frac{x}{2}\,\dot{y}'\right);$$

or, multiplying, as we may do, into $-2n$, and then replacing $-2nY$ by Y, we get
$$Y = x^2\dot{y}' - 2x\dot{y}, \quad Y' = x^2\dot{y}'' - 2\dot{y},$$
and
$$Y'' = x^2\dot{y}''' + 2x\dot{y}'' - 2\dot{y}' = (n+1)\,x\dot{y}'' + \tfrac{1}{2}x^{n+2}\dot{y}' - \left(x^{n+2} + 2\,\frac{n+1}{x}\right)\dot{y}$$
$$= \frac{n+1}{x}\,(x^2\dot{y}'' - 2\dot{y}) + \tfrac{1}{2}x^{n+1}\,(x^2\dot{y}' - 2x\dot{y}) = \frac{n+1}{x}\,Y' + \tfrac{1}{2}x^{n+1}\,Y.$$

163. And this is right; for if, as in Art. 111, we integrate the deformation and suppose the arbitrary constant to vanish, we are led to
$$Y'' - \frac{n+1}{x}\,Y' - \tfrac{1}{2}x^{n+1}\,Y = 0,$$
and so to a verification of preceding results. In dealing with this example no transcendent has been introduced.

Thursday, April 12*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. A. R. Johnson, M.A., Fellow of St. John's College, Cambridge, was elected a Member.

The following communications were made:—

Continuation of Former Paper on Simplicissima: W. J. C. Sharp, M.A.

Synthetical Solutions in the Conduction of Heat: E. W. Hobson, M.A.

Continuation of paper on Symmetric Functions: R. Lachlan, M.A.

On a Law of Attraction which might include both Gravitation and Cohesion: G. S. Carr, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLIII., No. 263.

"Educational Times," for April.

"Proceedings of the Manchester Literary and Philosophical Society," Vols. XXV., 1885—86, and XXVI., 1886—87.

"Memoirs of the Manchester Literary and Philosophical Society," Third Series, Tenth Volume.

"Annals of Mathematics," Vol. III., No. 6 (University of Virginia); Dec., 1887.

"Bulletin des Sciences Mathématiques," Tome XII.; March and April, 1888.

"Bulletin de la Société Mathématique de France," Tome XVI., No. 1.

"Journal de l'École Polytechnique," 57e cahier; Paris, 1887.

"Annales de l'École Polytechnique de Delft," Tome III., 4me Livraison; 1888.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 3; Leipzig, 1888.

"Jahrbuch über die Fortschritte der Mathematik," Band XVII., Heft 2, Jahrgang 1885; Berlin, 1888.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Math.-Phys. Classe, 1887, I., II.; Leipzig, 1888.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 8; März, 1888.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 10—13, Nov. 20—Dic. 18, 1887.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 52—54; Febb. 29—Marzo 31, 1888.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XL.—LIV., Oct. 20—Dec. 22, 1887, with Title, Index, &c.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo I., No. 8; México, 1888.

"Calendar, for the year 1887—88, of the Imperial University of Japan," 8vo; Tokio, 1888.

"Sur la détermination d'une Courbe algébrique par des Points donnés," par H. G. Zeuthen. (Excerpt from "Mathematische Annalen," Bd. XXXI.)

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Synthetical Solutions in the Conduction of Heat.

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[*Read April* 12*th*, 1888.]

The object of the present communication is to give the solutions expressed as definite integrals of certain problems in the variable motion of heat in two and three dimensions, in which the boundaries of the conducting body are straight edges or planes. The solutions are obtained by a method which has been applied by Sir W. Thomson,* to some cases of conduction; this method consists in superimposing

* See "Collected Works," Vol. II.

the temperatures due to suitable distributions of instantaneous or continuous sources. The effect of an initial distribution of heat in such a body is obtained by regarding such initial distribution as consisting of an infinite number of instantaneous point sources, the heat from which is left to diffuse throughout the body. If the straight boundaries of the given body are maintained at zero temperature, such a bounding condition must be taken account of by replacing the given body by one infinite in all directions, the initial distribution throughout the part of the second body which is beyond the boundaries of the first body being such that the conditions of zero temperature for all time over the given boundaries will be of itself fulfilled. I have shown that in order to solve the problem of conduction in cases in which the boundaries of the conductor are maintained not at zero temperature, but at any given temperature varying with the time and with the position of the point on the boundaries, distributions of doublet sources may be distributed over the boundaries in such a manner as to fulfil the given boundary condition; such a doublet consists of a source and a sink of equal magnitude indefinitely close to one another, the line joining them being perpendicular to the boundary, the length of this line being indefinitely small, but the product of its length into the magnitude of the source or sink being finite.

When the expressions for the temperature due to such distribution of doublets is added to that obtained on the supposition that under the given initial condition the temperatures of the boundaries are maintained at zero temperature, the resulting sum represents the solution of the problem of conduction through the given body when the initial temperature throughout is given, and also the given arbitrary boundary temperatures are maintained.

I have further considered the case in which radiation takes place across the boundary into a medium of arbitrarily given temperature which may vary not only with the time but also from point to point; it is shown that in certain cases the solution of this problem may be deduced from the preceding one, and that the effect of the radiation may be represented by means of infinite rows of sources, and of doublets distributed in an infinite conducting solid in the part of that solid beyond the given boundaries.

The comparison of the solutions obtained by this synthetical method, with those obtained by the use of Fourier's series and integrals, is interesting from an analytical point of view, but as the mode of reduction of the forms of solution obtained by the one method to those obtained by the other, is usually sufficiently obvious, I have not entered into those details.

1. Writing the equation of conduction in two dimensions in the form

$$\frac{dv}{dt} = \kappa\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right),$$

where v denotes the temperature at time t at a point (x, y), and κ denotes the conductivity divided by the product of the density and the specific heat, the temperature at time t due to a single instantaneous source at a time $t = 0$ at the point $(x'y')$ is

$$\frac{Q}{4\pi\kappa t}\, e^{-[(x-x')^2+(y-y')^2]/4\kappa t},$$

Q denoting the amount of heat generated. If such a source be at the point $(x', \frac{1}{2}\delta y')$, and a sink of the same strength at the point $(x', -\frac{1}{2}\delta y')$, the temperature at a time t afterwards is

$$\frac{Q}{4\pi\kappa t}\left\{e^{-[(x-x')^2+(y-\frac{1}{2}\delta y')^2]/4\kappa t} - e^{-[(x-x')^2+(y+\frac{1}{2}\delta y')^2]/4\kappa t}\right\}$$

$$= Q\,.\,\delta y\,\frac{y}{8\pi\kappa^2 t^2}\, e^{-[(x-x')^2+y^2]/4\kappa t},$$

when δy is very small; if, as δy is indefinitely diminished, $Q.\delta y$ remains finite and equal to Q_1, we obtain the expression

$$Q_1\,\frac{y}{8\pi\kappa^2 t^2}\, e^{-[(x-x')^2+y^2]/4\kappa t}$$

for the temperature due to an instantaneous doublet of strength Q_1, at time $t=0$, at the point $(x', 0)$, the axis of the doublet being parallel to the axis of y. The temperature due to a continuous doublet may be deduced from this expression: suppose $q.d\lambda$ the heat generated by the source in time $d\lambda$, then, supposing q independent of λ, we obtain the following expression for the temperature due to a continuous doublet of constant strength—

$$\frac{q}{8\pi\kappa}\int_0^t \frac{y}{(t-\lambda)^2}\, e^{-[(x-x')^2+y^2]/[4\kappa(t-\lambda)]}\, d\lambda,$$

which is found on performing the integration to be

$$\frac{q}{2\pi\kappa}\,\frac{y}{(x-x')^2+y^2}\, e^{-[(x-x')^2+y^2]/4\kappa t}.$$

Let
$$\frac{q}{2\kappa} = f(x')\, dx',$$

then this expression becomes

$$\frac{y}{(x-x')^2+y^2}\,\frac{dx'}{\pi}\, f(x')\, e^{-[(x-x')^2+y^2]/4\kappa t}.$$

If AA' be the element dx', and P be the point (xy), this expression is equal to

$$\angle APA' \frac{f(x')}{\pi} e^{-[(x-x')^2+y^2]/4\kappa t}.$$

We see that this expression vanishes for all points on the axis of x, except when P is in AA', and it is then equal to $\pm f(x')$ according to the side of the axis of x from which the point approaches the limiting value; the expression therefore represents the temperature at any point in the part of the plane of xy on the side of the axis of x, for which y is positive, when the initial temperature is zero, and the temperature over the boundary maintained at zero temperature, except for the element AA', which is maintained at temperature $f(x')$. It appears, then, that this boundary condition may be represented by a doublet of strength $2\kappa \cdot f(x')\,dx'$, at the point $(x', 0)$. I am not aware that the solution

$$v = \frac{y}{x^2+y^2} e^{-(x^2+y^2)/4\kappa t}$$

of the equation of conduction has been noticed before; it can, of course, be verified directly by differentiation.

The temperature due to a doublet of variable strength is

$$\int_0^t \frac{\chi(\lambda)}{8\pi\kappa^2} \frac{y}{(t-\lambda)^2} e^{-[(x-x')^2+y^2]/[4\kappa(t-\lambda)]} d\lambda,$$

or

$$\frac{\chi(0)}{2\pi\kappa} \frac{y}{(x-x')^2+y^2} e^{-[(x-x')^2+y^2]/4\kappa t}$$
$$+ \frac{1}{2\pi\kappa} \int_0^t \frac{\chi'(\lambda)\,y}{(x-x')^2+y^2} e^{-[(x-x')^2+y^2]/[4\kappa(t-\lambda)]} d\lambda,$$

where $\chi(\lambda)$ denotes the strength of the doublet at time λ. This expression vanishes as before when $y=0$, unless $x=x'$, and it is then equal to

$$\frac{1}{2\kappa}\chi(t);$$

thus a temperature $F(x', t)$ maintained over the element dx' can be represented by a variable doublet at the point $(x', 0)$ of strength

$$2\kappa F(x', t).$$

2. By proper distribution of the doublets described above, the solutions adapted to the case of portions of a plane bounded by straight edges may be written down.

(a) If the part of the plane for which y is positive be initially at zero temperature, and the boundary $y=0$ be maintained at temperatures given by

$$v=f(x'),$$

the solution is obtained by placing doublets of proper strength at every point of the axis of x, and integrating along that axis; the solution is

$$v=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-x')^2+y^2}\,e^{-[(x-x')^2+y^2]/4\kappa t}f(x')\,dx'.$$

This is zero initially and is equal to $f(x')$ when $y=0$. If the given temperatures be functions of the time, so that $F(x',t)$ is the temperature at $(x',0)$, the solution is

$$v=\frac{1}{\pi}\int_{-\infty}^{\infty}F(x',0)\,\frac{y}{(x-x')^2+y^2}\,e^{-[(x-x')^2+y^2]/4\kappa t}\,dx'$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^t\frac{y}{(x-x')^2+y^2}\,e^{-[(x-x')^2+y^2]/4\kappa t}\,\frac{dF}{d\lambda}\,dx'\,d\lambda.$$

If the temperature over the surface be initially

$$v=\dagger(x,y),$$

we must add the expression

$$\frac{1}{4\pi\kappa t}\int_{-\infty}^{\infty}\int_0^{\infty}\left\{e^{-[(x-x')^2+(y-y')^2]/4\kappa t}-e^{-[(x+x')^2+(y+y')^2]/4\kappa t}\right\}\dagger(x',y')\,dx'dy',$$

which is zero when $y=0$ and equals $\dagger(x,y)$ when $t=0$, to one of the above solutions.

(b) For the space bounded by the positive parts of the axes of x and y, the solution of the corresponding problem is obtained by means of the images of the doublets and sources; for example, a doublet at the point $(x,0)$ will have a doublet image of opposite sign at the point $(-x,0)$. The expression for the temperature at any point in the portion of the plane, under the conditions $v=f_1(x)$ when $y=0$, $v=f_2(y)$ when $x=0$, and $v=\dagger(x,y)$ when $t=0$, is thus seen to be

$$v=\frac{1}{\pi}\int_0^{\infty}\left\{\frac{y}{(x-x')^2+y^2}\,e^{-[(x-x')^2+y^2]/4\kappa t}\right.$$
$$\left.-\frac{y}{(x+x')^2+y^2}\,e^{-[(x+x')^2+y^2]/4\kappa t}\right\}f_1(x')\,dx'$$
$$+\frac{1}{\pi}\int_0^{\infty}\left\{\frac{x}{x^2+(y-y')^2}\,e^{-[x^2+(y-y')^2]/4\kappa t}-\frac{x}{x^2+(y+y')^2}\,e^{-[x^2+(y+y')^2]/4\kappa t}\right\}f_2(y')\,dy'$$
$$+\frac{1}{4\pi\kappa t}\int_0^{\infty}\int_0^{\infty}\left\{e^{-[(x-x')^2+(y-y')^2]/4\kappa t}+e^{-[(x+x')^2+(y+y')^2]/4\kappa t}-e^{-[(x+x')^2+(y-y')^2]/4\kappa t}\right.$$
$$\left.-e^{-[(x-x')^2+(y+y')^2]/4\kappa t}\right\}\dagger(x',y')\,dx'dy'.$$

The solution may be written down in a similar manner in the more general case in which the functions f_1, f_2 involve t.

By the method exemplified in (*a*) and (*b*), the corresponding problems may be solved for the space bounded by lines inclined at an angle $\frac{\pi}{n}$, and for an isosceles triangle of which $\frac{\pi}{n}$ is the vertical angle.

(*c*) Consider next the infinite rectangle bounded by $x=a$, $x=-a$, $y=0$. We shall first find the temperature when the boundaries are maintained at zero temperature, and the initial temperature over the plane is given $v = \dagger(x, y)$.

The images of a point $(x'y')$ are as follows :—positive images at the points $(x' \pm 4na,\ y')$, $(-x' \pm 4na - 2a,\ -y')$, and negative images at the points $(-x' \pm 4na - 2a,\ y')$, $(x' \pm 4na,\ -y')$, where n has all integral values; hence the value of v is

$$\frac{1}{4\pi\kappa t}\int_{-a}^{a}\int_{0}^{\infty}\left[e^{-(y-y')^2/4\kappa t} - e^{-(y+y')^2/4\kappa t}\right]$$

$$\left\{\sum_{0}^{\infty} e^{-(x-x'\pm 4na)^2/4\kappa t} - \sum_{0}^{\infty} e^{-(x+x'\pm 4na+2a)^2/4\kappa t}\right\}\dagger(x', y')\,dx'dy',$$

which may be written

$$\frac{1}{4\pi\kappa t}\int_{-a}^{a}\int_{0}^{\infty}\left\{e^{-(y-y')^2/4\kappa t} - e^{-(y+y')^2/4\kappa t}\right\}\left\{e^{-(x-x')^2/4\kappa t}\,\theta_3\left(\frac{\iota a(x-x')}{\kappa t},\ e^{-4a^2/\kappa t}\right)\right.$$

$$\left. - e^{-(x+x')^2/4\kappa t}\,\theta_3\left(\frac{\iota a(x+x'+2a)}{\kappa t},\ e^{-4a^2/\kappa t}\right)\right\}\dagger(x', y')\,dx'dy.$$

Now

$$\theta_3\left(\frac{\iota a(x-x')}{\kappa t},\ e^{-4a^2/\kappa t}\right) = \left(\frac{\pi\kappa t}{4a^2}\right)^{\frac{1}{2}} e^{(x-x')^2/4\kappa t}\,\theta_3\left(\frac{\pi(x-a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right),$$

and, using the corresponding transformation for the other θ-function, we obtain for the temperature the expression

$$\frac{1}{8a\sqrt{\pi\kappa t}}\int_{-a}^{a}\int_{0}^{\infty}\left[e^{-(y-y')^2/4\kappa t} - e^{-(y+y')^2/4\kappa t}\right]\left[\theta_3\left(\frac{\pi(x-a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)\right.$$

$$\left. - \theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)\right]\dagger(x', y')\,dx'dy'.$$

It order to obtain the expression which must be added in case the boundary $y=0$ is maintained at a temperature given by $F(x, \lambda)$, we must write $t-\lambda$ for t in the expression just obtained, suppress the integration with respect to y', and also let y' become indefinitely small,

and $\dagger(x', y)$ be such that in the limit $y'.\,\dagger(x', y') = \kappa F(x', \lambda)$, and integrate the result with respect to λ between the limits t and 0; we thus obtain the expression

$$\frac{1}{16a\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}}\int_0^t\int_{-a}^{a}\frac{y}{(t-\lambda)^{\frac{3}{2}}}e^{-y^2/4\kappa(t-\lambda)}\left[\theta_3\left(\pi\frac{x-a}{4a},\ e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right)\right.$$

$$\left.-\theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right)\right]F(x', t)\,d\lambda\,dx'.$$

This represents the effect of a row of doublets of strength $2\kappa^3.\,F(x', t)$ per unit of length of the axis of x, with their images.

(*d*) For the finite rectangle bounded by $x = \pm a$ and $y = \pm b$, the solution which is zero over the boundaries and is initially equal to $\dagger(x, y)$ is found, as in the last case, to be

$$\frac{1}{16ab}\int_{-a}^{a}\int_{-b}^{b}\left[\theta_3\left(\frac{\pi(x-x')}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)-\theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)\right]$$

$$\left[\theta_3\left(\frac{\pi(y-y')}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)-\theta_3\left(\frac{\pi(y+y'+2b)}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)\right]\dagger(x', y')\,dx'\,dy',$$

and the terms to be added when the temperature is given over the sides may be found by the same process as before.

3. I shall next proceed to consider the case of radiation from an infinite plane bounded by the axis of x across that axis. First, suppose that the initial temperature of the plane is zero, and let $f(x, t) = U$ denote the temperature of the external medium at any point of the axis of x; then, h denoting the external conductivity, the boundary equation is $\dfrac{dv}{dy} = h(v-U)$, when $y = 0$.

Let
$$u = \left(1-\frac{1}{h}\frac{d}{dy}\right)v,$$

then u satisfies the differential equation

$$\frac{du}{dt} = \kappa\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right),$$

and is to be equal to U when $y = 0$; hence u can be determined by the method in Art. 1. We have, then,

$$\frac{dv}{dy}-hv = -hu,\quad ve^{-hy} = -h\int_{\infty}^{y}u'.\,e^{-hy'}\,dy',$$

the lower limit being taken infinite, in order that v may vanish when y is infinite; putting $y' = y+z$, we have

$$v = h\int_0^\infty u''.e^{-hz}\,dz,$$

where u'' denotes the value of u when $y+z$ is written for y.

Now, $$u = \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty}\int_0^t f(x',\lambda)\,\frac{y}{(t-\lambda)^2}\,e^{-[(x-x')^2+y^2]/4\kappa(t-\lambda)}\,dx'\,d\lambda,$$
hence
$$v = \frac{h}{4\pi\kappa}\int_{-\infty}^{\infty}\int_0^t\int_0^\infty e^{-hz}.f(x',\lambda)\,\frac{y+z}{(t-\lambda)^2}e^{-[(x-x')^2+(y+z)^2]/4\kappa(t-\lambda)}\,dx'\,d\lambda\,dz;$$

in the case in which $f(x,t)$ does not involve t, this becomes

$$v = \frac{h}{\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{-hz}f(x')\,\frac{y+z}{(x-x')^2+(y+z)^2}\,e^{-[(x-x')^2+(y+z)^2]/4\kappa t}\,dx'\,dz.$$

These expressions show that the radiation over an element dx' of the boundary may be represented by means of an infinite row of doublets along an infinite line from the point $x=x'$, $y=0$ drawn parallel to $y=0$ in the negative direction, the strength of the doublet at the point $x=x'$, $y=-z$ being $he^{-hz}.2\pi\kappa$; the temperature at any point on the positive side of the axis of x is that due to this infinite distribution of doublets.

When $f(x,t)$ does not involve t, and the motion has become steady, the expression for v becomes

$$v = \frac{h}{\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{-hz}.f(x')\,\frac{y+z}{(x-x')^2+(y+z)^2}\,dx'\,dz.$$

Integrating by parts with respect to z, this becomes

$$v = \frac{h^2}{2}\int_{-\infty}^{\infty}\int_0^\infty f(x').e^{-hz}\log_e\frac{(x-x')^2+(y+z)^2}{(x-x')^2+y^2}\,dx'\,dz.$$

Next, let us suppose the initial temperature to be given, say $F(x,y)$; then we determine u under the conditions $u=0$, when $y=0$,

and $$u = F(x,y) - \frac{1}{h}\frac{d}{dy}F(x,y);$$

when $t=0$, the value of u is then

$$u = \frac{1}{4\pi\kappa t}\int_0^\infty\int_{-\infty}^{\infty}\left\{e^{-[(x-x')^2+(y-y')^2]/4\kappa t} - e^{-[(x-x')^2+(y+y')^2]/4\kappa t}\right\}$$
$$\times\left\{F(x',y') - \frac{1}{h}\frac{d}{dy'}F(x',y')\right\}dx'\,dy';$$

v is determined from u in the same way as before, and is equal to

$$\frac{h}{4\pi\kappa t}\int_0^\infty\int_{-\infty}^\infty\int_0^\infty e^{-hz}\left\{e^{-[(x-x')^2+(y-y'+z)^2]/4\kappa t}-e^{[(x-x')^2+(y+y'+z)^2]/4\kappa t}\right\}$$

$$\times\left\{\left(1-\frac{1}{h}\,\frac{d}{dy'}\right)\ F(x',\,y')\right\}dx'\,dy'\,dz.$$

This value of v satisfies the initial condition $v=F(x,y)$, and also $\frac{dv}{dy}-hv=0$, when $y=0$; hence, by adding this expression to the one obtained above, we have the complete solution of the problem of radiation under given initial conditions.

This expression shows that the effect of the initial distribution may be represented as follows—Corresponding to the initial heating $f(x',y')$, $dx'\,dy'$ of an element $dx'\,dy'$, we must suppose a series of sources along the infinite line commencing at (x',y') parallel to the axis of y in the negative direction, and such that the magnitude of the source at $(x',\,y'-z)$ is

$$he^{-hz}\left(1-\frac{1}{h}\,\frac{d}{dy}\right),\ F(x',y'),$$

and also a series of sinks commencing at the point $(x',\,-y)$, and of which the magnitude at the point $(x',\,-y'-z')$ is

$$he^{-hz_1}\left(1-\frac{1}{h}\,\frac{d}{dy'}\right),\ F(x',\,y').$$

This system of sources and sinks is equivalent to a line of sources from $(x',\,y')$, to $(x',\,-y')$, and a line of sinks from the latter point to infinity.

It does not appear to be possible to apply this method to problems in which radiation takes place over more than one boundary, but it may be applied to cases when there is radiation over one boundary, and the other boundaries are subject to temperature conditions. I shall apply this to one case.

Suppose radiation to take place over that point of the x axis which is bounded by the points $(0,\,\pm a)$, and suppose that the temperature over the lines $y=\pm a$ is maintained at zero, we can find the temperature at any point of the infinite rectangle. Using the expressions obtained in case (c) of Art. 2, we have for the part of the temperature

due to the initial distribution $\dagger(x, y)$,

$$v = \frac{h}{8a\sqrt{\pi\kappa t}} \int_{-a}^{a}\int_{0}^{\infty}\int_{0}^{\infty} \left[\theta_3\left(\frac{\pi(x-x')}{4a}, e^{-(\pi^2\kappa t)/4a^2}\right) - \theta_3\left(\frac{\pi(x+x'+2a)}{4a}, e^{-(\pi^2\kappa t)/4a^2}\right)\right]$$

$$\left[e^{-[(y+z-y')^2]/4\kappa t} - e^{-[(y+z+y')^2]/4\kappa t}\right] e^{-hz}\left\{1-\frac{1}{h}\frac{d}{dy'}\right\}\dagger(x', y')\, dx'dy'dz.$$

This expression is initially equal to $\dagger(x, y)$, vanishes when $y = \pm a$, and satisfies the condition of radiation into a medium at zero when $y = 0$.

The part of the temperature due to the radiation into a medium at temperature $F(x, t)$ is

$$v = \frac{h}{16a\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}} \int_0^t \int_{-a}^{a} \int_0^{\infty} \frac{y+z}{(t-\lambda)^{\frac{3}{2}}} e^{-(y^2)/[4\kappa(t-\lambda)]}$$

$$\times\left[\theta_3\left(\frac{\pi(x-a)}{4a}, e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right) - \theta_3\left(\frac{\pi(x+x'+2a)}{4a}, e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right)\right]$$

$$\times e^{-hz} F(x', t)\, d\lambda\, dx'\, dz.$$

4. All the preceding methods are applicable to problems in three dimensions. In this case the temperature due to an instantaneous source of strength Q at the point $(x'y'z')$ is

$$\frac{Q}{(2\sqrt{\pi\kappa t})^3} e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t},$$

and the temperature due to an instantaneous doublet of strength Q, at the point $(0, y', z')$ is

$$\frac{Qx}{16\pi^{\frac{3}{2}}(\kappa t)^{\frac{5}{2}}} e^{-[x^2+(y-y')^2+(z-z')^2]/4\kappa t}.$$

The temperature due to a continuous doublet of strength q is

$$\frac{qx}{16\pi^{\frac{3}{2}}\kappa^{\frac{5}{2}}} \int_0^t \frac{1}{(t-\lambda)^{\frac{5}{2}}} e^{-[x^2+(y-y')^2+(z-z')^2]/[4\kappa(t-\lambda)]}\, d\lambda.$$

Put
$$a = \frac{r}{2\sqrt{\kappa(t-\lambda)}},$$

and change to a as the independent variable; the temperature due to the doublet is then

$$\frac{qx}{\pi^{\frac{3}{2}}\kappa r^3} \int_{r/\sqrt{4\kappa t}}^{\infty} a^2 e^{-a^2}\, da,$$

where $$r^2 = x^2+(y-y')^2+(z-z')^2.$$

When r becomes zero, we have

$$\int_0^\infty a^2 e^{-a^2}\,da = \tfrac{1}{4}\sqrt{\pi},$$

thus the temperature becomes

$$\frac{q}{4\pi\kappa}\,L\,\frac{x}{r^3};$$

and since in the limit $$\frac{x}{r^3} = 2\pi,$$

the expression becomes equal to $\dfrac{q}{2\kappa}$, if then $q = 2\kappa V$, we see that the expression

$$\frac{2V}{\pi^{\frac{1}{2}}}\,\frac{x}{r^3}\int_{r/\sqrt{4\kappa t}}^\infty a^2 e^{-a^2}\,da$$

is equal to V when the point (x, y, z) approaches indefinitely near to the point $(0, y', z')$, and is zero when $x = 0$ for all other values of y and z.

Since $$\int a^2 e^{-a^2}\,da = -\tfrac{1}{2}a e^{-a^2}+\tfrac{1}{2}\int e^{-a^2}\,da,$$

the above solution becomes

$$\frac{V}{\pi^{\frac{1}{2}}}\,\frac{x}{r^3}\left\{\frac{r}{\sqrt{4\kappa t}}e^{-r^2/4\kappa t}+\int_{r/\sqrt{4\kappa t}}^\infty e^{-a^2}\,da\right\}.$$

Using the notation $$\int_c^\infty e^{-a^2}\,da = Erf\,c,\text{*}$$

we obtain the expression

$$\frac{V}{\pi^{\frac{1}{2}}}\,\frac{x}{r^3}\left\{Erf\left(\frac{r}{\sqrt{4\kappa t}}\right)+\frac{r}{\sqrt{4\kappa t}}e^{-r^2/4\kappa t}\right\},$$

where $$r^2 = x^2+(y-y')^2+(z-z')^2,$$

for the temperature due to a continuous doublet of strength $2\kappa V$ at the point $(0, y', z')$; and this expression equals V for $x = 0$, $y = y'$, $z = z'$, and equals zero for $x = 0$ when y and z have any other values. The quantity $\dfrac{x}{r^3}$ is proportional to the solid angle subtended by a

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* See Articles by Glaisher in the *Phil. Trans.* for 1871.

small area about the point $(0, y', z')$ in the plane $x = 0$ at any point of the solid. It may, of course, be verified by differentiation that the expression just given satisfies the differential equation

$$\frac{dv}{dt} = \kappa\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right).$$

5. We can now apply the expressions in Art. 4 to obtain solutions of problems in three dimensions corresponding to those in Art. 3.

(*a*) Consider the solid bounded by the plane $x = 0$, so that conduction takes place on the positive side of this plane; if the temperature throughout the solid is initially zero, and the plane $x = 0$ is maintained at temperature $f(y, z)$, the temperature at any time t in the interior of the solid is

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{r^3}f(y', z')\left\{Erf\left(\frac{r}{2\sqrt{\kappa t}}\right) + \frac{r}{2\sqrt{\kappa t}}\,e^{-r^2/4\kappa t}\right\}dy'dz',$$

where $$r^2 = x^2 + (y-y')^2 + (z-z')^2,$$

for this is zero initially, and is equal to $f(y', z')$ when $x = 0$, and satisfies the equation of conduction.

If the given temperature over the plane $x = 0$ is $f(y, z, t)$, the temperature at any time is

$$\frac{1}{8\kappa\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}}\int_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{(t-\lambda)^{\frac{5}{2}}}f(y', z', \lambda)\,e^{-[x^2+(y-y')^2+(z-z')^2]/[4\kappa(t-\lambda)]}\,d\lambda\,dy'\,dz'.$$

If the initial temperature throughout the solid is $\dagger(x, y, z)$ we must, in order to obtain the complete solution, add to one of the above expressions, the expression

$$\frac{1}{8(\pi\kappa t)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dagger(x', y', z')\left\{e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t}\right.$$
$$\left.-e^{-[(x+x')^2+(y-y')^2+(z-z')^2]/4\kappa t}\right\}dx'\,dy'\,dz',$$

which vanishes when $x = 0$, and is equal to $\dagger(x, y, z)$ when $t = 0$; we thus have the complete solution for the case when the initial temperature throughout the solid and the temperature over the boundary $x = 0$ are arbitrarily given.

(*b*) Next, consider the solid bounded by the three coordinate planes, the conduction taking place through the space for which all the coordinates are positive, and suppose the temperatures over the

boundaries to be given as follows—

$$v = f_1(y, z) \text{ when } x = 0 \text{ for all positive values of } y \text{ and } z,$$
$$v = f_2(z, x) \text{ when } y = 0 \quad ,, \quad ,, \quad z \text{ and } x,$$
$$v = f_3(x, y) \text{ when } z = 0 \quad ,, \quad ,, \quad x \text{ and } y.$$

Each doublet will now have three images; for instance, the doublet at the point $(0, y', z')$ will have negative images at $(0, -y', z')$ and $(0, y', -z')$ and a positive image at $(0, -y', -z')$; the effect of these doublets will be to produce zero temperature over the planes of (zx) and (xy). The temperature at any point of the solid, supposing first the initial temperature throughout the interior to be zero, will be

$$\frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{r_1^3}f_1(y', z')\left\{Erf\left(\frac{r_1}{2\sqrt{\kappa t}}\right)+\frac{r_1}{2\sqrt{\kappa t}}e^{-r_1^2/4\kappa t}\right\}dy'\,dz'$$
$$+\frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y}{r_2^3}f_2(z', x')\left\{Erf\left(\frac{r_2}{2\sqrt{\kappa t}}\right)+\frac{r_2}{2\sqrt{\kappa t}}e^{-r_2^2/4\kappa t}\right\}dz'\,dx$$
$$+\frac{1}{\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{r_3^3}f_3(x', y')\left\{Erf\left(\frac{r_3}{2\sqrt{\kappa t}}\right)+\frac{r_3}{2\sqrt{\kappa t}}e^{-r_3^2/4\kappa t}\right\}dx'\,dy',$$

where $r_1^2 = x^2+(y-y')^2+(z-z')^2$, $r_2^2 = (x-x')^2+y^2+(z-z')^2$,

$r_3^2 = (x-x')^2+(y-y')^2+z^2$, and also $f_1(-y', z') = -f_1(y', z')$, &c.

If $\dagger(x, y, z)$ is the initial temperature, the expression to be added to the above expression in order to obtain the complete temperature is

$$\frac{1}{8(\pi\kappa t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dagger(x', y', z')\Big[e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t}$$
$$-e^{-[(x-x')^2+(y+y')^2+(z+z')^2]/4\kappa t}-e^{-[(x+x')^2+(y-y')^2+(z-z')^2]/4\kappa t}$$
$$+e^{-[(x+x')^2+(y+y')^2+(z-z')^2]/4\kappa t}+\ldots\ldots\ldots\Big]\,dx'\,dy'\,dz',$$

where there are eight exponentials within the square bracket. This expression vanishes over each coordinate plane, and is equal to $\dagger(x, y, z)$ when $t = 0$.

(*c*) Consider the infinite prism on the positive side of the plane $z = 0$, bounded by the planes $x = \pm a$, $y = \pm b$. If $\dagger(x, y, z)$ is the initial temperature, and the bounding planes are maintained at zero temperature, the temperature at any point at time t is, as in the case (*c*) in Art. 3,

$$\frac{1}{(2\sqrt{\pi\kappa t})^{\frac{3}{2}}}\int_{-a}^{a}\int_{-b}^{b}\left\{e^{-(z-z')^2/4\kappa t}-e^{-(z+z')^2/4\kappa t}\right\}$$
$$\left\{\Sigma\, e^{-(x-x'\pm 4na)^2/4\kappa t}-\Sigma\, e^{-(x+x'\pm 4na+2a)^2/4\kappa t}\right\}$$
$$\left\{\Sigma\, e^{-(y-y'\pm 4nb)^2/4\kappa t}-\Sigma\, e^{-(y+y'\pm 4nb+2b)^2/4\kappa t}\right\}\dagger(x', y', z')\,dx'\,dy'\,dz'.$$

By the same transformation as before, this becomes

$$\frac{1}{8\sqrt{2\pi\kappa t}\,ab}\int_{-a}^{a}\int_{-b}^{b}\left\{e^{-(z-z')^2/4\kappa t}-e^{-(z+z')^2/4\kappa t}\right\}$$

$$\left[\theta_3\left(\frac{\pi(x-a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)-\theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)\right]$$

$$\left[\theta_3\left(\frac{\pi(y-b)}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)-\theta_3\left(\frac{\pi(y+y'+2b)}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)\right]$$

$$f(x',y',z')\,dx'\,dy'\,dz'.$$

In order to find the expression to be added to this, if the plane $z=0$ is maintained at temperature $F_1(x, y, t)$, by the same process as in Art. 3 (c), we get

$$\frac{1}{16\sqrt{2\pi}\,ab}\int_0^t\int_{-a}^{a}\int_{-b}^{b}\frac{z}{(t-\lambda)^{\frac{3}{2}}}e^{-z^2/4\kappa t}$$

$$\left[\theta_3\left(\frac{\pi(x-a)}{4a},\ e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right)-\theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-[\pi^2\kappa(t-\lambda)]/4a^2}\right)\right]$$

$$\left[\theta_3\left(\frac{\pi(y-b)}{4b},\ e^{-[\pi^2\kappa(t-\lambda)]/4b^2}\right)-\theta_3\left(\frac{\pi(y+y'+2b)}{4b},\ e^{-[\pi^2\kappa(t-\lambda)]/4b^2}\right)\right]$$

$$F_1(x',y',\lambda)\,dt\,dx'\,dy';$$

this vanishes initially throughout the solid, is zero over the planes (zx), (zy), and has the given value over the plane (xy).

(d) For the rectangular parallelepiped bounded by $x=\pm a$, $y=\pm b$, $z=\pm c$, and of which the initial temperature is $f(x, y, z)$, the temperature when the boundaries are maintained at zero temperature is

$$\frac{1}{16\sqrt{2}\,abc}\int_{-}^{a}\int_{-b}^{b}\int_{-c}^{c}$$

$$\left\{\theta_3\left(\frac{\pi(x-a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)-\theta_3\left(\frac{\pi(x+x'+2a)}{4a},\ e^{-(\pi^2\kappa t)/4a^2}\right)\right\}$$

$$\left\{\theta_3\left(\frac{\pi(y-b)}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)-\theta_3\left(\frac{\pi(y+y'+2b)}{4b},\ e^{-(\pi^2\kappa t)/4b^2}\right)\right\}$$

$$\left\{\theta_3\left(\frac{\pi(z-c)}{4c},\ e^{-(\pi^2\kappa t)/4c^2}\right)-\theta_3\left(\frac{\pi(z+z'+2c)}{4c},\ e^{-(\pi^2\kappa t)/4c^2}\right)\right\}$$

$$f(x',y',z')\,dx'\,dy'\,dz'.$$

The expressions for the temperature due to given arbitrary tempera-

tures over the boundaries may be obtained as in the other cases, but the expressions become very complicated.

6. The method of Art. 3 can be applied to deduce the solution of the corresponding radiation problems in three dimensions.

Consider the case in which conduction is taking place in the infinite solid on the positive side of the plane of xy, and radiation is taking place across this plane into a medium of temperature $f(x, y, t)$.

Writing
$$u = \left(1 - \frac{1}{h}\frac{d}{dz}\right) v,$$
we have as in Art. 3,
$$v = h\int_0^\infty e^{-h\zeta} \,.\, u''\, d\zeta,$$
where u'' denotes the value of u when $z+\zeta$ is written for z.

Now, by Art. 5,
$$u = \frac{1}{8\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}}\int_0^t\int_{-\infty}^\infty\int_{-\infty}^x \frac{zf(x', y', \lambda)}{(t-\lambda)^{\frac{5}{2}}} e^{-[x-x^2+(y-y')^2+z^2]/4\kappa(t-\lambda)}\, d\lambda\, dx'\, dy'.$$
Hence
$$v = \frac{h}{8\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}}\int_0^\infty e^{-h\zeta}\, d\zeta \int_0^t\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{(z+\zeta) f(x', y', \lambda)}{(t-\lambda)^{\frac{5}{2}}}$$
$$e^{-[(x-x')^2+(y-y')^2+(z+\zeta)^2]/4}\, d\lambda\, dx'dy'.$$

In the case in which $f(x, y, t)$ is independent of t, we have
$$v = \frac{h}{\pi^{\frac{3}{2}}}\int_0^\infty e^{-h\zeta}\, d\zeta \,.\, \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{z+\zeta}{r^3} f(x', y')$$
$$\left\{ Erf\left(\frac{r}{2\sqrt{\kappa t}}\right) + \frac{r}{2\sqrt{\kappa t}}\, e^{-r^2/4\kappa t} \right\} dx'dy',$$
where
$$r^2 = (x-x')^2 + (y-y')^2 + z^2.$$

These formulæ give the temperature of the solid initially at zero temperature, radiating into a medium of given temperature. When the motion of the heat becomes steady the above expressions reduce to
$$v = \frac{h}{\pi}\int_0^\infty e^{-h\zeta}\, d\zeta \int_{-\infty}^\infty\int_{-\infty}^\infty f(x', y')$$
$$\frac{z+T}{\{(x-x')^2+(y-y')^2+(z+\zeta)^2\}}\, dx'dy',$$

an expression given by Poisson,* who obtained it as a limiting case of the problem of radiation over a spherical surface.

* Théorie de la Chaleur.

As in the two-dimensional problem, these formulæ represent the temperature due to an infinite series of doublets on the negative side of the bounding plane of magnitude diminishing in geometrical ratio with the distance, there being an infinite row of such doublets corresponding to each element of the plane $z = 0$, on the line through the element drawn in the negative direction parallel to the axis of z;

$$2\pi\kappa \, . \, he^{-h\zeta} . f(x', y') \, dx' \, dy'$$

is the magnitude of the doublet at the point

$$z = -\zeta, \quad x = x', \quad y = y'.$$

If $F(x, y, z)$ is the initial temperature of the solid, we must as in Art. 3, add the expression

$$v = \frac{h}{(2\sqrt{\pi\kappa t})^3} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-h\zeta} \left\{ e^{-[(x-x')^2+(y-y')^2+(z+\zeta-z')^2]/4\kappa t} \right.$$
$$\left. - e^{-[(x-x')^2+(y-y')^2+(z+\zeta+z')^2]/4\kappa t} \right\} \left\{ 1 - \frac{1}{h} \frac{d}{dz'} \right\} F(x', y', z') \, dx' \, dy' \, dz'$$

this satisfies the conditions

$$v = F(x, y, z) \text{ when } t = 0,$$

and
$$\frac{dv}{dz} - hv = 0 \text{ when } z = 0.$$

If we add this expression to either of those obtained above, we have the complete solution of the problem for the case in which the initial temperature $F(x, y, z)$ is arbitrarily given and radiation is taking place across the plane $z = 0$ into a medium at arbitrarily given temperature $f(x, y, t)$.

On Certain Operators in connection with Symmetric Functions.

(*Supplementary Note.*)

By R. LACHLAN, M.A.

[*Read April 12th,* 1888.]

This note contains a few further results obtained by the use of the operators which were employed in a former paper (*Proceedings*, Vol. XVIII., pp. 39—48); and, to facilitate reference, the sections are numbered continuously with that paper. The object of the present

paper is to obtain symbolical expressions for coefficients of an equation, or for the sums of the symmetric functions of the same weight, in terms of the sums of the powers of the roots. A symbolical expression for the sums of the powers of the roots in terms of the coefficients is also given. I am indebted to one of the referees for pointing out that the formulæ given for the coefficients in terms of the sums of the powers of the roots, and the analogous formulæ, are most easily derived by a direct application of the Multinomial Theorem, which method I have indicated at the end of the Note.

19. From § 4, we have

$$\sigma e^u = \Sigma \frac{\sigma u^n}{n!} = \Sigma \frac{u^{n-1}}{(n-1)!} \sigma u$$

$$= e^u \sigma u;$$

and further,

$$\sigma(e^u v) = e^u \sigma v + v\sigma e^u$$

$$= e^u(\sigma v + v\sigma u).$$

Hence, since by definition, § 3,

$$\pi = s_1 - \sigma, \quad \vartheta = s_1 + \sigma;$$

if we define s to be such a symbol that $\sigma s = s_1$, we shall have

$$\sigma(e^{-s} v) = e^{-s}(\sigma v - s_1 v) = -e^{-s}\pi v,$$

and $$\sigma(e^s v) = e^s(\sigma v + s_1 v) = e^s \vartheta v.$$

So that $$-\pi v = e^s \sigma e^{-s} v,$$

$$\vartheta v = e^{-s}\sigma e^s v;$$

consequently $$(-)^r \pi^r v = e^s \sigma^r e^{-s} v \quad \text{.......................} \quad (26)$$

$$\vartheta^r v = e^{-s}\sigma^r e^s v \quad \text{.......................} \quad (27)$$

20. From (26), we obtain,

$$(-)^r\, r!\, p_r = (-)^r \pi^{r-1} p_1 = -e^s \sigma^{r-1} e^{-s} s_1$$

$$= e^s \sigma^r e^{-s} \quad \text{.................................} \quad (28)$$

To express p_r in terms of s_1, s_2, &c., we may expand this formula, or we may shorten the process by observing that the result will be the same if we write

$$(-)^r\, r!\, p_r = \sigma^r e^{-s},$$

and then, after the operations are performed, put s zero.

Consider now the expression $\sigma^r u^m$. By § 4, we may write

$$\sigma^r u^m = (\sigma_1 + \sigma_2 + \ldots + \sigma_m)^r u_1 u_2 \ldots u_m,$$

where σ_κ operates only on u_κ, and after expansion and performing the operations, the suffixes are to be dropped.

Hence we have

$$\sigma^r u^m = \Sigma \frac{m!\ u^{\kappa-i}}{a!\ \beta! \ldots} N(\sigma^a u)(\sigma^\beta u)\ldots,$$

where i is the number of the indices $a, \beta, \ldots$, and N is the number of terms in the symmetric function $\Sigma\sigma_1^a \sigma_2^\beta \ldots$; so that

$$N = \frac{\kappa(\kappa-1)\ldots(\kappa-i+1)}{\lambda!\ \mu!\ldots},$$

λ being the number of times the index a recurs, μ the number of times the index β recurs, and so on. Hence we deduce that, when $s = 0$,

$$\sigma^m s^\kappa = \Sigma \frac{m!\ \kappa!}{\lambda!\ \mu!\ldots} \left(\frac{s_a}{a}\right)^\lambda \left(\frac{s_\beta}{\beta}\right)^\mu \ldots,$$

where $\kappa = i = \lambda + \mu + \ldots,$

and $m = a\lambda + \beta\mu + \ldots.$

Hence we have

$$(-)^r p_r = \frac{\sigma^r}{r!} e^{-s} = \Sigma \frac{\sigma^r}{r!} \frac{(-)^\kappa}{\kappa!} s^\kappa$$

$$= \Sigma \frac{(-)^{\lambda+\mu+\ldots}}{\lambda!\ \mu!\ldots} \left(\frac{Sa}{a}\right)^\lambda \left(\frac{S\beta}{\beta}\right)^\mu \quad \ldots\ldots\ldots (29),$$

where $r = a\lambda + \beta\mu + \ldots.$

21. Similarly, from (27), we may deduce

$$r!\ h_r = \vartheta^{r-1} h_1 = e^{-s} \sigma^{r-1} e^s s_1$$

$$= e^{-s} \sigma^r e^s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

or $r!\ h_r = \sigma^r e^s$, when $s = 0$;

therefore $$h_r = \Sigma \frac{1}{\lambda!\ \mu!\ldots} \left(\frac{Sa}{a}\right)^\lambda \left(\frac{S\beta}{\beta}\right)^\mu \quad \ldots\ldots\ldots\ldots\ldots (31),$$

where $r = a\lambda + \beta\mu + \ldots.$

22. If $S^\kappa_{m+\kappa}$ denote the sum of the symmetric functions of weight $m+\kappa$, each containing κ, and only κ, roots, we have by (6),

$$S^\kappa_{m+\kappa} = \frac{\sigma^m}{m!} p_\kappa;$$

hence, by (28), we have

$$S^{\kappa}_{m+\kappa} = \frac{(-)^{\kappa}}{m!\ \kappa!}\,\sigma^{m}e^{s}\sigma^{\kappa}e^{-s} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (32).$$

23. *To express s_r in terms of $p_1, p_2, \ldots p_r$.*

Since $$\pi\,(1-ax) = 1-p_1x+p_2x^2-\ldots,$$

we may write $$\pi\,(1-ax) = e^{-\pi x},$$

where π^{κ} is to be replaced by $\kappa\,!\,p_{\kappa}$.

Hence, if $\omega_1, \omega_2, \ldots \omega_r$ be the imaginary r^{th} roots of unity, we have

$$\pi\,(1-a^r x^r) = e^{-(\pi_1\omega_1+\pi_2\omega_2+\ldots+\pi_r\omega_r)\,x},$$

where after expansion π_i^{κ} is to be replaced by $\kappa\,!\,p_{\kappa}$.

Equating coefficients of $x^{r\kappa}$, we have

$$(-)^{\kappa}\Sigma\, a_1^r a_2^r \ldots a_{\kappa}^r = \frac{(-)^{r\kappa}}{(r\kappa)\,!}(\pi_1\omega_1+\ldots+\pi_r\omega_r)^{r\kappa}$$

$$= (-)^{r\kappa}\Sigma\, p_{\alpha}p_{\beta}p_{\gamma}\ldots\Sigma\omega_1^{\alpha}\omega_2^{\beta}\omega_3^{\gamma}\ldots,$$

where $$\alpha+\beta+\gamma+\ldots = r\kappa.$$

When $\kappa = 1$, we have

$$\Sigma\,\omega_1^{\alpha}\omega_2^{\beta}\omega_3^{\gamma}\ldots = \frac{r\,(m-1)\,!\,(-)^{m-1}}{\lambda\,!\,\mu\,!\,\nu\,!},$$

where λ is the number of times the index α recurs, μ the number of times the index β recurs, and so on; and where $\lambda+\mu+\ldots = m$.

Hence we have

$$(-)^{r-1}s_r = \frac{1}{r\,!}(\pi_1\omega_1+\ldots+\pi_r\omega_r)^r$$

$$= \Sigma\frac{(-)^{m-1}\,r\,(m-1)\,!}{\lambda\,!\,\mu\,!\,\nu\,!\ldots}\,p_{\alpha}^{\lambda}p_{\beta}^{\mu}p_{\gamma}^{\nu} \ \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

where $$\alpha\lambda+\beta\mu+\ldots = r.$$

Or, again, we have

$$(-)^{\kappa}\Sigma\, a_1^2 a_2^2 \ldots a_{\kappa}^2 = \frac{1}{2\kappa\,!}(\pi_1-\pi_2)^{2\kappa}$$

$$= 2p_{2\kappa}-2p_1p_{2\kappa-1}+\ldots+(-)^{\kappa-1}\,2p_{\kappa-1}p_{\kappa+1}+(-)^{\kappa}p_{\kappa}^2$$

$$\ldots\ldots\ldots\ldots (34).$$

These formulæ, however, are merely what would be obtained at once by the law of symmetry.

24. Similarly, since

$$\frac{1}{\pi\,(1-ax)} = 1+h_1x+h_2x^2+\ldots = e^{\vartheta x},$$

where after expansion ϑ^κ is to be replaced by $\kappa\,!\,h_\kappa$, we have

$$s_r = \frac{1}{r\,!}\,(\vartheta_1\,\omega_1+\ldots+\vartheta_r\,\omega_r)^r,$$

and so

$$s_r = \Sigma\frac{(-)^{m-1}\,r\,(m-1)\,!}{\lambda\,!\,\mu\,!\ldots}\,h_\alpha^\lambda\,h_\beta^\mu \;\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Or again, the sum of the homogeneous products of α_1^2, α_2^2, &c., whose weight is

$$2\kappa = \frac{1}{2\kappa\,!}\,(\vartheta_1-\vartheta_2)^{2\kappa}$$

$$= 2h_{2\kappa}-2h_1h_{2\kappa-1}\ldots+(-1)^{\kappa-1}\,2h_{\kappa-1}\,h_{\kappa+1}+(-)^\kappa h_\kappa^2\;\ldots\ldots(36).$$

25. In a similar manner we may derive the symmetric function

$$\Sigma\alpha_1^2\alpha_2^2\ldots\alpha_\mu^2\,\alpha_{\mu+1}\ldots\alpha_{\mu+\nu}\,;$$

for this is clearly the coefficient of

$$(-)^\nu\,q_1^\nu q_2^\mu\,x^{2\mu+\nu},$$

in

$$\pi\,(1-q_1\,ax+q_2\,a^2x^2) = e^{-(\pi_1\omega_1+\pi_2\omega_2)}\,,$$

where ω_1, ω_2 are the roots of the equation

$$x^2-q_1x+q_2 = 0.$$

Hence the required symmetric function is the coefficient of $q_1^\nu q_2^\mu$ in

$$\frac{(-)^{2\mu+\nu}}{(2\mu+\nu)\,!}\,(\pi_1\,\omega_1+\pi_2\,\omega_2)^{2\mu+\nu},$$

e.g., in

$$(-)^{2\mu+\nu}\,\Sigma p_\alpha\,p_\beta\,\Sigma\omega_1^\alpha\,\omega_2^\beta,$$

where

$$\alpha+\beta = 2\mu+\nu.$$

But

$$\Sigma\omega_1^\alpha\omega_2^\beta = q_2^\beta\,\Sigma\omega_1^{\alpha-\beta}$$

$$= q_2^\beta\,\Sigma(-)^{\alpha-\beta}q_1^\nu q_2^{\mu-\beta}\,\frac{(-)^{\mu+\nu-\beta}(\mu+\nu-\beta-1)\,!\,(\alpha-\beta)}{\nu\,!\,(\mu-\beta)\,!}\,;$$

so that the coefficient of $q_1^\nu q_2^\mu$

$$= (-)^{a+\mu+\nu}\frac{(a-\mu-1)!\,(2a-2\mu-\nu)}{\nu!\,(a-\mu-\nu)!}.$$

Putting r for $a-\mu-\nu$, we have

$$\Sigma a_1^2 a_2^2 \dots a_\mu^2 a_{\mu+1} \dots a_{\mu+\nu}$$

$$= \Sigma p_{\nu+\mu+r}\, p_{\mu-r}\, \frac{(-)^r\,(r+\nu-1)!\,(\nu+2r)}{\nu!\,r!} \quad\dots\dots\dots\dots(37),$$

which agrees with the formula given by Serret (*Cours d'Algèbre Supérieure*, § 176).

26. It should be stated that the formulæ (29), (31), (33), and (35) are most easily obtained by the application of the multinomial theorem.

Thus we have

$$1-p_1x+p_2x^2-\dots = e^{-s_1x-\frac{s_2x^2}{2}-\dots}$$

$$= \sum_{i=0}^{i=\infty}\frac{(-)^i}{i!}\left(s_1x+\frac{s_2x^2}{2}+\dots\right)^i;$$

hence $$(-)^r p_r = \Sigma\frac{(-)^{\lambda+\mu+\nu+\dots}}{\lambda!\,\mu!\,\nu!\dots}\left(\frac{s_\alpha}{\alpha}\right)^\lambda\left(\frac{s_\beta}{\beta}\right)^\mu\left(\frac{s_\gamma}{\gamma}\right)^\nu,$$

where $$\alpha\lambda+\beta\mu+\gamma\nu+\dots = r.$$

Or, again, since

$$\Sigma\frac{s_r}{r}x^r = -\log(1-p_1x+p_2x^2-\dots)$$

$$= \sum_{i=1}^{i=\infty}\frac{1}{i}(p_1x-p_2x^2+\dots)^i,$$

we have

$$\frac{s_r}{r} = \Sigma\frac{(\lambda+\mu+\nu+\dots-1)!\,(-)^{r+\lambda+\mu+\dots}}{\lambda!\,\mu!\,\nu!\dots}\,p_\alpha^\lambda p_\beta^\mu\dots.$$

Similarly the other formulæ may be derived, as also an expression for p_r, in terms of h_1, h_2, &c.; or for h_r in terms of p_1, p_2, &c.

A Case of Complex Multiplication with Imaginary Modulus arising out of the Cubic Transformation in Elliptic Functions.

By Professor CAYLEY, F.R.S.*

The case in question is referred to in my "Note on the Theory of Elliptic Integrals" (*Math. Ann.*, XII. (1877), pp. 143—146); but I here work it out directly.

In the cubic transformation the modular equation is

$$u^4-v^4+2uv\,(1-u^2v^2)=0,$$

and we have

$$y=\frac{\left(1+\frac{2u^3}{v}\right)x+\frac{u^6}{v^2}\,x^3}{1+vu^2\,(v+2u^3)\,x^2},\ \text{giving}\ \frac{dy}{\sqrt{1-y^2\,.\,1-v^8y^2}}=\frac{\left(1+\frac{2u^3}{v}\right)dx}{\sqrt{1-x^2\,.\,1-u^8x^2}}.$$

We thus have a case of complex multiplication if $v^8=u^8$, or say $v=\gamma u$, where $\gamma^8=1$, or γ denotes an eighth root of unity. Substituting in the modular equation, this becomes

$$u^4\,(1-\gamma^4)+2\gamma u^2\,(1-\gamma^2u^4)=0,$$

or, throwing out the factor u^2 and reducing,

$$u^4-\tfrac{1}{2}u^2\,(\gamma^5-\gamma)-\gamma^6=0,$$

that is,
$$\frac{u^2}{\gamma}=\tfrac{1}{4}\,(\gamma^4-1\pm\sqrt{\gamma^8+14\gamma^4+1}),$$

or, what is the same thing,

$$=\tfrac{1}{4}\,\{\gamma^4-1\pm\sqrt{14\gamma^4+2}\}.$$

We have $\gamma^8=1$, that is, $\gamma^4=\pm 1$. Considering first the case $\gamma^4=1$, here

$$\frac{u^2}{\gamma}=\pm 1,$$

and thence
$$1+\frac{2u^3}{v}=1+\frac{2u^2}{\gamma},\ =1\pm 2,\ =3\ \text{or}\ -1;$$

moreover, $u^8=v^8=1$. We have thus only the non-elliptic formulæ

$$\frac{dy}{1-y^2}=\frac{-dx}{1-x^2},\ \text{satisfied by}\ y=-x,$$

and
$$\frac{dy}{1-y^2}=\frac{3dx}{1-x^2},\qquad \text{by}\ y=\frac{3x+x^3}{1+3x^2}.$$

* [This Note was written by way of illustration of Mr. Greenhill's paper, and is printed here at Prof. Cayley's suggestion.]

If however, $\gamma^4 = -1$, then

$$\frac{u^2}{\gamma} = \tfrac{1}{4}(-2 \pm \sqrt{-12}),$$

viz., this is $$\frac{u^2}{\gamma} = \tfrac{1}{2}(-1 \pm i\sqrt{3}) = \omega$$

if ω be an imaginary cube root of unity $(\omega^2+\omega+1 = 0)$; hence

$$u^8 = (\gamma\omega)^4 = -\omega.$$

Moreover, $$1+\frac{2u^2}{v} = 1+\frac{2u^2}{\gamma}, \quad = 1+2\omega,$$

or say, $$= \omega-\omega^2, \quad [= \sqrt{-3} \text{ if } \omega = \tfrac{1}{2}(-1+i\sqrt{3})];$$

and we thus have, as in the above-mentioned Note,

$$y = \frac{(\omega-\omega^2)\,x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)\,x^2}, \text{ giving } \frac{dy}{\sqrt{1-y^2\,.\,1+\omega y^2}} = \frac{(\omega-\omega^2)\,dx}{\sqrt{1-x^2\,.\,1+\omega x^2}};$$

or, what is the same thing, for the modulus $k^2 = -\omega$, we have

$$\operatorname{sn}(\omega-\omega^2)\,\theta = \frac{(\omega-\omega^2)\operatorname{sn}\theta+\omega^2\operatorname{sn}^3\theta}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

the values of $\operatorname{cn}(\omega-\omega^2)\,\theta$ and $\operatorname{dn}(\omega-\omega^2)\,\theta$ are thence found to be

$$\operatorname{cn}(\omega-\omega^2)\,\theta = \frac{\operatorname{cn}\theta\,(1-\omega^2\operatorname{sn}^2\theta)}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

and $$\operatorname{dn}(\omega-\omega^2)\,\theta = \frac{\operatorname{dn}\theta\,(1+\omega^2\operatorname{sn}^2\theta)}{1-\omega^2(\omega-\omega^2)\operatorname{sn}^2\theta};$$

which are the formulæ of transformation for the elliptic functions.

Complex Multiplication Moduli of Elliptic Functions.

By A. G. Greenhill.

[*Read March 8th*, 1888.]

The problem of the Complex Multiplication of Elliptic Functions is the determination of the elliptic functions of the complex argument $(a+b\sqrt{\Delta}\,i)\,u$, in terms of the elliptic functions of the argument u, where the ratio of the periods $K'/K = \sqrt{\Delta}$, and Δ is a prime number;

but, if Δ is a composite number mn, then we can have

$$K'/K = \surd(m/n);$$

but in this case b must contain the factor n.

The coefficients in the expression of an elliptic function of the argument $(a+b\surd\Delta\, i)\, u$ in terms of the elliptic functions of u will involve the values of the *modular functions* corresponding to

$$K'/K = \surd\Delta,$$

and thus the *modular equation* in some shape requires solution; and it is the chief object of this paper to make a collection of all the numerical solutions hitherto obtained, for integral values of Δ.

According to a remark of Abel (*Œuvres*, t. 1, p. 272, 1st edition), quoted by Kronecker (*Berlin Sitz*, 1857), the modular equation in such cases is always soluble by radicals.

A few numerical cases are given by Legendre and Abel, but the first important collection of results is due to Kronecker (*Berlin Sitz*, 1862), who gives the numerical values of Legendre's modulus κ, or in some cases of $\kappa\kappa'$, for a series of values of Δ, and promises a more complete collection, which has not yet appeared.

According to the form of Δ with respect to the modulus 4 or 8, it will be convenient to consider four classes, and to choose the *absolutely simplest numerical invariant* appropriate to each class, which classes are distinguished as follows—

Class *A*. $\Delta \equiv 3$, mod. 8.

Class *B*. $\Delta \equiv 7$, mod. 8.

Class *C*. $\Delta \equiv 1$, mod. 4.

Class *D*. $\Delta \equiv 2$, mod. 4.

The class for $\Delta \equiv 0$, mod. 4, does not require special treatment, as it can be made to depend on one of the previous classes by means of the quadric transformation.

The article, "*Neue Untersuchungen im Gebiete der elliptischen Functionen*," of F. Klein (*Math. Ann.*, Bd. xxvi., 1886), gives references to the most recent researches on modular equations; and in the course of this paper great use will be made of the following articles:—

Sohnke, "*Æquationes modulares pro transformatione Functionum Ellipticarum*," Crelle, 16.

Schröter, "*Dissertatio inauguralis de Æquationibus modularibus*," *Regiomonti*, 1854; also *Liouville*, 1858; and *Acta Mathematica*, 1882.

Hermite, "*Théorie des Équations modulaires*;" Paris, 1859.

Klein and Kiepert, "*Ueber die Transformation der elliptischen Functionen*," *Math. Ann.*, xiv., p. 111; xxvi., p. 369; xxxii., p. 1.

G. H. Stuart, "*Complex Multiplication of Elliptic Functions*," *Quar. Jour. of Math.*, Vol. xx., p. 18.

E. W. Fielder, "*Ueber eine besondere Classe irrationalem Modulargleichungen*;" Zürich, 1885.

R. Russell, "*On* $\kappa\lambda$, $\kappa'\lambda'$ *Modular Equations*," *Proc. of the Lond. Math. Soc.*, Nov. 10, 1887.

The general expressions for the formulas of Complex Multiplication are also given by the author in an article in the *Quarterly Journal of Mathematics*, Vol. xxii.

CLASS *A*.

$$\Delta \equiv 3, \text{ mod. } 8.$$

The absolutely simplest numerical invariant to choose for this class is Klein's *absolute invariant* J, the same as Dedekind's *Valenz* (*Crelle*, 83), and connected with Hermite's a by the equation

$$J = -\tfrac{4}{27}a$$

(*Théorie des Équations modulaires*); but it is convenient to use Kiepert's form in terms of Legendre's moduli κ and κ' (*Math. Ann.*, Vol. xxvi.),

$$J = -\frac{(1-16\kappa^2\kappa'^2)^3}{108\kappa^2\kappa'^4},$$

obtained by a quadric transformation from Klein's form

$$J = \frac{4}{27}\,\frac{(1-\kappa^2\kappa'^2)^3}{\kappa^4\kappa'^4},$$

so that Kiepert's J is a "*Modul-function zweiter Stufe*."

Then, if we work with Weierstrass's canonical first elliptic integral

$$\int_x^\infty \frac{dx}{\sqrt{(4x^3-g_2x-g_3)}},$$

and *normalize* it by multiplying by the twelfth root of the discriminant

$$D = g_2^3-27g_3^2,$$

so that it becomes $$\int \frac{dy}{\sqrt{(4y^3-\gamma_2y-\gamma_3)}},$$

we can make the new discriminant

$$\gamma_2^3-27\gamma_3^2 = -1,$$

and thus the absolute invariant

$$J = \frac{g_2^3}{D} = -\gamma_2^3,$$

$$J-1 = \frac{27g_3^2}{D} = -27\gamma_3^2.$$

In this Class A, the simplest formula of complex multiplication connects

$$x = \wp u, \text{ and } y = \wp \frac{u}{M},$$

where

$$\frac{1}{M} = \tfrac{1}{2}(-1 + \sqrt{\Delta}\, i),$$

leading to the differential relation

$$\frac{M\,dy}{\sqrt{(4y^3 - g_2 y - g_3)}} = \frac{dx}{\sqrt{(4x^3 - g_2 x - g_3)}},$$

by an equation of the form (*Quar. Jour. of Math.*, xxii., p. 127)

$$y = M^2 \frac{x^n - A_1 x^{n-1} + A_2 x^{n-2} \dots}{(x^m - G_1 x^{m-1} + G_2 x^{m-2} \dots)^2},$$

where $\qquad n = 2m+1, \quad \Delta = 4n - 1 = 8n+3,$

and $\qquad A_1 = 2G_1;$

the A's and G's being certain modular functions, to be subsequently determined. (Kiepert, *Math. Ann.* XXVI., p. 398.)

The determination of G_1 is the most trouble, so it is important to notice that it is a numerical factor of $\sqrt{\Delta} + i$.

For, if we denote by M' and G_1' the conjugate imaginaries of M and G_1, and if we put

$$z = \wp n u = \wp \frac{u}{MM'},$$

then

$$z = M'^2 \frac{y^n - 2G_1' y^{n-1}}{(y^m - G_1' y^{m-1} \dots)^2}$$

$$= MM'^2 \frac{(x^n - 2G_1 x^{n-1} \dots)^n - 2G_1' M^{-2} (x^n \dots)^{n-1} (x^m \dots)^2 \dots}{(x^m - G_1 x^{m-1} \dots)^2 \left\{ (x^n - 2G_1 x^{n-1} \dots)^m - G_1' M^{-2} (x^n \dots)^{m-1} (x^n \dots)^2 \right\}^2}$$

$$= \frac{1}{n^2} \; \frac{x^{n^2} - 2\,(nG_1 + G_1' M^{-2})\, x^{n^2-1} \dots}{x^{n^2-1} - 2\,(nG_1 + G_1' M^{-1})\, x^{n^2-1} \dots};$$

and then, from the known expansion of $\wp u$ in powers of u,

$$\wp u = \frac{1}{u^2} + * + \frac{g_2}{20} u^2 + \dots$$

The absent terms denoted by the $*$ require that

$$nG_1 + G_1' M^{-2} = 0;$$

so that, if we put $\; G_1 = a\sqrt{\Delta} + bi, \quad G_1' = a\sqrt{\Delta} - bi,$

then $$na\sqrt{\Delta}+nbi+(a\sqrt{\Delta}-bi)\,\tfrac{1}{2}\,(-2n+1+\sqrt{\Delta}\,i) = 0,$$
or $$(a-b)\,\{\sqrt{\Delta}-(4n-1)\,i\} = 0,$$
or $$a = b;$$
but, to determine a, the expansions of y and z in terms of x must be carried out further.

When once G_1, and therefore $A_1 = 2G_1$, has been determined, the remaining G's and A's are found by the recurring formulas given by Kiepert, *Math. Ann.*, xxvi., p. 399.

The preceding equation connecting y and x is equivalent to any one of the three following equations (*Quar. Jour. of Math.*, XXII., p. 125)—

$$y-e_1 = M^2\,(x-e_2)\prod_{r=1}^{r=m}\{x-\wp\,(\omega_2+2r\omega_1/n)\}^2 \div D,$$

$$y-e_2 = M^2\,(x-e_3)\,\Pi\,\{x-\wp\,[(n-2r)\,\omega_1/n]\}^2 \div D,$$

$$y-e_3 = M^2\,(x-e_1)\,\Pi\,\{x-\wp\,[\omega_2+(n-2r)\,\omega_1/n]\}^2 \div D,$$

$$D = \Pi\,\{x-\wp\,(2r\,\omega_1/n)\}^2;$$

where $$\frac{\omega_2'}{\omega_2} = \frac{K'}{K}\,i = \sqrt{\Delta}\,i,$$

and $$\omega_1 = \tfrac{1}{2}\,(\omega_2+\omega_2'),\quad \omega_3 = \tfrac{1}{2}\,(\omega_2-\omega_2');$$

so that $$G_1 = \sum_{r=1}^{r=m}\wp\,(2r\omega_1/n).$$

Thus, for example, when $\Delta = 51$,

$$G_1 = \wp\,\frac{2\omega_1}{13}+\wp\,\frac{4\omega_1}{13}+\wp\,\frac{8\omega_1}{13}+\wp\,\frac{16\omega_1}{13}+\wp\,\frac{32\omega_1}{13}+\wp\,\frac{64\omega_1}{13}$$

(Kiepert, *Math. Ann.*, XXVI., p. 381).

But suppose the complex multiplier $\dfrac{1}{M}$, instead of being

$$\tfrac{1}{2}\,(-1+\sqrt{\Delta}\,i),$$

had been $$\tfrac{1}{2}\,(-\rho+\sqrt{\Delta}\,i),$$

where ρ is an odd integer; then we should have to put

$$n = \tfrac{1}{4}\,(\Delta+\rho^2)$$

in the above formulas; and this explains why in Hermite's *Équations Modulaires*, p. 44, Class 3° (our Class A), Δ has the values

$$4n-\rho^2 = 4n-1,\quad 4n-9,\quad 4n-25,\ \ldots.$$

CLASS *A*.

$\Delta \equiv 3$, mod. 8.

$\Delta = 3$. From Jacobi's modular equation of the third order,

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = 1,$$

we obtain, putting $\kappa = \lambda'$, $\kappa' = \lambda$,

$$2\sqrt{\kappa\kappa'} = 1,$$

or

$$2\kappa\kappa' = \tfrac{1}{2} = \sin 30^\circ,$$

so that the modular angle is 15°, and

$$\kappa = \sin 15^\circ, \qquad \kappa' = \cos 15^\circ.$$

Then the absolute invariant $J = 0$, and

$$\gamma_2 = 0, \quad 27\gamma_3^2 = 1, \quad \text{or} \quad 3\sqrt{3}\,\gamma_3 = 1.$$

Also

$$\frac{1}{M} = \omega,$$

an imaginary cube root of unity; and

$$\wp\,\omega u = \omega \wp u,$$

the simplest case of *Complex Multiplication*, required in the reduction of the elliptic integrals considered by Legendre (*Fonctions Elliptiques*, t. I., cap. XXVI.).

$\Delta = 11$. Taking Schröter's or Russell's form of the modular equation of the 11th order,

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} + 2\sqrt[6]{4\kappa\lambda\kappa'\lambda'} = 1,$$

and putting $\kappa = \lambda'$, $\kappa' = \lambda$; then

$$2\sqrt{\kappa\kappa'} + 2\sqrt[3]{2\kappa\kappa'} = 1.$$

Forming the equation in $\kappa^2\kappa'^2$, and putting

$$J = -\frac{(1-16\kappa^2\kappa'^2)^3}{108\kappa^2\kappa'^2},$$

we find

$$J = -\frac{2^9}{3^3}, \quad J-1 = -\frac{7^2 \times 11}{3^3},$$

$$\gamma_2 = \frac{8}{3}, \quad \gamma_3 = \frac{7\sqrt{11}}{27}.$$

Here Hermite's $a = 2^7$, the value of which could be inferred from his equations at the foot of p. 47 of the *Équations Modulaires.*

Also $G_1 = -\frac{1}{6}(\sqrt{11}+i)$, $A_1 = -\frac{1}{3}(\sqrt{11}+i)$,

and the values of A_2 and A_3 are given in the *Quar. Jour. of Math.*, XXII., p. 134.

$\Delta = 19$. We shall find

$$J = -2^9, \quad J-1 = -3^3 \times 19;$$

$$\gamma_2 = 8, \quad \gamma_3 = \sqrt{19}, \quad \gamma_2+1 = 3^2, \quad \gamma_2^2-\gamma_2+1 = 3\times 19.$$

These values are obtained from Hermite's *Théorie des Équations Modulaires*, p. 47, where it is shown that, a being the equivalent of $-\frac{27}{4}J$,

$$\Delta = 3, \quad a = 0;$$

and

$$\Delta = 11, \quad a = 2^7, \text{ as before};$$

$$\Delta = 19, \quad a = 2^7\times 3^3;$$

$$\Delta = 27, \quad a = 2^7\times 3\times 5^3;$$

$$\Delta = 43, \quad a = 2^{10}\times 3^3\times 5^3;$$

and by inference from the approximate equation (*Équations Modulaires*, p. 48),

$$2^8 a = e^{\pi\sqrt{\Delta}} - 744 + 196880 e^{-\pi\sqrt{\Delta}} + \ldots,$$

we obtain

$$\Delta = 67, \quad a = 2^7\times 3^3\times 5^3\times 11^3,$$

$$\Delta = 163, \quad a = 2^{10}\times 3^3\times 5^3\times 23^3\times 29^3.$$

According to Hermite (*Equations Modulaires*, p. 47) the value of a is an integer when there is only one improperly primitive class of the determinant $-\Delta$; and $\Delta = 163$ is probably the highest number of this nature.

Hermite points out that in these cases $e^{\pi\sqrt{\Delta}}$ is very nearly an integer; for instance, in $e^{\pi\sqrt{163}}$ the decimal part begins with a series of twelve 9's.

Using the symbol $\approx$ for approximate equality,

$$1728J \approx -\frac{1}{q} \approx -e^{\pi\sqrt{\Delta}},$$

so that

$$12\gamma_2 \approx e^{\frac{1}{3}\pi\sqrt{\Delta}}, \quad \text{also} \quad 216\gamma_3 \approx e^{\frac{1}{2}\pi\sqrt{\Delta}};$$

therefore $e^{\frac{1}{3}\pi\sqrt{\Delta}}$ is also very nearly an integer, a multiple of 12; while $e^{\frac{1}{2}\pi\sqrt{\Delta}} \div \sqrt{\Delta}$ is also very nearly an integer, a multiple of 216. (H. J. S. Smith, *Report on the Theory of Numbers to the British Association*, 1865, p. 374.)

Thus
$$e^{\frac{1}{3}\pi\sqrt{19}} \approx 96 \quad = 12\,(3^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{43}} \approx 960 \quad = 12\,(9^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{67}} \approx 5280 \quad = 12\,(21^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{163}} \approx 640320 = 12\,(231^2-1);$$

while
$$e^{\frac{1}{2}\pi\sqrt{19}} \div \sqrt{19} \approx 216,$$
$$e^{\frac{1}{2}\pi\sqrt{43}} \div \sqrt{43} \approx 216\times 21 \quad = 216\times 7\times 3,$$
$$e^{\frac{1}{2}\pi\sqrt{67}} \div \sqrt{67} \approx 216\times 217 \quad = 216\times 7\times 31,$$
$$e^{\frac{1}{2}\pi\sqrt{163}} \div \sqrt{163} \approx 216\times 185801 = 216\times 7\times 11\times 19\times 127.$$

The values of J corresponding to $\Delta=3$, 11, and 19 afford interesting numerical applications of Klein's *ikosahedron equation*, the corresponding r resolvent equation (*Ikosaëder*, p. 102) having a root $r=3$, 11, 19, respectively. The determination of z, the corresponding *ikosahedron irrationality*, is then an interesting numerical exercise.

$$\Delta = 27.\ \text{Here}\quad J = -\frac{2^9\times 5^3}{3^2},\quad J-1 = -\frac{11^2\times 23^2}{3^2};$$
$$\gamma_2 = \frac{40}{3}\,3^{\frac{1}{3}},\quad \gamma_3 = \frac{253}{27}\,3^{\frac{1}{2}}.$$

These values can be obtained by the cubic transformation of Klein (*Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (\tau-1)(9\tau-1)^3 : (27\tau^2-18\tau-1)^2 : -64\tau,$$

and J' the same function of τ', with $\tau\tau'=1$.

Putting
$$J' = 0,$$
then
$$9\tau' = 1,\quad \tau = 9;$$
and
$$J = -\frac{2^9\times 5^3}{3^2};$$
also
$$G_1 = -\tfrac{1}{2}\,3^{\frac{1}{4}}\,(\sqrt{27}+i),$$
(*Quar. Jour. of Math.*, XXII., p. 136).

$$\Delta = 35.\ \text{Here}\quad J = -\gamma_2^3,\quad J-1 = -27\gamma_3^2,$$
where
$$\gamma_2 = \tfrac{8}{3}\sqrt{5}\,\{\tfrac{1}{2}(\sqrt{5}+1)\}^4,$$
$$\gamma_3 = \frac{256+115\sqrt{5}}{27}\sqrt{7},$$

(*Quar. Jour. of Math.*, Vol. XXII., p. 137, 1887);

also $$G_1 = -\tfrac{1}{6}\left\{\tfrac{1}{2}(\sqrt{5}+1)\right\}^6(\sqrt{35}+i).$$

The manner in which these numerical values were obtained from Kiepert's L-equation for $n=9$ is there explained.

The values of γ_2 and γ_3 above correspond to the case of
$$K'/K = \sqrt{35}\,; \quad \text{but when } K'/K = \sqrt{(7 \div 5)},$$
we must change the sign of $\sqrt{5}$.

We might have obtained the same value of J by employing Fiedler's modular equation of the 35th order (*Irrationale Modulargleichungen*, p. 97), by putting $\lambda = \kappa'$, $\lambda' = \kappa$, and $x = 2\sqrt{\kappa\kappa'}$; then, in Fiedler's notation,
$$Z_1' = x-1, \quad Z_2' = \tfrac{1}{4}x^2 - x, \quad Z_3' = -\tfrac{1}{4}x^3,$$
and $$Z_0^{(4)} = -\tfrac{1}{2}x + \sqrt{(2x-x^3)}\,;$$
so that, substituting in his equation, we obtain
$$x^3-5x^2+3x+1+4\sqrt{(2x-x^3)} = 0,$$
$$x^6-10x^5+31x^4-12x^3-x^2-26x+1 = 0,$$
$$(x^3-5x^2+13x-1)^2-20(x^2-3x)^2 = 0;$$
$$x^3-(5+2\sqrt{5})\,x^2+(13+6\sqrt{5})\,x-1 = 0;$$
and forming the equations for x^2 and x^4,
$$x^6-(19+8\sqrt{5})\,x^4+(339+152\sqrt{5})\,x^2-1 = 0,$$
$$x^{12}-3x^8+(230403+103040\sqrt{5})\,x^4-1 = 0,$$
$$(x^4-1)^3+(230400+103040\sqrt{5})\,x^4 = 0.$$
Then $$J = -\frac{4}{27}\frac{(1-x^4)^3}{x^4} = -\frac{4}{27}(230400+103040\sqrt{5})$$
$$= -\frac{2^9\times 5\sqrt{5}}{3^3}\left(\frac{\sqrt{5}+1}{2}\right)^{12}.$$

The same values could also have been obtained by combining Schröter's or Russell's modular equation of the 5th order with Gutzlaff's of the 7th order, but examples of this method will occur hereafter.

$\Delta = 43$. Here $J = -2^{12}\times 5^3$, $J-1 = -3^3\times 21^2\times 43$;

and $$\gamma_2 = -\sqrt[3]{J} = 80, \quad \gamma_3 = 21\sqrt{43}\,;$$
$$\gamma_2+1 = 3^4; \quad \gamma_2^2-\gamma_2+1 = 3\times 7^2\times 43\,;$$

obtained in Hermite's manner by approximate numerical calculation, or obtainable in his manner from the modular equation for $n = 11$.

We notice that when J is an integer, then γ_2+1 is the square of a number which is a multiple of 3, while γ_3 has a factor 7; these considerations are useful in determining the value of J by approximate numerical calculation for high values of Δ.

For $$\Delta = 43, \quad G_1 = -3(\sqrt{43}+i),$$

(*Quar. Jour. of Math.*, XXII., p. 171).

$\Delta = 51$. The value obtained by Dr. L. Kiepert for J is

$$J = -64(5+\sqrt{17})^2(\sqrt{17}+4)^2$$

$$= -256(3\sqrt{17}+11)(\sqrt{17}+4)^2,$$

and then $$J-1 = -7^2(128+31\sqrt{17})^2,$$

$$\gamma_3 = \frac{7\sqrt{3}}{3^2}(128+31\sqrt{17}).$$

The modular functions for this transformation are intimately associated with Kiepert's functions for $n = 13$ (*Math. Ann.*, XXVI., p. 381); Kiepert's L-equation (p. 425) having a factor of the form

$$L^4+aL^2+13 = 0,$$

where, according to Kiepert,

$$a = -\tfrac{1}{2}(3\sqrt{17}+1);$$

so that $$L^2 = \tfrac{1}{2}(1+\sqrt{51}\,i)\,\omega,$$

where ω is an imaginary cube root of unity; and the corresponding modular functions will depend on arguments of the 13th part of multiples of the periods.

$\Delta = 59$. Here we shall find it most convenient to employ Hermite's method in *Équations Modulaires*, p. 44, for class 3°, with $n = 17$.

The number of improperly primitive classes, which we shall denote henceforth by the letter p, for the determinant $-\Delta$, is in this case of $\Delta = 59$ equal to 3 (Gauss, *Werke*, t. II., p. 287), so that a cubic equation for a must be expected.

Putting $u/v = t$, we shall find, with Hermite's notation,

$$u^8 = \frac{1}{1-v^8}, \quad u^8 = x,$$

$$a = -\frac{(1-x+x^2)^3}{(x-x^2)^2} = \frac{(1-t^8)^3}{t^8}.$$

Then Sohnke's modular equation of the 17^{th} order (*Crelle*, t. 16),

$$(v-u)^{18} - 16uv\,(1-u^8)(1-v^8)\,\{17uv\,(v-u)^6 - (v^4-u^4)^2 + 16\,(1+u^4v^4)^2\} = 0,$$

becomes an equation of the 18^{th} order in t,

$$(t-1)^{18} + 16\times 17t^3\,(t-1)^6 + 15\times 16t^9 + 16\times 34t^6 - 16t = 0.$$

The corresponding values of Δ are $4n-\rho^2 = 67$, 59, 43, and 19; and from the known integral values of a for $\Delta = 19$; 43, and 67 given previously, we infer the corresponding factors, for

$$\Delta = 19, \quad t^3 - t^2 + 3t - 1 = 0,$$
$$\Delta = 43, \quad t^3 - 3t^2 + 7t - 1 = 0,$$
$$\Delta = 67, \quad t^3 - 7t^2 + 13t - 1 = 0;$$

leaving the factor of the 9^{th} degree

$$t^9 - 7t^8 + 22t^7 - 34t^6 + 40t^5 - 28t^4 + 22t^3 - 10t^2 + 11t - 1 = 0,$$

for $\Delta = 59$.

Forming from this equation the corresponding equation in t^8, we shall find on putting

$$\frac{(1-t^8)^3}{t^8} = a,$$

a cubic equation for a.

We might also have employed Fiedler's modular equation for $n = 15$, and then the corresponding values of Δ are 59, 51, 35, and 11; and the factors for 11, 35, and 51 can be inferred from the preceding values of a.

Also the modular equation for $n = 13$ might have been employed in Hermite's manner for the case of $\Delta = 51$, solved above by Kiepert, the extraneous factors corresponding to $\Delta = 43$, 27 and 3 being known, and easily divided out.

$\Delta = 67$. Here

$$J = -2^9\times 5^3\times 11^3, \quad J-1 = -27\times 7^2\times 31^2\times 67;$$

obtained from Hermite's *Équations Modulaires*, p. 48;

then $\gamma_2 = 2^3\times 5\times 11, \quad \gamma_3 = 217\sqrt{67},$

$$\gamma_2 + 1 = 3^2\times 7^2, \quad \gamma_2^2 - \gamma_2 + 1 = 3\times 7^2\times 31^2\times 67.$$

The modular functions in this transformation correspond to Kiepert's case of $n = 17$ (*Math. Ann.*, XXVI., p. 428), the corresponding L-equation having the factor

$$L^4+L^3+17;$$

and the associated modular functions have as arguments the 17th parts of multiples of the periods.

$\Delta = 75 = 3\times 5^2$. Here

$$J = -64\sqrt{5}\,(31\sqrt{5}+69)^2.$$

This is obtained by Klein's quintic transformation (*Math. Ann.*, XIV., p. 143; *Proc. Lond. Math. Soc.*, Vol. IX., p. 126),

$$J : J-1 : 1 = (\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau;$$

and J' the same function of τ', with $\tau\tau' = 125$.

Putting $J' = 0$, then

$$\tau'^2-10\tau'+5 = 0,$$

$$\tau' = 5-2\sqrt{5},$$

$$\tau = 25\sqrt{5}\,(\sqrt{5}+2),$$

leading to the value of J above.

Then $$\gamma_2 = 4\times 5^3\,(69+31\sqrt{5}),$$

$$\gamma_3 = \tfrac{1}{9}\sqrt{3}\,(4352\sqrt{5}+9729).$$

This transformation is associated with Kiepert's transformation for $n = 19$ (*Math. Ann.*, XXVI., p. 428), and the corresponding L-equation has a factor of the form

$$L^4+aL^3+19 = 0.$$

$\Delta = 83$. Employing Hermite's method with $n=23$, and Schröter's or Russell's modular equation

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1,$$

then, putting $$\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\,\kappa'\lambda')} = 2s,$$

from Hermite's equations (*Équations Modulaires*, p. 44),

$$u^8 = \kappa^2 = x,\quad 1-v^8 = \lambda'^2 = \frac{1}{x},$$

κ'^2

$$\kappa'^2 = 1-x,\quad \lambda^2 = \frac{x-1}{x},$$

so that $$\kappa^2\lambda'^2 = 1,$$

we find $$\kappa^2\lambda^2+\kappa'^2\lambda'^2 = x+\frac{1}{x}-2 = -\kappa^2\lambda^2\kappa'^2\lambda'^2$$
$$= -256s^{24};$$
$$\kappa\lambda+\kappa'\lambda' = \sqrt{(32s^{12}-256s^{24})};$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = \sqrt{\left[4s^3+\sqrt{\{8s^6+\sqrt{(32s^{12}-256s^{24})}\}}\right]};$$
so that $$\sqrt{\{8s^6+\sqrt{(32s^{12}-256s^{24})}\}} = 1-4s+4s^2-4s^3,$$
or $$\sqrt{(32s^{12}-256s^{24})} = 1-8s+24s^2-40s^3+48s^4-32s^5+8s^6,$$
or $$\begin{aligned}1-16s+112s^2-464s^3+1312s^4-2752s^5+4432s^6\\ -5504s^7+5248s^8-3712s^9+1792s^{10}-512s^{11}\\ +32s^{12}+256s^{24} = 0,\end{aligned}$$
an equation of the 24th degree in s, for $\Delta = 11$, 43, 67, 83, and 91.

Putting $$\beta = \frac{1-2s-2s^2-2s^3}{2s^2},$$
we shall find that this becomes an equation of the 8th degree in β, and that

$$\begin{aligned}\beta &= 0, &&\text{for } \Delta = 67;\\ \beta &= -1, &&\text{for } \Delta = 43;\\ \beta &= -2, &&\text{for } \Delta = 11;\\ \beta &= \tfrac{1}{2}(\sqrt{13}-1), &&\text{for } \Delta = 91.\end{aligned}$$

The equation in β will therefore be of the form
$$\beta(\beta+1)(\beta+2)(\beta^2+\beta-3)(\beta^3+A\beta^2+B\beta+C) = 0;$$
and we easily find $A = 4$, $B = 2$, $C = -5$;
so that the cubic equation
$$\beta^3+4\beta^2+2\beta-5 = 0$$
having the discriminant $83 \div 27$, gives the value of β for $\Delta = 83$; and, forming the equation in t^8 or s^{24}, t being connected with s by the equations
$$x+\frac{1}{x}-2 = -256s^{24} = -t^8,$$
$$x+\frac{1}{x}-1 = 1-256s^{24} = 1-t^8,$$
we obtain a cubic equation for
$$a = \frac{(1-t^8)^3}{t^8} = \frac{(1-256s^{24})^3}{256s^{24}}.$$
A cubic for a was to be expected, as $p = 3$ for the determinant -83.

$\Delta = 91 = 7\times 13$. Here

$$J = -\gamma_2^3,\quad J-1 = -27\gamma_3^2,$$

where $\gamma_2 = 908+252\sqrt{13}$, $\gamma_3 = 11\sqrt{7}\,(2\sqrt{13}+7)(5\sqrt{13}+18)$.

These values were obtained originally by calculating the approximate values of

$$\gamma_2+1 = (6\sqrt{13}+21)^2 = 9\,(2\sqrt{13}+7)^2,$$

$$\gamma_2^2-\gamma_2+1 = 3\times 7\times 11^2\left(\frac{\sqrt{13}+3}{2}\right)^6,$$

and the values of γ_2' corresponding to a change of sign of $\sqrt{13}$, and

$$K'/K = \sqrt{(13\div 7)}.$$

Calculating the approximate values of

$$12\gamma_2 \approx e^{\frac{1}{3}\pi\sqrt{(91)}},\quad 12\gamma_2' \approx e^{\frac{1}{3}\pi\sqrt{(13\div 7)}},$$

we find
$$\gamma_2+\gamma_2' \approx 1816,$$
$$\gamma_2\gamma_2' \approx -1088,$$

so that we may guess that γ_2, γ_2' are the roots of the quadratic

$$\gamma^2-1816\gamma-1088 = 0.$$

These values of γ_2 or γ_2' in Kiepert's L-equation for $n = 23$ will make the equation have a factor of the form

$$L^2+aL+23 = 0.$$

$\Delta = 99 = 3^2\times 11$. Here J is obtained by performing the cubic transformation on

$$J' = -\frac{2^9}{3^3},$$

corresponding to $\Delta = 11$.

With Klein's form, putting $\tau' = \dfrac{x}{27}$,

$$\frac{(x-27)(x-243)^3}{2^6\times 3^3x^3} = \frac{2^9}{3^3},$$

or
$$(x-27)(x-243)^3 = 2^{15}x^3.$$

Put
$$x-27 = y^3,$$

and extract the cube root; then

$$y^4-32y^3-216y-864 = 0,$$

or
$$(y^2-16y-30)^2 = 196\,(y+3)^2,$$
$$y^2-16y-30 = \pm 14\,(y+3),$$
$$(y^2-2y+12)(y^2-30y-72) = 0,$$
$$y = 1+\sqrt{11}\,i, \quad \text{or} \quad 3\,(\sqrt{33}+5)\,;$$
$$r' = \frac{x}{27} = 1+\frac{y^3}{27} = 1+(\sqrt{33}+5)^3 = 27\,(23+4\sqrt{33})\,;$$
$$r = \frac{23-4\sqrt{33}}{27} = \frac{(2\sqrt{3}-\sqrt{11})^2}{27},$$

and these values of r will presumably give the required value of J.

$\Delta = 107$, a prime, notyet solved, but depending on $n = 27$ (Kiepert, *Math. Ann.*, XXXII., p. 67).

$\Delta = 115 = 5\times 23$. Here
$$J = -\gamma_2^3, \quad J-1 = -27\gamma_3^2,$$
where
$$\gamma_2 = 3140+1404\sqrt{5},$$
$$\gamma_3 = (6\sqrt{5}+13)(378+169\sqrt{5})\sqrt{23},$$
$$\gamma_2+1 = (18\sqrt{5}+39)^2 = 9\,(6\sqrt{5}+13)^2,$$
$$\gamma_2^2-\gamma_2+1 = 3\times 23\,(378+169\sqrt{5})^2.$$

These numerical values are obtained by the combination of the modular equations of the 5th and 23rd order, as explained below.

Combine Mr. Russell's modular equations of the 5th and 23rd order,
$$\kappa\lambda'+\kappa'\lambda+2\sqrt[3]{(4\kappa\lambda\kappa'\lambda')} = 1,$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1\,;$$
putting
$$4\kappa\lambda\kappa'\lambda' = x^{12}.$$

Then, from the equation of the 5th order,
$$\kappa\lambda'+\kappa'\lambda = 1-2x^4,$$
$$(\kappa'\lambda+\kappa'\lambda')^2 = 1+4\kappa\lambda\kappa'\lambda'-(\kappa\lambda'+\kappa'\lambda)^2$$
$$= 1+x^{12}-(1-2x^4)^2 = 4x^4-4x^8+x^{12}\,;$$
$$\kappa\lambda+\kappa'\lambda' = 2x^2-x^6,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=\sqrt{2}\,x,$$

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=\sqrt[4]{2}\,\sqrt{(x+x^3)}=1-\sqrt{2}\,x,$$

from the equation of the 23^{rd} order.

Therefore $$\sqrt{2}\,x^3-2x^2+3\sqrt{2}\,x-1=0.$$

Now, $$\kappa'\lambda+\kappa\lambda'=1-2x^4,$$

$$\kappa\lambda+\kappa'\lambda'=2x^2-x^6;$$

so that, by multiplication,

$$\kappa\kappa'+\lambda\lambda'=(1-2x^4)(2x^2-x^6)=x^2(2-5x^4+2x^8);$$

$$(\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'})^2=2x^2(1-x^4)^2,$$

$$(-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'})^2=2x^2\{(1-x^4)^2-x^4\}.$$

Therefore $$\sqrt{2\kappa\kappa'}=x-x^5-\sqrt{(x^2-3x^6+x^{10})},$$

$$\sqrt{2\lambda\lambda'}=x-x^5+\sqrt{(x^2-3x^6+x^{10})};$$

$$\sqrt[4]{2\kappa\kappa'}=\sqrt{\frac{x+x^3-x^5}{2}}-\sqrt{\frac{x-x^3-x^5}{2}},$$

$$\sqrt[4]{2\lambda\lambda'}=\sqrt{\frac{x+x^3-x^5}{2}}+\sqrt{\frac{x-x^3-x^5}{2}}.$$

Now, since $$\sqrt{2}\,x^3-2x^2+3\sqrt{2}\,x-1=0,$$

therefore we shall obtain the cubic equations for $t=\sqrt[4]{4\kappa\kappa'}=2s^3$,

$$t^3-\sqrt{5}\,t^2+(18+7\sqrt{5})\,t-1=0,$$

$$2s^3-(3+\sqrt{5})\,s^2+(3+\sqrt{5})\,s-1=0;$$

also $$\sqrt{2}\,x=\left(\frac{\sqrt{5}+1}{2}\right)^2(1+\sqrt[4]{4\kappa\kappa'})=\left(\frac{\sqrt{5}-1}{2}\right)^2(1+\sqrt[4]{4\lambda\lambda'}).$$

Forming the equations for t^2, t^4, and t^8,

$$t^6+(31+14\sqrt{5})\,t^4+(569+254\sqrt{5})\,t^2-1=0,$$

$$t^{12}-(803+360\sqrt{5})\,t^8+(646403+28908\sqrt{5})\,t^4-1=0,$$

$$t^{24}-3t^{16}+(835673068803+373724357760\sqrt{5})\,t^8-1=0;$$

so that $$J=-\frac{4}{27}\,\frac{(1-t^8)^3}{t^8}$$

$$=-\frac{4}{27}(835673068800+373724357760\sqrt{5})$$

$$=-2^6\times5\sqrt{5}\,(157\sqrt{5}+351)^2,$$

$$\gamma_2 = 3140+1404\sqrt{5},$$
$$\gamma_2+1 = 9\,(6\sqrt{5}+13)^2,$$
$$\gamma_2^2-\gamma_2+1 = 3\times 23\,(378+169\sqrt{5})^2,$$
$$\gamma_3 = (6\sqrt{5}+13)(378+169\sqrt{5})\sqrt{23}.$$

The corresponding modular functions are those of the 29th part of multiples of the periods, and Kiepert's L-equation for $n = 29$ with these values of γ_2 and γ_3 has a factor of the form L^4+aL^2+29.

$\Delta = 123 = 3\times 41$ can be solved by a combination of the modular equations of the 3rd and 41st order, or by using Hermite's method with $n = 23$.

$\Delta = 131$, a prime, not yet solved.

$\Delta = 139$, a prime, not yet solved.

$\Delta = 147 = 3\times 7^2$ can be solved by employing Klein's transformation of the 7th order (*Proc. Lond. Math. Soc.*, Vol. IX., p. 125; *Math. Ann.*, XIV., p. 143) with $J' = 0$.

$\Delta = 155 = 5\times 31$; combine the modular equations of the 5th and 31st order. Then, as for $\Delta = 115$, if

$$x^{12} = 4\kappa\lambda\,\kappa'\lambda',$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = \sqrt[4]{2}\,\sqrt{(x+x^3)}.$$

Then, in Russell's modular equation of the 31st order (*Proc. Lond. Math. Soc.*, Nov. 10, 1887),

$$(P^2-4Q)^2-4PR = 0,$$

we must put
$$P = \sqrt[4]{2}\,\sqrt{(x+x^3)}+1,$$
$$Q = \frac{x^3}{\sqrt{2}}+\sqrt[4]{2}\,\sqrt{(x+x^3)},$$
$$R = \frac{x^3}{\sqrt{2}};$$

which will lead to the required result.

$\Delta = 163$. Here $J = -\gamma_2^3,\quad J-1 = -27\gamma_3^2$,

where
$$\gamma_2 = 53360,\quad \gamma_3 = 185801\sqrt{163},$$
$$\gamma_2+1 = 3^2\times 7^2\times 11^2,\quad \gamma_2^2-\gamma_2+1 = 3\times 19^2\times 127^2\times 163;$$

and then
$$2s^3-4s^2+6s-1 = 0.$$

These values were inferred by approximate calculation from

Hermite's formula (*Équations Modulaires*, p. 48; H. J. S. Smith, *Report* (1865) *on the Theory of Numbers*, p. 374), the calculation being very much abbreviated from the consideration that γ_2+1 is the square of a number which is a multiple of 3.

The value of $\Delta=163$ appears to be the highest for which, according to Hermite's canon, the absolute invariant J is an integer, and so for the present we terminate at this point the series of values of Δ in Class A.

CLASS *B*.

$\Delta \equiv 7$, mod. 8.

This is the class, Hermite's class 4°, for which no simple numerical invariant has yet been discovered; and, according to Hermite and Joubert, the only modular function to seek to determine numerically is $\sqrt[4]{(\kappa\kappa')}$, or sometimes $\sqrt[12]{(\kappa\kappa')}$.

Jacobi's modular equations between u and v are not suitable for this purpose, but the $\kappa\lambda-\kappa'\lambda'$ equations of Mr. Robert Russell become immediately of the requisite form on putting $\lambda=\kappa'$, $\lambda'=\kappa$.

The corresponding complex multiplication formulas are given in the *Quar. Jour. of Math.*, Vol. XXII., p. 143, where Weierstrass's notation is employed.

Guided, however, by Hermite's *Équations Modulaires*, p. 44, we may in this Class 4°, employ a complex multiplier,

$$\frac{1}{M}=\tfrac{1}{4}(-\rho+\sqrt{\Delta}\,i),$$

where ρ is an odd integer, and then, following Mr. G. H. Stuart's method (*Quar. Journal of Math.*, Vol. XX., p. 38), we can express

$$y=\wp\frac{u}{M}\ \text{ in terms of }\ x=\wp u,$$

by means of an irrational formula; or, with Jacobi's notation, we can express

$$y=\operatorname{cn}\frac{u}{M}\ \text{ in terms of }\ x=\operatorname{cn}u;$$

which, in the simplest case of $\Delta=7$, and $\rho=1$, becomes

$$\frac{\sqrt{i}\,c-y}{\sqrt{i}\,c+y}=\sqrt[4]{(-ic)}\sqrt{\frac{1+x}{ic-x}};$$

where $$c=8+3\sqrt{7};$$

leading to the differential relation

$$\frac{dy}{\sqrt{(1-y^2\,.\,y^2+c^2)}}=\frac{\tfrac{1}{2}(-1+\sqrt{7}\,i)\,dx}{\sqrt{(1-x^2\,.\,x^2+c^2)}}$$

(*Proc. Camb. Phil. Soc.*, Vol. IV.); and generally, for any value of $\Delta = 8n-1$,

$$\frac{\sqrt{ic-y}}{\sqrt{ic+y}} = A\left(\frac{1+x}{ic-x}\right)^{\frac{1}{2}} \Pi \left\{\frac{\operatorname{cn} 2s\omega+x}{\operatorname{cn}(2s+1)\,\omega+x}\right\}^{\frac{1}{2}};$$

where $$\omega = (K+iK')/n,$$

connecting $x = \operatorname{cn} u$, and $y = \operatorname{cn} \frac{1}{4}(-1+\sqrt{\Delta}\, i)\, u$, and leading to the differential relation

$$\frac{dy}{\sqrt{(1-y^2 \,.\, y^2+c^2)}} = \frac{\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, dx}{\sqrt{(1-x^2 \,.\, x^2+c^2)}},$$

where $c = \kappa'/\kappa$; or, in Weierstrass's notation,

$$\frac{y-\wp(\omega_2+\frac{1}{2}\omega_3)}{y-\wp\,\frac{1}{2}\omega_3} = \Pi\left[\frac{x-\wp\left\{\omega_2+(2s+1)\,\omega_1/n\right\}}{x-\wp(\omega_2+2s\omega_1/n)}\right]^{\frac{1}{2}},$$

connecting $x = \wp u$, and $y = \wp\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, u$, and leading to the differential relation

$$\frac{dy}{\sqrt{(4y^3-g_2 y-g_3)}} = \frac{\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, dx}{\sqrt{(4x^3-g_2 x-g_3)}}.$$

The modular functions required in the general case are then the n^{th} parts of multiples of the periods, where

$$n = \tfrac{1}{8}(\Delta+\rho^2),$$

an integer; and then $$\Delta = 8n-\rho^2,$$

thus giving the interpretation of the formula for class 4°, p. 44, of the *Équations Modulaires*, in which

$$u^2 = \frac{1-v^4}{2iv^2}, \quad u^8 = 1-x,$$

in the modular equation of the n^{th} degree, connecting Jacobi's u and v.

$\Delta = 7$. From Gutzlaff's modular equation of the 7^{th} order,

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = 1,$$

we obtain, putting $\kappa = \lambda'$, $\kappa' = \lambda$,

$$2\sqrt[4]{\kappa\kappa'} = 1.$$

$\Delta = 15$. In this case Joubert (*Comptes Rendus*, t. 50) gives the value

$$\sqrt[4]{\kappa\kappa'} = \sin 18°,$$

for $$K'/K = \sqrt{15};$$

while $$\sqrt[4]{\kappa\kappa'} = \sin 54°,$$

for $$K'/K = \sqrt{(5 \div 3)}.$$

These values of $\sqrt[4]{(\kappa\kappa')}$ can also be obtained from Fiedler's modular equation for $n = 15$.

$\Delta = 23$. Putting $\lambda = \kappa'$, $\lambda' = \kappa$ in Mr. Russell's modular equation of the 23rd order, $$P^3 - 4R = 0,$$

or $$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + \sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'} - 1 = 0,$$

then $$2\sqrt[4]{\kappa\kappa'} + \sqrt[3]{4}\sqrt[6]{\kappa\kappa'} - 1 = 0,$$

or $$x^3 + x^2 - 1 = 0,$$

where $$x^{12} = 16\kappa\kappa';$$

the real root of this cubic being given by
$$\frac{1}{x} = \sqrt[3]{\left(\frac{3\sqrt{3}+\sqrt{23}}{6\sqrt{3}}\right)} + \sqrt[3]{\left(\frac{3\sqrt{3}-\sqrt{23}}{6\sqrt{3}}\right)}.$$

$\Delta = 31$. Putting $\lambda = \kappa'$, $\lambda' = \kappa$ in Mr. Russell's modular equation of the 31st order,
$$(P^2 - 4Q)^2 - 4PR = 0,$$

then $$P = 2\sqrt[4]{\kappa\kappa'} + 1, \quad Q = \sqrt{\kappa\kappa'} + 2\sqrt[4]{\kappa\kappa'}, \quad R = \sqrt{\kappa\kappa'};$$

and $$P = x^3 + 1, \quad Q = \tfrac{1}{4}x^6 + x^3, \quad R = \tfrac{1}{4}x^6,$$

if $$x^{12} = 16\kappa\kappa';$$

so that $$x^9 - 3x^6 + 4x^3 - 1 = 0,$$
$$(x^3 - 1)^3 = -x^3,$$
$$x^3 - 1 = -x,$$
$$x^3 + x - 1 = 0,$$

a cubic for x, the real root being
$$x = \sqrt[3]{\left(\frac{31+3\sqrt{3}}{6\sqrt{3}^{*}}\right)} - \sqrt[3]{\left(\frac{31-3\sqrt{3}}{6\sqrt{3}}\right)}.$$

$\Delta = 39 = 3\times 13$. The equation for $x = 2\sqrt[4]{\kappa\kappa'}$ is given by Joubert in the *Comptes Rendus*, t. 50, in the form
$$x^4 + 2x^3 + 4x^2 + 3x - 1 = 0,$$

so that $$(x^2 + x + \tfrac{3}{2})^2 = \tfrac{13}{4},$$

or $$2x = -1 + \sqrt{(2\sqrt{13} - 5)} = -1 + \{\tfrac{1}{2}(\sqrt{13} - 1)\}^{\frac{3}{2}}.$$

$\Delta = 47$. The modular equation of the 47^{th} order has been given by Hurwitz in the *Math. Ann.*, Vol. XVII., p. 69, in the form

$$\left\{2\left(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1\right)-\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\,\kappa'\lambda')}\right\}^2$$
$$= 8\left(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1\right)-7\sqrt[3]{16}\,\sqrt[6]{(\kappa\lambda\,\kappa'\lambda')}\,;$$

and Russell's form, given in the *Proc. Lond. Math. Soc.*, Vol. XIX., p. 111, is

$$(P^2-4Q)^3-4PR\,(7P^2+24Q)-128R^2 = 0.$$

Hence, if $K'/K = \sqrt{47}$, we have, putting

$$\lambda = \kappa', \quad \lambda' = \kappa,$$

$$\left(4\sqrt[4]{\kappa\kappa'}-\sqrt[3]{4}\,\sqrt[6]{\kappa\kappa'}-2\right)^2 = 16\sqrt{\kappa\kappa'}-7\sqrt[3]{16}\,\sqrt[3]{\kappa\kappa'}+8\,;$$

or, if $16\kappa\kappa' = x^{12}$,

$$(2x^3-x^2-2)^2 = 4x^6-7x^4+8,$$

or

$$x^5-2x^4+2x^3-x^2+1 = 0.$$

$$16\,\kappa\kappa' = y^6,$$

then

$$x = \sqrt{y},$$

and the quintic for y is

$$y^5+3y^2+2y-1 = 0,$$

a *Hauptgleichung* (Klein, *Icosaëder*) which has been solved by Prof. G. Paxton Young, the solution being given in the *American Journal of Mathematics*, Vol. x., p. 108.

The quintic has only one real root, which is

$$u_1+u_2+u_3+u_4,$$

where

$$u_1^5 = \tfrac{13}{500}(15+7\sqrt{5})+\tfrac{1}{625}\sqrt{\left\{\tfrac{47}{8}(21125+9439\sqrt{5})\right\}},$$
$$u_2^5 = \tfrac{13}{500}(15-7\sqrt{5})-\tfrac{1}{625}\sqrt{\left\{\tfrac{47}{8}(21125-9439\sqrt{5})\right\}},$$
$$u_3^5 = \tfrac{13}{500}(15-7\sqrt{5})+\tfrac{1}{625}\sqrt{\left\{\tfrac{47}{8}(21125-9439\sqrt{5})\right\}},$$
$$u_4^5 = \tfrac{13}{500}(15+7\sqrt{5})-\tfrac{1}{625}\sqrt{\left\{\tfrac{47}{8}(21125+9439\sqrt{5})\right\}}\,;$$

the other imaginary roots of the quintic being of the form

$$\epsilon u_1+\epsilon^2 u_2+\epsilon^3 u_3+\epsilon^4 u_4,$$

where ϵ is an imaginary fifth root of unity.

We might have employed the modular equation of the 47^{th} order,

given by Hurwitz, in the *Math. Ann.*, xvii., p. 69, of the form

$$\{2(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1)-\sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'}\}^2$$
$$= 8(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1)-7\sqrt[3]{16}\sqrt[6]{\kappa\lambda\kappa'\lambda'}.$$

Putting $\lambda = \kappa', \quad \lambda' = \kappa, \quad \text{or} \quad \kappa'\lambda' = \kappa\lambda = \kappa\kappa'$,

in this equation, then $K'/K = \sqrt{47}$;

and $(4\sqrt[4]{\kappa\kappa'}-\sqrt[3]{4}\sqrt[6]{\kappa\kappa'}-2)^2 = 16\sqrt{\kappa\kappa'}-7\sqrt[3]{16}\,\kappa\kappa'+8$;

or, if $16\kappa\kappa' = x^{12}$,

$$(2x^3-x^2-2)^2 = 4x^6-7x^4+8,$$

or

$$x^6-2x^4+2x^3-x^2+1=0;$$

and if $x = \sqrt{y}$,

$$y^5+3y^2+2y-1=0,$$

as before.

$\Delta = 55 = 5\times 11$. Combining Schröter's or Russell's modular equations of the 5^{th} and 11^{th} orders,

$$\kappa'\lambda+\kappa\lambda'+2\sqrt[3]{(4\kappa\lambda\,\kappa'\lambda')} = 1,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')} = 1;$$

putting

$$4\kappa\lambda\,\kappa'\lambda' = x^{12},$$

then, from the equation of the 5^{th} order,

$$\kappa'\lambda+\kappa\lambda' = 1-2x^4,$$

$$\kappa\lambda+\kappa'\lambda' = 2x^2-x^6;$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \sqrt{2}\,x$$
$$= 1-2x^2,$$

from the equation of the 11^{th} order; so that

$$2x^2+\sqrt{2}\,x-1=0,$$

$$x = \frac{\sqrt{5}-1}{2\sqrt{2}}.$$

Then, as before, in $\Delta = 115$,

$$\kappa\kappa'+\lambda\lambda' = x^2(2-5x^4+2x^8),$$

$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \sqrt{2}\,x\,(1-x^4)$$
$$=\frac{7-\sqrt{5}}{8}$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \frac{\sqrt{5}\sqrt{(10\sqrt{5}-18)}}{8}.$$

$\Delta = 63 = 3^2 \times 7$. Performing the cubic transformation on the modulus for

$$K'/K = \sqrt{7},$$

we shall obtain the required modulus.

Putting $$x = 2\sqrt[4]{\kappa\kappa'},$$

the equation is written by Joubert (*Comptes Rendus*, t. 50)

$$(x^2-x+5)^2-21\,(x-1)^2 = 0,$$

whence x can readily be determined.

For $\Delta = 71$ and 79, consult Dr. E. W. Fiedler, *Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen*; Zürich, 1885.

Putting $$2\sqrt[4]{\kappa\kappa'} = x,$$

then in Fiedler's notation

$$Z_1 = x \mp 1,$$
$$Z_2 = \tfrac{1}{4}x^2 \mp x, \quad Z_2' = \pm 2x+1,$$
$$Z_3 = \mp \tfrac{1}{4}x^3;$$

and for $\Delta = 71$, Fiedler's equation for x is

$$(x-1)^9+x^3\left\{(2x+1)^3+9\,(x+1)^2\,(2x+1)^2+21\,(x+1)^4\,(2x+1)\right.$$
$$\left.+12\,(2x+1)^6\right\}-(x-1)^2\,x^4\left\{6\,(2x+1)+7\,(x-1)^2\right\}+x^6 = 0;$$

while for $\Delta = 79$, the equation for x is

$$(-2x+1)^5-(x+1)\,x^2\left\{(x+1)^6+10\,(x+1)^4\,(-2x+1)\right.$$
$$\left.+28\,(x+1)^2\,(-2x+1)^2+21\,(-2x+1)^3\right\}$$
$$-x^4\left\{7\,(x+1)^4+26\,(x+1)^2\,(-2x+1)+24\,(-2x+1)^2\right\}$$
$$+8\,(x+1)\,x^6 = 0;$$

equations of the 9th and 8th degree respectively.

$\Delta = 87 = 3\times 29$ can be solved by the combination of the modular equations of the 3rd and 29th order, but this last modular equation has not yet been calculated by Mr. Russell or others in a convenient form, the form given by Schröter being unsuitable for our purposes.

$\Delta = 95 = 5\times 19$.

Putting, as before $$4\kappa\lambda\kappa'\lambda' = x^{12},$$

and $$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \sqrt{2}\,x,$$

from the modular equation of the 5th order, then, in Mr. Russell's notation,

$$P = \sqrt{2}\, x - 1,$$
$$Q = \tfrac{1}{2}x^6 - \sqrt{2}\, x,$$
$$R = -\tfrac{1}{2}x^6,$$

which, substituted in his modular equation of the 19th order, will give an equation of the 12th degree for x.

Class C.

$$\Delta \equiv 1, \text{ mod. } 4.$$

This is Hermite's Class 1° (*Équations Modulaires*, p. 44), and the absolutely simplest numerical invariant, according to Hermite, is

$$a = -\frac{(x+1)^4}{x\,(x-1)^2},$$

which, on replacing x by $1-\kappa^{-2}$, becomes

$$a = \frac{(1-4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2}, \quad a+16 = \frac{(1+4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2};$$

so that, putting $\beta = \tfrac{1}{2}\sqrt{a}, \quad \gamma = \tfrac{1}{2}\sqrt{(a+16)},$

$$\beta = \frac{1}{2\kappa\kappa'} - 2\kappa\kappa', \quad \gamma = \frac{1}{2\kappa\kappa'} + 2\kappa\kappa';$$

and, according to Hermite, β or γ are in a great many cases integers.

With the complex multiplier

$$\frac{1}{M} = \tfrac{1}{2}(-\rho + \sqrt{\Delta}\, i),$$

where ρ is an odd integer, we can connect

$$y = \text{cn}\,\frac{u}{M} \text{ with } x = \text{cn}\, u,$$

by means of an irrational relation in Mr. G. H. Stuart's manner, as explained in the *Quarterly Journal of Mathematics*, Vol. XXII., p. 147; and then the modular functions involved in these relations are functions of the n^{th} part of multiple of the periods, where

$$n = \tfrac{1}{2}(\Delta + \rho^2),$$

an integer; so that $\Delta = 2n - \rho^2,$

as in Hermite's formulas (*Équations Modulaires*, p. 44).

Or, in Weierstrass's notation, we can connect

$$x = \wp u \text{ and } y = \wp \tfrac{1}{2}(-1+\sqrt{\Delta}\, i),$$

where $\Delta = 4n+1, \quad m = 2n+1,$

by means of the relation

$$\frac{y-\wp\frac{1}{2}\omega_2'}{y-\wp\frac{1}{2}\omega_2} = \left(\frac{x-e_1}{x-e_2}\right)^2 \Pi\, \frac{x-\wp\{\omega_2-(4r+1)\,\omega_2/m\}}{x-\wp\,(4r+1)\,\omega_2/m},$$

a transformation of the order $n+\frac{1}{2} = \frac{1}{2}m$.

$\Delta = 1$. Then $\kappa' = \kappa$, and $\kappa' = \kappa = \sin 45^\circ$;

also $J = 1, \quad J-1 = 0$, so that $g_3 = 0$.

Then $\beta = 0, \quad \alpha = 0, \quad 2\kappa\kappa' = 1.$

$\Delta = 5$. Here $\beta = 2^2$, $\alpha = 2^6$; obtained from Russell's modular equation of the 5th order, with $\lambda = \kappa'$, $\lambda' = \kappa$. Then

$$2\kappa\kappa' + 2\sqrt[3]{4\kappa^2\kappa'^2} - 1 = 0,$$

or, putting $\sqrt[3]{2\kappa\kappa'} = x,$

$$x^3+2x^2-1 = 0, \quad (x+1)(x^2+x-1) = 0, \quad x = \tfrac{1}{2}(\sqrt{5}-1),$$

$$2\kappa\kappa' = \sqrt{5}-2 = \left(\frac{\sqrt{5}-1}{2}\right)^3 \text{ (Abel).}$$

$\Delta = 9 = 3^2$. Here

$$\alpha = 2^8 \times 3 \text{ (Kronecker)},$$

$$\beta = 8\sqrt{3}, \quad \gamma = 14,$$

or, putting $2\kappa\kappa' = z,$

$$z^2-14z-1 = 0, \quad z = 7-4\sqrt{3} = (2-\sqrt{3})^2 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4.$$

$\Delta = 13$. Here $\beta = 36$, $\alpha = 2^6 \times 3^4$ (Kronecker);

$$2\kappa\kappa' = 5\sqrt{13}-18 = \left(\frac{\sqrt{13}-3}{2}\right)^3.$$

$\Delta = 17$. Taking Mr. Russell's modular equation for the 17th order, and putting $\lambda = \kappa'$, $\lambda' = \kappa$, there results the equation in $z = 2\kappa\kappa'$,

$$(z-1)(z^2-36z-1)^2\,(z^4-80z^3-98z^2-80z+1) = 0.$$

The factor $z-1 = 0$ corresponds to $\Delta = 1$, and the factor

$z^2-36z-1 = 0$ to $\Delta = 13$; so that

$$z^4-80z^3-98z^2-80z+1 = 0;$$

$$\gamma^2 - 80\gamma - 100 = 0,$$

$$\gamma = 10\sqrt{17} + 40,$$

$$\beta = \frac{1}{z} - z = 4\sqrt{(206 + 50\sqrt{17})},$$

$$\alpha = 4\beta^2 = 2^7(25\sqrt{17} + 103).$$

$\Delta = 21 = 3 \times 7$. Then (Kronecker)

$$\alpha = 2^2 \times 3^3 (\sqrt{3} + 1)^8,$$

$$2\kappa\kappa' = \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^4 \left(\frac{3 - \sqrt{7}}{\sqrt{2}}\right)^2, \text{ for } K'/K = \sqrt{21},$$

$$2\lambda\lambda' = \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^4 \left(\frac{3 + \sqrt{7}}{2}\right)^2, \text{ for } \Lambda'/\Lambda = \sqrt{7} \div 3.$$

$\Delta = 25 = 5^2$. Then

$$\beta = 2^4 \times 3^2 \times \sqrt{5}, \quad \gamma = 322 = 2 \times 7 \times 23;$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5} - 1}{2}\right)^{12}.$$

$\Delta = 29$. The form of the $(\kappa\lambda, \kappa'\lambda')$ modular equation, according to Mr. Russell, will be

$$P^{15} + R\,(AP^{12} + BP^{10}Q + CP^8Q^2 + DP^6Q^3 + EP^4Q^4 + FP^2Q^5 + GQ^6)$$
$$+ R^2\,(HP^9 + JP^7Q + KP^5Q^2 + LP^3Q^3 + MPQ^4)$$
$$+ R^3\,(NP^6 + OP^4Q + SP^2Q^2 + TQ^3)$$
$$+ R^4\,(UP^3 + VPQ) + WR^5 = 0,$$

connecting $\quad P = x + y - 1, \quad Q = xy - x - y, \quad R = -xy,$

where $\quad x = \kappa\lambda, \quad y = \kappa'\lambda';$

and $A, B, C, \ldots U, V, W$ are numerical coefficients, the values of which have not yet been determined.

Putting $\lambda = \kappa'$, $\lambda' = \kappa$, and then $z = 2\kappa\kappa'$; $P = z - 1$, $Q = \frac{1}{4}z^2 - z$, $R = -\frac{1}{4}z^2$; and, by analogy with the preceding cases, the resulting equation will have a factor $z + 1$, and other factors corresponding to previous values of Δ.

Pending the determination of the numerical coefficients in Russell's modular equation of the 29th order, let us determine the numerical values of the modular functions for $\Delta = 29$ in Hermite's manner for his Class 1°, by means of the modular equation for $n = 19$.

Then the corresponding values of Δ are given by

$$2n-\rho^2 = 37,\ 29,\ 13;$$

of which the solutions for $\Delta = 37$ and 13 are simple and well known.

Putting, in Hermite's notation,

$$\kappa\lambda = u^4v^4 = w^4,$$

then, since
$$\kappa^2 = u^8 = x,$$

and
$$u^4 = \frac{v^4-1}{v^4+1},$$

$$\kappa = \frac{\lambda-1}{\lambda+1},\quad \lambda = \frac{1+\kappa}{1-\kappa};$$

$$w^4 = \kappa\lambda = \sqrt{x}\,\frac{1+\sqrt{x}}{1-\sqrt{x}};$$

$$\kappa'\lambda' = 2i\sqrt[4]{x}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} = 2i\sqrt{\kappa\lambda}.$$

Then, in Russell's notation, with

$$\kappa\lambda = w^4,\quad \kappa'\lambda' = 2iw^2,\quad \epsilon^4 = -1,$$

$$P = w\,(t+\epsilon\sqrt{2}),$$

$$Q = w^2\,(\epsilon\sqrt{2}\,t-1),$$

$$R = -\epsilon\sqrt{2}\,w^2;$$

where
$$t = w-\frac{1}{w} = \sqrt[4]{\kappa\lambda}-\frac{1}{\sqrt[4]{\kappa\lambda}}.$$

Then
$$t^2+2 = w^2+w^{-2}$$

$$= \frac{1+x}{\sqrt[4]{x}\,\sqrt{(1-x)}} = \epsilon\sqrt[4]{\alpha} = (1+i)\sqrt{\beta},\quad \epsilon^4 = -1.$$

With Russell's notation, for $n = 19$,

$$P = w^2+\epsilon\sqrt{2}\,w-1 \quad = \quad w\,(t+1+i),$$

$$Q = \epsilon\sqrt{2}\,w^3-w^2-\epsilon\sqrt{2}\,w = \quad w^2\,(t+it-1),$$

$$R = \quad -\epsilon\sqrt{2}\,w^3 \quad = -w^3\,(1+i);$$

and substituting in

$$P^6-112P^3R+256QR = 0,$$

we obtain

$$(t+1+i)^6+112\,(1+i)\,(t+1+i)^3-256\,(1+i)\,(t+i\ \ -1) = 0,$$

or

$$t^6+5t^4+92t^3-20t+28+i\,(5t^4+20t^3+132t^2-64t+476)=0.$$

This equation has the factor

$$t^2+8t+16-18i=0,$$

giving $$t=-4\pm 3\sqrt{2}\,\epsilon$$

$$=-1+3i, \text{ or } -7-3i;$$

and therefore $$\beta=6^2 \text{ or } 42^2,$$

corresponding to $$\Delta=13 \text{ or } 37.$$

The remaining cubic factor is

$$t^3-(3-5i)\,t^2+(8-2i)\,t-14+14i=0;$$

so that

$$t^6+(32+26i)\,t^4+(116+192i)\,t^2+392i=0;$$

and since $$t^2=(1+i)\sqrt{\beta-2},$$

therefore $$\beta^{\frac{3}{2}}+26\beta+44\beta^{\frac{1}{2}}+56=0,$$

a cubic equation with discriminant $32^2\times 29\div 27$; then

$$\beta^3-588\beta^2-976\beta-3136=0;$$

giving $$\beta=\frac{1}{2\kappa\kappa'}-2\kappa\kappa';$$

and putting $z=2\kappa\kappa'$, the equation for z is

$$z^6+588z^5-979z^4+1960z^3+979z^2+588z-1=0.$$

The knowledge of this factor will be of great assistance in the determination of the numerical factors A, B, C, ... U, V, W in the modular equation of the 29$^{\text{th}}$ order.

$\Delta=33=3\times 11$. Here

$$a=2^4\times 3\,(300+52\sqrt{33})^2,$$

$$2\kappa\kappa'=\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6.$$

These values are obtained by the combination of Schröter's or Russell's modular equation of the 11$^{\text{th}}$ order,

$$\sqrt{\kappa\lambda'}+\sqrt{\kappa'\lambda}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')}=1 \quad\text{.................(1)}$$

with the equation of the 3$^{\text{rd}}$ order,

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1 \quad\text{..........................(2).}$$

Putting $$4\kappa\lambda\,\kappa'\lambda' = x^6,$$

then $$\sqrt{\kappa\lambda'} + \sqrt{\kappa'\lambda} = 1-2x,$$

$$\kappa\lambda' + \kappa'\lambda = 1-4x+4x^2-x^3,$$

and $$\kappa\lambda + \kappa'\lambda' = 1-x^3,$$

$$(\kappa'\lambda' - \kappa\lambda)^2 = 1-2x^3\,;$$

so that $$1-2x^3+(1-4x+4x^2-x^3)^2 = (\kappa'\lambda'-\kappa\lambda)^2+(\kappa\lambda'+\kappa'\lambda)^2 = 1,$$

or $$(1-4x+4x^2-x^3)^2 = 2x^3,$$

or $$x^6-8x^5+24x^4-36x^3+24x^2-8x+1 = 0,$$

a reciprocal sextic, having the factors

$$(x^2-4x+1)(x^4-4x^3+7x^2-4x+1) = 0.$$

Putting $$x+\frac{1}{x} = y,$$

then $$y^3-8y^2+21y-20 = 0,$$

$$(y-4)(y^2-4y+5) = 0\,;$$

so that $y = 4,\ 2\pm i$.

Taking the real root $$x+\frac{1}{x} = 4,$$

$$x = 2-\sqrt{3} = \tfrac{1}{2}(\sqrt{3}-1)^2\,;$$

then $$\kappa\lambda'+\kappa'\lambda = \sqrt{2}\,x^{\frac{3}{2}} = \tfrac{1}{2}(\sqrt{3}-1)^3,$$

$$\kappa\lambda+\kappa'\lambda' = 1-x^3 = \tfrac{5}{2}(\sqrt{3}-1)^3,$$

$$(\kappa'+\kappa)(\lambda'+\lambda) = 3(\sqrt{3}-1)^3,$$

$$(\kappa'-\kappa)(\lambda'-\lambda) = 2(\sqrt{3}-1)^3\,;$$

$$(1+2\kappa\kappa')(1+2\lambda\lambda') = 9(\sqrt{3}-1)^6,$$

$$(1-2\kappa\kappa')(1-2\lambda\lambda') = 4(\sqrt{3}-1)^6,$$

$$4(\kappa\kappa'+\lambda\lambda') = 5(\sqrt{3}-1)^6,$$

$$64\,\kappa\lambda\,\kappa'\lambda' = \tfrac{1}{4}(\sqrt{3}-1)^{12},$$

$$16(\kappa\kappa'-\lambda\lambda')^2 = \tfrac{99}{4}(\sqrt{3}-1)^{12},$$

$$4(\lambda\lambda'-\kappa\kappa') = \tfrac{3}{2}\sqrt{11}(\sqrt{3}-1)^6,$$

$$8\kappa\kappa' = \tfrac{1}{2}(10-3\sqrt{11})(\sqrt{3}-1)^6,$$

$$2\kappa\kappa' = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \text{ for } K'/K = \sqrt{33}\,;$$

$$2\lambda\lambda' = \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \text{ for } \Lambda'/\Lambda = \sqrt{(11 \div 3)}.$$

$$\begin{aligned}
\beta = \tfrac{1}{2}\sqrt{a} &= \frac{1}{2\kappa\kappa'} - 2\kappa\kappa' \\
&= (10+3\sqrt{11})\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 - (10-3\sqrt{11})\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \\
&= 156\sqrt{11}+300\sqrt{3} \\
&= 4\sqrt{3}\,(75+13\sqrt{33});
\end{aligned}$$

$$a = 2^4 \times 3\,(75+13\sqrt{33})^2;$$

$$\begin{aligned}
\gamma &= \frac{1}{2\kappa\kappa'} + 2\kappa\kappa' \\
&= 520+90\sqrt{33} \\
&= 10\,(52+9\sqrt{33});
\end{aligned}$$

$$a+16 = 2^4 \times 5^2\,(52+9\sqrt{33})^2.$$

Similarly for $\lambda\lambda'$ and the values of a', corresponding to

$$\Lambda'/\Lambda = \sqrt{(11 \div 3)},$$
$$a' = 2^4 \times 3\,(75-13\sqrt{33})^2,$$
$$a'+16 = 2^4 \times 5^2\,(52-9\sqrt{33}).$$

$\Delta = 37$. Here $a = 2^6 \times 3^4 \times 7^4$,

$$\beta = 2^3 \times 3^2 \times 7^2, \qquad \gamma = 2 \times 5 \times 29\sqrt{37};$$

obtained from Hermite's *Théorie des Équations Modulaires*, Note, p. 50; also by Kronecker, *Berlin Sitz.* 1862; then

$$2\kappa\kappa' = (\sqrt{37}-6)^2, \quad \frac{1}{2\kappa\kappa'} = (\sqrt{37}+6)^2.$$

$\Delta = 41$. Not yet solved; but a quartic equation for a, β, or γ must be expected, as $p = 4$.

$\Delta = 45 = 3^2 \times 5$. Here, by means of the cubic transformation in $\Delta = 5$,

$$a = 2^6\,(17+10\sqrt{3})^4, \quad a+16 = 80\,(527+304\sqrt{3})^2,$$
$$\beta = 2^3\,(17+10\sqrt{3})^2, \quad \gamma = 2\sqrt{5}\,(527+304\sqrt{3});$$
$$2\kappa\kappa = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4, \quad K'/K = \sqrt{(45)};$$
$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4, \quad \Lambda'/\Lambda = \sqrt{(9 \div 5)}.$$

$\Delta = 49 = 7^2$. Here, by the 7^{ic} transformation on $\Delta = 1$,

$$\alpha = 2^5 \times 3^4 (3+\sqrt{7})^6 \sqrt{7},$$

Then
$$\sqrt[4]{\kappa}\,\sqrt[4]{\kappa'} = \sqrt[8]{2},$$

$$\kappa\kappa' = \left(\frac{\sqrt{7}+1-\sqrt{2}\,\sqrt[4]{7}}{2\sqrt{2}}\right)^{12}$$

(Kronecker, *Berlin Sitz.* 1862 : G. H. Stuart, *Quart. Journal o Math.*, Vol. xx.).

$\Delta = 53$, a prime number not yet solved. Here $p = 3$, so that a cubic for α must be expected.

$\Delta = 57 = 3 \times 19$. We combine the modular equations of the 3rd and 19th order, and put

$$4\kappa\lambda\,\kappa'\lambda' = y^4.$$

Then from the equation of the 3rd order

$$\kappa'\lambda + \kappa\lambda' = 1 - y^2,$$

$$\kappa\lambda + \kappa'\lambda' = \sqrt{2}\,y,$$

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = \sqrt{(y^2 + \sqrt{2}\,y)}.$$

With Russell's notation, we have

$$P = \sqrt{(y^2+\sqrt{2}\,y)} - 1,$$

$$Q = \tfrac{1}{2}y^2 - \sqrt{(y^2+\sqrt{2}\,y)},$$

$$R = -\tfrac{1}{2}y^2;$$

and then, substituting in his modular equation of the 19th order, we obtain an equation for y, which is reciprocal when rationalised; and then putting

$$y - \frac{1}{y} = \sqrt{2}\,v,$$

we obtain
$$v^5 + 5v^4 - 46v^3 + 706v^2 - 611v + 169 = 0,$$

or
$$(v+13)(v^4 - 8v^3 + 58v^2 - 48v + 13) = 0,$$

or
$$(v+13)\left\{(v^2 - 4v + 3)^2 + (6v+2)^2\right\} = 0.$$

Taking
$$v = -13,$$

$$\frac{1}{y} - y = 13\sqrt{2},$$

$$\frac{1}{y} + y = 3\sqrt{38};$$

$$\kappa\kappa'+\lambda\lambda' = \sqrt{2}\, y\,(1-y^2) = -\,2y^2 v$$
$$= 26y^2;$$
$$2\sqrt{\kappa\lambda\,\kappa'\lambda'} = y^2;$$
$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = 3\sqrt{3}\,y,$$
$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = 5y.$$

But $$2y = \sqrt{2}\,(3\sqrt{19}-13),$$

so that $$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^2, \quad K'/K = \sqrt{57},$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^2, \quad \Lambda'/\Lambda = \sqrt{(19\div 3.)}$$

$\Delta = 61$. A prime number, not yet solved, but depending in Hermite's manner on $n = 31$.

$\Delta = 65 = 5\times 13$. Combine the modular equations of the 5th and 13th orders; then, if we put

$$4\kappa\lambda\kappa'\lambda' = x^6,$$

we obtain from the equation of the 5th order,

$$\kappa'\lambda+\kappa\lambda' = 1-2x^3,$$
$$\kappa\lambda+\kappa'\lambda' = 2x-x^4;$$

and in the equation of the 13th order, with Russell's notation,

$$P = -\,1+2x-x^4,$$
$$Q = -\,2x+x^4-\tfrac{1}{4}x^6,$$
$$R = -\,\tfrac{1}{4}x^6.$$

Substituting these values of P, Q, and R in the modular equation of the 13th order, and dividing out the factors $x+1$, $x^2\pm x+1$, we are left with a reciprocal equation for x, which, on putting

$$\frac{1}{x}+x = y,$$

becomes $$y^2-5y-10 = 0,$$

so that $$y = \tfrac{1}{2}\,(\sqrt{65}+5);$$

whence $\kappa\kappa'$ and $\lambda\lambda'$ can be determined.

$\Delta = 69 = 3\times 23$. Combine Schröter's, Hurwitz's, or Russell's modular equation of the 23rd order,

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[12]{(256\kappa\lambda\kappa'\lambda')} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

with Jacobi's equation of the 3rd order,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

Putting $\qquad 4\kappa\lambda\kappa'\lambda'=x^{12},$

then $\qquad \sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=1-\sqrt{2}\,x,$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3,$$

$$\kappa\lambda+\kappa'\lambda'=(1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3)^2-x,$$

and $\qquad \kappa'\lambda+\kappa\lambda'=1-x^3,$

$$(\kappa'\lambda-\kappa\lambda')^2=1-2x^6,$$

therefore $\qquad \kappa\lambda+\kappa'\lambda'=\sqrt{2}\,x^3,$

or $\qquad (1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3)^2-x^6=\sqrt{2}\,x^3,$

or $\qquad x^6-4\sqrt{2}\,x^5+12x^4-10\sqrt{2}\,x^3+12x^2-4\sqrt{2}\,x+1=\sqrt{2}\,x^3,$

a reciprocal sextic for x.

Put $\qquad \dfrac{1}{x}+x=y,$

then $\qquad y^3-4\sqrt{2}\,y^2+9y=3\sqrt{2}.$

The equation

$$y^3-4\sqrt{2}\,y^2+9y-3\sqrt{2}=0$$

has the factors

$$(y-\sqrt{2})\,(y^2-3\sqrt{2}\,y+3)=0,$$

and therefore the roots $\sqrt{2}$, $\dfrac{3\pm\sqrt{3}}{\sqrt{2}}$.

For x to be real, y must be greater than 2, and therefore we must put

$$\frac{1}{x}+x=\frac{3+\sqrt{3}}{\sqrt{2}}.$$

Then $\qquad \kappa\lambda+\kappa'\lambda'=\sqrt{2}\,x^3,$

$$\kappa'\lambda+\kappa\lambda'=1-x^6;$$

and, multiplying these equations together,

$$\kappa\kappa'+\lambda\lambda'=\sqrt{2}\,x^3\,(1-x^6),$$

also $\qquad 2\sqrt{\kappa\lambda\kappa'\lambda'}=x^6,$

so that $$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = x^3\sqrt{\left\{\sqrt{2}\left(\frac{1}{x^3}-x^3\right)+1\right\}},$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = x^3\sqrt{\left\{\sqrt{2}\left(\frac{1}{x^3}-x^3\right)-1\right\}}.$$

$\Delta = 73$. A prime number not yet solved; but depending in Hermite's manner on $n = 37$. Since $p = 2$, we must expect a quadratic for a.

$\Delta = 77 = 7\times 11$. Combine the modular equation of the 11th order,

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')} = 1,$$

with Gutzlaff's equation, of the 7th order,

$$\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'} = 1,$$

by putting $$4\kappa\lambda\kappa'\lambda' = x^{12}.$$

Then $$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = 1-2x^2,$$

$$\kappa\lambda+\kappa'\lambda' = 1-4x^2+4x^4-x^6,$$

and $$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'} = 1-\sqrt{2}\,x^3,$$

$$\kappa'\lambda+\kappa\lambda' = 1-2\sqrt{2}\,x^3+x^6.$$

But $$(\kappa\lambda+\kappa'\lambda')^2+(\kappa'\lambda+\kappa\lambda')^2 = 1+4\kappa\lambda\kappa'\lambda',$$

and therefore

$$(1-4x^2+4x^4-x^6)^2+(1-2\sqrt{2}\,x^3+x^6) = 1+x^{12},$$

or $$(1-2\sqrt{2}\,x^3+x^6)^2 = 1+x^{12}-(1-4x^2+4x^4-x^6)^2$$

$$= 8x^2-24x^4+34x^6-24x^8+8x^{10}$$

$$= 2x^2(2-3x^2+2x^4)^2.$$

$$1-2\sqrt{2}\,x^3+x^6 = 2\sqrt{2}\,x-3\sqrt{2}\,x^3+2\sqrt{2}\,x^5,$$

or $$x^6-2\sqrt{2}\,x^5+\sqrt{2}\,x^3-2\sqrt{2}\,x+1 = 0,$$

a reciprocal sextic in x, which, putting

$$\frac{1}{x}+x = y,$$

becomes $$y^3-2\sqrt{2}\,y^2-3y+5\sqrt{2} = 0,$$

$$(y-\sqrt{2})\,(y^2-\sqrt{2}\,y-5) = 0,$$

having roots $\sqrt{2}$, $\dfrac{1 \pm \sqrt{11}}{\sqrt{2}}$, of which the root greater than 2 must be chosen.

Then
$$\kappa\lambda + \kappa'\lambda' = 1 - 4x^2 + 4x^4 - x^6,$$
$$\kappa'\lambda + \kappa\lambda' = 2\sqrt{2}\,x - 3\sqrt{2}\,x^3 + 2\sqrt{2}\,x^5,$$
$$\kappa\kappa' + \lambda\lambda' = \sqrt{2}\,x^6 \left\{ \frac{1}{x^3} - x^3 - 4\left(\frac{1}{x} - x\right) \right\} \left\{ 2\left(\frac{1}{x^2} + x^2\right) - 3 \right\},$$
$$2\sqrt{\kappa\lambda\kappa'\lambda'} = x^6;$$

whence $\sqrt{\kappa\kappa'}$ and $\sqrt{\lambda\lambda'}$; and then $2\kappa\kappa'$ and $2\lambda\lambda'$.

$\Delta = 81 = 3^4$. Obtained from $\Delta = 3$.

$\Delta = 85 = 5 \times 17$. Taking the modular equation of the 5th order, and putting
$$4\kappa\lambda\kappa'\lambda' = x^6,$$
then
$$x = \tfrac{1}{2}(\sqrt{85} - 9)$$
is a value implied from the approximate solutions given by Professor H. J. S. Smith, in the *Report on the Theory of Numbers*, 1865, *to the British Association*, p. 374.

Then
$$\kappa'\lambda + \kappa\lambda' = 1 - 2x^2,$$
$$\kappa\lambda + \kappa'\lambda' = 2x - x^3;$$

so that, by multiplication,
$$\kappa\kappa' + \lambda\lambda' = x(2 - x^2)(1 - 2x^2)$$
$$= x^3(2x^{-2} - 5 + 2x^2)$$
$$= 161\,x^3;$$
$$-\kappa\kappa' + \lambda\lambda' = 72\sqrt{5}\,x^3;$$
$$2\kappa\kappa' = \left(\frac{\sqrt{5} - 1}{2}\right)^{12} x^3;$$
$$2\lambda\lambda' = \left(\frac{\sqrt{5} + 1}{2}\right)^{12} x^3;$$
$$2\kappa\kappa' = \left(\frac{\sqrt{5} - 1}{2}\right)^{12} \left(\frac{\sqrt{85} - 9}{2}\right)^3, \quad K'/K = \sqrt{85},$$
$$2\lambda\lambda' = \left(\frac{\sqrt{5} + 1}{2}\right)^{12} \left(\frac{\sqrt{85} - 9}{2}\right)^3, \quad \Lambda'/\Lambda = \sqrt{(17 \div 5)}.$$

$\Delta = 89$. A prime number, not yet solved.

$\Delta = 93 = 3 \times 31$. From the modular equation of the 3rd order,

$$\sqrt{\kappa'\lambda} + \sqrt{\kappa\lambda'} = 1,$$

putting

$$\sqrt[4]{4\kappa\lambda\,\kappa'\lambda'} = x,$$

$$\kappa'\lambda + \kappa\lambda' = 1 - 2x^2,$$

$$(\kappa\lambda + \kappa'\lambda')^2 = 1 + 4\kappa\lambda\kappa'\lambda' - (\kappa'\lambda + \kappa\lambda')^2$$

$$= 1 + 4x^4 - (1 - 2x^2)^2 = 4x^2,$$

$$\kappa\lambda + \kappa'\lambda' = 2x;$$

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = \sqrt{(2x + 2x^2)}$$

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = \sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}}.$$

Then, in Russell's notation,

$$P = 1 + \sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}},$$

$$Q = x + \sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}},$$

$$R = x,$$

$$P^2 - 4Q = 1 - 2x + \sqrt{(2x + 2x^2)} - 2\sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}};$$

and the modular equation of the 31st order,

$$(P^2 - 4Q)^2 - 4PR = 0,$$

becomes

$$\left[1 - 2x + \sqrt{(2x + 2x^2)} - 2\sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}}\right]^2$$
$$- 4x - 4x\sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}} = 0;$$

and, rationalising,

$$1 + 2x + 6x^2 + 2(3 - 2x)\sqrt{(2x + 2x^2)}$$
$$= 4\{1 - x + \sqrt{(2x + 2x^2)}\}\sqrt{\{\sqrt{(2x + 2x^2)} + 2x\}},$$

or

$$1 - 20x - 8x^2 - 72x^3 + 68x^4 = 4(1 + 12x - 18x^2 + 12x^3)\sqrt{(2x + 2x^2)},$$

or

$$1 - 72x - 416x^2 - 4048x^3 + 12680x^4 - 8096x^5 - 1664x^6 - 576x^7 + 16x^8 = 0,$$

which, on putting $\sqrt{2}\,x = y$, becomes

$$1-36\sqrt{2}\,y-208y^2-1012\sqrt{2}\,y^3+3170y^4 -1012\sqrt{2}\,y^5-208y^6-36\sqrt{2}\,y^7+y^8 = 0,$$

a reciprocal equation for y.

Now, put
$$y+\frac{1}{y} = \sqrt{2}\,v,$$
then
$$y^2+\frac{1}{y^2} = 2v^2-2,$$
$$y^3+\frac{1}{y^3} = 2\sqrt{2}\,v^3-3\sqrt{2}\,v,$$
$$y^4+\frac{1}{y^4} = 4v^4-8v^2+2;$$
so that
$$v^4-36v^3-106v^2-452v+897 = 0;$$
$$(v-39)(v^3+3v^2+11v-23) = 0.$$

Taking $v = 39$, then
$$\frac{1}{y}+y = 39\sqrt{2},$$
$$\frac{1}{y}-y = 7\sqrt{2}\sqrt{31},$$
$$2y = \sqrt{2}\,(39-7\sqrt{31}),$$
$$2x = 39-7\sqrt{31};$$
$$\kappa\kappa'+\lambda\lambda' = 2x\,(1-2x^2)$$
$$= 2xy\left(\frac{1}{y}-y\right)$$
$$= 14\sqrt{31}\,y^2,$$
$$2\sqrt{\kappa\lambda\,\kappa'\lambda'} = \qquad y^2;$$
$$-\kappa\kappa'+\lambda\lambda' = 45\sqrt{3}\,y^2,$$
$$2\kappa\kappa' = (14\sqrt{31}-45\sqrt{3})\,y^2$$
$$= \left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^3\left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2,$$

for $$\mathrm{K}'/\mathrm{K} = \sqrt{93};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^2 \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2.$$

for $$\Lambda'/\Lambda = \sqrt{(31 \div 3)}.$$

$\Delta = 97$. Guided by the approximate numerical values given by Kronecker, and quoted by Smith in the 1865 *Report on the Theory of Numbers*, p. 374, we infer that

$$a = 33210\sqrt{97}+327078;$$

a being given by a quadratic equation, since $p = 2$ for the determinant -97.

$\Delta = 101$, a prime number. Since $p = 7$ for the determinant -101, we must expect an irreducible equation of the 7^{th} degree for the determination of a.

$\Delta = 105 = 3 \times 5 \times 7$, a number composed of three prime factors, the earliest number of the kind to be encountered.

The solution of the modular equation in this case has been given by Kronecker in the *Berlin Sitzungsberichte*, 1862, but the method by which the solution was obtained is very briefly indicated, and the results contain numerous misprints.

We shall obtain the solution by combining Gutzlaff's equation,

$$\sqrt[4]{\kappa'\lambda} + \sqrt[4]{\kappa\lambda'} = 1,$$

with Fiedler's modular equation of the 15^{th} order,

$$Z_1 Z_{2'} + 4Z_3 = 0,$$

where
$$Z_1 = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'}+1,$$
$$Z_2 = \sqrt[4]{\kappa\lambda\kappa'\lambda'} + \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'},$$
$$Z_3 = +\sqrt[4]{\kappa\lambda\kappa'\lambda'},$$
and
$$Z_{2'} = Z_1^2 - 4Z_2.$$

Write x for $\sqrt[4]{\kappa\lambda\kappa'\lambda'}$ and w for $\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'}$; then

$$Z_1 = w+1, \quad Z_2 = w+x, \quad Z_3 = x,$$
$$Z_{2'} = (w+1)^2 - 4(w+x),$$
$$= (w-1)^2 - 4x,$$

and Fiedler's equation becomes

$$(w+1)(w-1)^2-4x\,(w+1)+4x = 0,$$

or $$(w+1)(w-1)^2 = 4wx,$$

or $$w^3-w^2-w+1 = 4wx.$$

Now, from Gutzlaff's equation,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'} = 1-2x,$$

$$\kappa'\lambda+\quad\kappa\lambda' = 1-4x+2x^2,$$

and therefore $$(\kappa\lambda+\kappa'\lambda')^2 = 1+4x^4-(1-4x+2x^2)^2$$
$$= 8x-20x^2+16x^3.$$

But $$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = w,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = w^2-2x,$$

$$\kappa\lambda+\quad\kappa'\lambda' = w^4-4w^2x+2x^2$$
$$= w^3+w^2-w+2x^2 = 2w^2+4wx+2x^2-1$$
$$= 2\,(w+x)^2-1.$$

Therefore, putting $w+x = z$,

$$2z^2-1 = \sqrt{(16x^3-20x^2+8x)},$$

and $$z^3-(3x+1)\,z^2+(3x^2-2x-1)\,z-x^3+3x^2+x+1 = 0,$$

and z is to be eliminated between these two equations.

Putting $$2z^2-1 = t,\quad z^2 = \tfrac{1}{2}\,(t+1),$$

then, since

$$\{z^3+(3x^2-2x-1)\,z\}^2-\{(3x+1)\,z^2-x^3+3x^2+x+1\}^2 = 0,$$

or $$z^6-(3x^2+10x+3)\,z^4+(3x^4+4x^3+10x^2+12x+3)\,z^2$$
$$-(x^3-3x^2-x-1)^2 = 0,$$

therefore

$$\left(\frac{t+1}{2}\right)^3-(3x^2+10x+3)\left(\frac{t+1}{2}\right)^2+(3x^4+4x^3+10x^2+12x+3)\,\frac{t+1}{2}$$
$$-(x^3-3x^2-x-1)^2 = 0,$$

or $$t^3-(6x^2+20x+3)\,t^2+(12x^4+16x^3+28x^2+8x+3)\,t$$
$$-(8x^6-48x^5+44x^4+16x^3+22x^2-12x+1) = 0,$$

or $$t\,(t^3+12x^4+16x^3+28x^2+8x+3)$$

$$\equiv \sqrt{(16x^3-20x^2+8x)}(12x^4+32x^3+8x^2+16x+3)$$

$$= 8x^6+48x^5+244x^4-288x^3+122x^2+12x+1,$$

a reciprocal equation in $\sqrt{2}\,x$.

Putting $\sqrt{2}\,x = y$, and squaring,

$$(4\sqrt{2}\,y^3-10y^2+4\sqrt{2}\,y)(3y^4+8\sqrt{2}\,y^3+4y^2+8\sqrt{2}\,y+3)^2$$

$$= (y^6+6\sqrt{2}\,y^5+61y^4-72\sqrt{2}\,y^3+61y^2+6\sqrt{2}\,y+1)^2;$$

and putting $$y+\frac{1}{y} = \sqrt{2}\,v,$$

$$y^2+\frac{1}{y^2} = 2v^2-2,\quad y^3+\frac{1}{y^3} = 2\sqrt{2}\,v^3-3\sqrt{2}\,v,$$

$$(8v-10)(6v^2+16v-2)^2 = 2\,(2v^3+12v^2+58v-84)^2,$$

or $$(4v-5)(3v^2+8v-1)^2 = (v^3+6v^2+29v-42)^2,$$

or $$v^6-24v^5-53v^4+272v^3+691v^2-2520v+1769 = 0.$$

A quadratic factor of this equation, $v^2-28v+61$, was discovered by calculating the approximate numerical values of x in a manner to be explained subsequently.

The remaining quartic factor of the sextic

$$v^4+4v^3-2v^2-28v+29 = (v^2+2v-5)^2+4\,(v-1)^2 = 0,$$

has only imaginary roots.

Taking v as determined by the quadratic

$$v^2-28v+61 = 0,$$

then $$v = 14+3\sqrt{15}.$$

The reciprocal 12^{ic} for y, expanded at full length, would be

$$y^{12}-24\sqrt{2}\,y^{11}-100y^{10}+424\sqrt{2}\,y^9+2355y^8-8688\sqrt{2}\,y^7$$
$$+19064y^6-8688\sqrt{2}\,y^5\ldots\ldots-24\sqrt{2}\,y+1 = 0,$$

which has the reciprocal factor,

$$y^4-28\sqrt{2}\,y^3+124y^2-28\sqrt{2}\,y+1,$$

obtained from $$v^2-28v+61.$$

Now $$\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'} = 1,$$

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'} = 1-\sqrt{2}\,y,$$

$$\kappa'\lambda+\kappa\lambda' = 1-2\sqrt{2}\,y+y^2,$$

and $$\kappa\lambda+\kappa'\lambda' = \sqrt{(4\sqrt{2}\,y-10y^2+4\sqrt{2}\,y^3)}.$$

Now, if $$y^4 = 4\kappa\lambda\kappa'\lambda',$$

$$\frac{1}{y}+y = \sqrt{2}\,v,$$

where $$v = 14+3\sqrt{15},$$

$$\frac{1}{y}-y = \sqrt{(2v^2-4)}$$
$$= \sqrt{(658+168\sqrt{15})}$$
$$= \sqrt{2}\,(3\sqrt{21}+2\sqrt{35}).$$

Therefore $$\frac{2}{y} = \sqrt{2}\,(14+3\sqrt{15}+3\sqrt{21}+2\sqrt{35})$$
$$= \sqrt{2}\,(2\sqrt{7}+3\sqrt{3})(\sqrt{7}+\sqrt{5}),$$

$$\frac{1}{y} = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^2\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

$$y = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^2\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}.$$

Now $$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = 1,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = 1-\sqrt{2}\,y,$$

$$\kappa\lambda+\kappa'\lambda' = 1-2\sqrt{2}\,y+y^2,$$

$$\kappa'\lambda+\kappa\lambda' = \sqrt{\{1+y^4-(1-2\sqrt{2}\,y+y^2)^2\}}$$
$$= \sqrt{(4\sqrt{2}\,y-10y^2+4\sqrt{2}\,y^3)}.$$

Therefore

$$\kappa\kappa'+\lambda\lambda' = y^2\left(\frac{1}{y}+y-2\sqrt{2}\right)\sqrt{\left\{4\sqrt{2}\left(\frac{1}{y}+y\right)-10\right\}}$$
$$= y^2\,(\sqrt{2}v-2\sqrt{2})\sqrt{(8v-10)}$$
$$= 2y^2\,(v-2)\sqrt{(4v-5)}$$
$$= 2y^2\,(12+3\sqrt{15})(6+\sqrt{15})$$
$$= y^2\,(234+60\sqrt{15}),$$

and $2\sqrt{\kappa\lambda\kappa'\lambda'} = y^2$;

therefore
$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = y\sqrt{(235+60\sqrt{15})}$$
$$= y\,(3\sqrt{15}+10),$$
$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = y\sqrt{(233+60\sqrt{15})}$$
$$= y\,(5\sqrt{5}+6\sqrt{3}).$$

Therefore
$$2\sqrt{\kappa\kappa'} = y\,(3\sqrt{15}+10-5\sqrt{5}-6\sqrt{3})$$
$$= y\,(3\sqrt{3}-5)(\sqrt{5}-2)$$
$$= 4y\left(\frac{\sqrt{3}-1}{2}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3,$$
$$2\sqrt{\lambda\lambda'} = y\left(\frac{\sqrt{3}+1}{2}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3;$$

or
$$\sqrt{2\kappa\kappa'} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

when
$$K'/K = \sqrt{(105)};$$

and
$$\sqrt{2\lambda\lambda'} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

when
$$K'/K = \sqrt{(15\div 7)}.$$

Therefore
$$\frac{1}{\sqrt{2\kappa\kappa'}} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}};$$

and therefore
$$\beta = \tfrac{1}{2}\sqrt{\alpha} = \frac{1}{2\kappa\kappa'}-2\kappa\kappa'$$
$$= \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}+1}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2$$
$$-\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}-1}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2;$$

and
$$\beta' = \tfrac{1}{2}\sqrt{\alpha'} = \frac{1}{2\lambda\lambda'}-2\lambda\lambda'$$
$$= \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}-1}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2$$
$$-\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}+1}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2;$$

so that β and β' are the roots of a quadratic equation.

If we had taken the other root of the quadratic

$$v^2-28v+61=0,$$

$$v=14-3\sqrt{15},$$

we should find

$$\frac{2}{y}=\sqrt{2}\,(14-3\sqrt{15}+3\sqrt{21}-2\sqrt{35})$$

$$=\sqrt{2}\,(2\sqrt{7}+3\sqrt{3})(\sqrt{7}-\sqrt{5}),$$

$$\frac{1}{y}=\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

$$y=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}.$$

Also

$$\kappa\kappa'+\lambda\lambda'=y^2\,(234-60\sqrt{15}),$$

and therefore

$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=y\,(3\sqrt{15}-10),$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=y\,(5\sqrt{5}-6\sqrt{3}),$$

and

$$\sqrt{2\kappa\kappa'}=\frac{y}{\sqrt{2}}\,(3\sqrt{15}-10-5\sqrt{5}+6\sqrt{3})$$

$$=(3\sqrt{3}-5)(\sqrt{5}+2)\,\frac{y}{\sqrt{2}}$$

$$=\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

when

$$K'/K=\sqrt{(35\div 3)}\,;$$

and

$$\sqrt{2\lambda\lambda'}=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

when

$$\Lambda'/\Lambda=\sqrt{(21\div 5)}.$$

According to Kronecker, *Berlin Sitz.*, 1862,

$$2\kappa\kappa'=(2\gamma-3\alpha)^2\,(5+9\alpha+16\beta+4\gamma+7\beta\gamma+12\alpha\beta+3\alpha\beta\gamma),$$

where α, β, γ denote $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, respectively.

The factor

$$2\gamma-3\alpha=\left(\frac{\gamma-\alpha}{2}\right)^3,$$

but the remaining factor cannot be made to agree with

$$(2+\alpha)^3\left(\frac{\beta-1}{2}\right)^6(6+\beta\gamma),$$

the result obtained above.

The approximate numerical values of x and y were obtained from the formulas

$$\sqrt[4]{\kappa\kappa'} = \sqrt{2}\, q^{\frac{1}{8}}, \quad \sqrt[4]{2\kappa\kappa'} = 2^{\frac{3}{4}} q^{\frac{1}{8}}.$$

Now, if $K'/K = \sqrt{105}$,

$$\begin{aligned}
\log 105 &= 2{\cdot}0211893, \\
\log K'/K = \log \sqrt{105} &= 1{\cdot}01059465, \\
\log \pi \log e &= {\cdot}1349342, \\
\log\log\left(\frac{1}{q}\right) &= 1{\cdot}14552885, \\
\log\left(\frac{1}{q}\right) &= 13{\cdot}9806984, \\
\log\left(\frac{1}{q}\right)^{\frac{1}{8}} &= 1{\cdot}7475873, \\
\log 2^{\frac{3}{4}} &= {\cdot}2257725, \\
\log \frac{1}{\sqrt[4]{2\kappa\kappa'}} &= 1{\cdot}5218148, \\
\log \sqrt[4]{2\kappa\kappa'} &= \bar{2}{\cdot}4781852.
\end{aligned}$$

Again, if $\Lambda'/\Lambda = \sqrt{(15 \div 7)}$,

$$\begin{aligned}
\log 15 &= 1{\cdot}1760913, \\
\log 7 &= {\cdot}8450980, \\
\log \tfrac{15}{7} &= {\cdot}3309933, \\
\log \Lambda'/\Lambda = \log \sqrt{\tfrac{15}{7}} &= {\cdot}16549665, \\
\log \pi \log e &= {\cdot}1349342, \\
\log\log\left(\frac{1}{q}\right) &= {\cdot}30043085, \\
\log\left(\frac{1}{q}\right) &= 1{\cdot}9972425, \\
\log\left(\frac{1}{q}\right)^{\frac{1}{8}} &= {\cdot}2496553, \\
\log 2^{\frac{3}{4}} &= {\cdot}2257725, \\
\log \frac{1}{\sqrt[4]{2\lambda\lambda'}} &= {\cdot}0238828, \\
\log \sqrt[4]{2\lambda\lambda'} &= \bar{1}{\cdot}9761172.
\end{aligned}$$

Combining these transformations when the modular equations of the 7[th] and 15[th] order are employed,

$$\log\frac{1}{y} = 1{\cdot}5456976, \qquad \frac{1}{y} = 35{\cdot}131633,$$

$$\log y = \bar{2}{\cdot}4543024, \qquad y = {\cdot}028464,$$

$$\frac{1}{y} + y = 35{\cdot}161097,$$

$$\log\sqrt{2}\,v = 1{\cdot}5460624,$$

$$\log\sqrt{2} = {\cdot}1505150,$$

$$\log v = 1{\cdot}3955474, \qquad v = 24{\cdot}862.$$

In a similar manner, if $K'/K = \sqrt{(25 \div 3)}$,

$$\log\sqrt[4]{2\kappa\kappa'} = \bar{1}{\cdot}64324355,$$

and if $\Lambda'/\Lambda = \sqrt{(21 \div 5)}$,

$$\log\sqrt[4]{2\lambda\lambda'} = \bar{1}{\cdot}87623125,$$

$$\log y' = \bar{1}{\cdot}5194748,$$

$$\log\frac{1}{y'} = {\cdot}4805252,$$

$$y' = {\cdot}33073,$$

$$\frac{1}{y'} = 3{\cdot}02360,$$

$$\sqrt{2}\,v' = 3{\cdot}35433,$$

$$\log\sqrt{2}\,v' = {\cdot}5256058,$$

$$\log\sqrt{2} = {\cdot}1505150,$$

$$\log v' = {\cdot}3750908, \qquad v' = 2{\cdot}37187,$$

$$\log v = 1{\cdot}3955474,$$

$$\log vv' = 1{\cdot}7706382, \qquad vv' = 58{\cdot}971,$$

and

$$\begin{aligned} v + v' &= 24{\cdot}862 \\ &\ \ 2{\cdot}372 \\ &= 27{\cdot}234\,; \end{aligned}$$

indicating, as the approximations are rather rough for the ratios of the periods

$$K'/K = \sqrt{(15 \div 7)}, \quad \sqrt{(21 \div 5)}, \quad \text{and} \quad \sqrt{(35 \div 3)},$$

the true values of $v+v'$ and vv', namely

$$v+v' = 28 \quad \text{and} \quad vv' = 61.$$

Taking, however, the values from Legendre's tables, we find that

$$K'/K = \sqrt{(15\div 7)}, \quad 2\kappa\kappa' \approx \sin 45^\circ\, 30';$$

$$K'/K = \sqrt{(21\div 5)}, \quad 2\kappa\kappa' \approx \sin 18^\circ\, 40';$$

$$K'/K = \sqrt{(35\div 3)}, \quad 2\kappa\kappa' \approx \sin 2^\circ\, 8'.$$

If we had combined Jacobi's modular equation of the 3rd order,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'} = 1,$$

with Fiedler's form of the 35th order,

$$Z_1'^3 - 8Z'Z_2' + 8Z_3' - 4Z_0^{(4)} = 0,$$

where $Z_1' = \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1,$

$$Z' = \sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'},$$

$$Z_3' = -\sqrt{\kappa\lambda\kappa'\lambda'},$$

and $Z_0^{(4)} = -(\kappa'\lambda+\kappa\lambda')\sqrt[4]{\kappa\lambda\kappa'\lambda'}+(\kappa'-\lambda')\sqrt[4]{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda};$

then, putting

$$\kappa\lambda\kappa'\lambda' = x^4,$$

$$\kappa'\lambda+\kappa\lambda' = 1-2x^2,$$

$$(\kappa\lambda+\kappa'\lambda')^2 = 1+4x^4-(1-2x^2)^2$$

$$= 4x^2,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \sqrt{(2x+2x^2)};$$

so that $Z_1' = \sqrt{(2x+2x^2)}-1,$

$$Z_2' = x^2-\sqrt{(2x+2x^2)},$$

$$Z_3' = -x^2,$$

$$Z_0^{(4)} = -(x-2x^3)+(\kappa'-\lambda')\sqrt[4]{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda}.$$

Then

$$\left\{\sqrt{(2x+2x^2)}-1\right\}^3 - 8\left\{\sqrt{(2x+2x^2)}-1\right\}\left\{x^2-\sqrt{(2x+2x^2)}\right\}$$
$$+4x-8x^2-8x^3$$
$$= 4\left\{(\kappa'-\lambda')\sqrt{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda}\right\}$$
$$= 4\sqrt{(1-2x)}\sqrt{\left\{(1-2x+2x^2)\sqrt{(2x+2x^2)}+2x-4x^3\right\}}.$$

Squaring and rearranging,

$$1-10x+250x^2+448x^3-172x^4-136x^5+136x^6$$
$$=(6+80x+128x^2-16x^3+152x^4-96x^5)\sqrt{(2x+2x^3)}\,;$$

and squaring again, and reducing,

$$1-92x-1392x^2-21896x^3-3252x^4+155296x^5+82976x^6$$
$$-310592x^7-13008x^8+175168x^9-22272x^{10}+2944x^{11}+64x^{12}=0,$$

a reciprocal equation in $\sqrt{2}\,x=y$, so that

$$y^{12}+46\sqrt{2}\,y^{11}-696y^{10}+5474\sqrt{2}\,y^9-813y^8-19412\sqrt{2}\,y^7+6032y^6$$
$$+19412\sqrt{2}\,y^5-813y^4-5474\sqrt{2}\,y^3-696y^2-46\sqrt{2}\,y+1=0.$$

Putting, as before, $\quad y+\dfrac{1}{y}=\sqrt{2}\,v,$

then $\quad v^6-351v^4+495v^2+783+46\sqrt{(v^2-2)}(v^4+58v^2-135)=0.$

But, putting

$$\frac{1}{y}-y=\sqrt{2}\,u,$$
$$\frac{1}{y^2}+y^2=2u^2+2,$$
$$\frac{1}{y^3}-y^3=\sqrt{2}\,(2u^3+3u),$$
$$\frac{1}{y^4}+y^4=4u^4+8u^2+2,$$
$$\frac{1}{y^5}-y^5=\sqrt{2}\,(4u^5+10u^3+5u),$$
$$\frac{1}{y^6}+y^6=8u^6+24u^4+18u^2+2\,;$$

then $\quad u^6-46u^5-345u^4-2852u^3-897u^2+690u+377=0,$

having a quadratic factor $\quad u^2-54u+29,$

inferred as before from the approximate numerical values of the moduli.

This sextic then splits into the factors

$$(u^2-54u+29)(u^4+8u^3+58u^2+48u+13)=0,$$

and the quartic factor

$$=(u^2+4u+3)^2+(6u+2)^2,$$

and therefore gives imaginary roots only.

But if $\frac{1}{y}-y = \sqrt{2}\,u$, and $u = 27+10\sqrt{7}$, from

$$u^2-54u+29 = 0;$$

$$\begin{aligned}
\frac{1}{y}+y &= \sqrt{(2u^2+4)} \\
&= \sqrt{(2862+1080\sqrt{7})} \\
&= \sqrt{2}\,(6\sqrt{21}+15\sqrt{3}); \\
\frac{2}{y} &= \sqrt{2}\,(27+10\sqrt{7}+6\sqrt{21}+15\sqrt{3}) \\
&= \sqrt{2}\,(3\sqrt{3}+5)(2\sqrt{7}+3\sqrt{3}), \\
\frac{1}{y} &= \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3, \\
y &= \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3.
\end{aligned}$$

But

$$\begin{aligned}
\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} &= 1, \\
\kappa\lambda + \kappa'\lambda' &= 1-y^2, \\
\kappa'\lambda + \kappa\lambda' &= \sqrt{\{1+y^4-(1-y^2)^2\}} = \sqrt{2}\,y, \\
\kappa\kappa' + \lambda\lambda' &= \sqrt{2}\,y\,(1-y^2) = 2y^2u, \\
2\sqrt{\kappa\lambda\kappa'\lambda'} &= y^2;
\end{aligned}$$

therefore

$$\begin{aligned}
\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} &= y\sqrt{(2u+1)} \\
&= y\sqrt{(55+20\sqrt{7})} \\
&= y\,(\sqrt{35}+2\sqrt{5}); \\
-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} &= y\sqrt{(2u-1)} \\
&= y\sqrt{(53+20\sqrt{7})} \\
&= y\,(2\sqrt{7}+5); \\
2\sqrt{\kappa\kappa'} &= y\,(\sqrt{35}+2\sqrt{5}-2\sqrt{7}-5) \\
&= y\,(\sqrt{5}-2)(\sqrt{7}-\sqrt{5}); \\
\sqrt{2\kappa\kappa'} &= y\left(\frac{\sqrt{5}-1}{2}\right)^3 \frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}} \\
&= \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},
\end{aligned}$$

where $K'/K = \sqrt{105}$, as before; and

$$\sqrt{2\lambda\lambda'} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{5}+1}{2}\right)^2 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^2 \frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

when $\Lambda'/\Lambda = \sqrt{(35 \div 3)}$, as before.

$\Delta = 161 = 7 \times 23$. Combine Schröter's or Russell's modular equation of the 23^{rd} order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + \sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1,$$

with Gutzlaff's of the 7^{th},

$$\sqrt[4]{\kappa'\lambda} + \sqrt[4]{\kappa\lambda'} = 1,$$

by putting $4\kappa\lambda\,\kappa'\lambda' = x^{12}$.

Then, for the equation of the 23^{rd} order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = 1 - \sqrt{2}\,x,$$

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = 1 - 2\sqrt{2}\,x + 2x^2 - \sqrt{2}\,x^3,$$

$$\begin{aligned}\kappa\lambda + \kappa'\lambda' &= (1 - 2\sqrt{2}\,x + 2x^2 - \sqrt{2}\,x^3)^2 - x^6 \\ &= 1 - 4\sqrt{2}\,x + 12x^2 - 10\sqrt{2}\,x^3 + 12x^4 - 4\sqrt{2}\,x^5 + x^6,\end{aligned}$$

a reciprocal expression.

Again, from the equation of the 7^{th} order,

$$\sqrt{\kappa'\lambda} + \sqrt{\kappa\lambda'} = 1 - \sqrt{2}\,x^3,$$

$$\kappa'\lambda + \kappa\lambda' = 1 - 2\sqrt{2}\,x^3 + x^6.$$

But $$(\kappa\lambda + \kappa'\lambda')^2 + (\kappa'\lambda + \kappa\lambda')^2 = 1 + 4\kappa\lambda\,\kappa'\lambda';$$

therefore

$$(1 - 4\sqrt{2}\,x + 12x^2 - 10\sqrt{2}\,x^3 + 12x^4 - 4\sqrt{2}\,x^5 + x^6)^2 + (1 - 2\sqrt{2}\,x^3 + x^6)^2 = 1 + x^{12},$$

a reciprocal equation of the 12^{th} degree for x.

Put $$\frac{1}{x} + x = \sqrt{2}\,y,$$

then $$4y^6 - 32y^5 + 100y^4 - 160y^3 + 113y^2 - 24y + 9 = 0,$$

a sextic equation for y, of which a quadratic factor can be discovered by calculating as before the approximate numerical values of a pair of the roots.

$\Delta = 193$. This is a prime number, for which, according to Gauss, $p = 2$; so that a should be of the form $M\sqrt{193}+N$, when M and N are integers; and the values of M and N can be determined by approximate numerical calculation.

Class D.

$$\Delta \equiv 2, \text{ mod. } 4.$$

This is the same as Hermite's class 2° (*Équations modulaires*, p. 44).

To solve the modular equation according to Hermite, we put

$$u^4 = -\frac{v^4-1}{v^4+1}, \quad u^8 = x;$$

then
$$\kappa' = u^4 = \frac{1-\lambda}{1+\lambda},$$

$$\kappa' = \frac{2\sqrt{\lambda}}{1+\lambda};$$

so that
$$\kappa'\lambda' = \frac{2\sqrt{\lambda}}{1+\lambda}\sqrt{(1-\lambda^2)} = 2\sqrt{\kappa\lambda},$$

equivalent to the modular queation of the quadric transformation.

Then, if we put

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = \frac{1}{u^2v^2} - u^2v^2,$$

$$\beta = \frac{1}{\sqrt{x}}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} - \sqrt{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

$$= \frac{1+x}{\sqrt{x}\sqrt{(1-x)}}$$

$$= \sqrt[4]{(-a)},$$

where a denotes Hermite's absolute invariant, given by

$$a = -\frac{(1+x)^4}{x(1-x)^2},$$

and then
$$x = \kappa^2.$$

If we put
$$v/u = e^{\phi}, \quad u/v = e^{-\phi},$$

then, if $w = uv$,

$$v^2 = we^{i\phi}, \quad u^2 = we^{-i\phi};$$

so that
$$v^{2n}+u^{2n} = 2w^n \cosh n\phi,$$
$$v^{2n}-u^{2n} = 2w^n \sinh n\phi;$$
and, since
$$u^4+v^4 = 1-u^4v^4,$$
therefore
$$2w^2 \cosh 2\phi = 1-w^4,$$
or
$$\beta = \frac{1}{w^2}-w^2 = 2\cosh 2\phi,$$
$$\beta+2 = 4\cosh^2\phi, \quad \beta-2 = 4\sinh^2\phi;$$
so that
$$\kappa+\lambda = 2\sqrt{\kappa\lambda}\cosh 2\phi,$$
$$-\kappa+\lambda = 2\sqrt{\kappa\lambda}\sinh 2\phi.$$

In the following numerical illustrations we shall find it convenient to put
$$\sqrt{(\beta^2+4)} = \frac{1}{\sqrt{\kappa\lambda}}+\sqrt{\kappa\lambda} = \gamma,$$
and then
$$\sqrt{\kappa\lambda} = w^2 = \tfrac{1}{2}(\gamma-\beta).$$

$\Delta = 2.$ $\quad \kappa = \sqrt{2}-1, \quad \alpha = -2^4, \quad \beta = 2, \quad \gamma = 2\sqrt{2};$
$$\cosh\phi = 1, \quad \phi = 0, \quad v = u, \quad \kappa = \lambda.$$

$\Delta = 6.$ Putting $\kappa'\lambda' = 2\sqrt{\kappa\lambda}$ in the modular equation of the 3rd order,
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = 1,$$
then
$$\sqrt{\kappa\lambda}+\sqrt{2}\sqrt[4]{\kappa\lambda} = 1,$$
or
$$\frac{1}{\sqrt[4]{\kappa\lambda}}-\sqrt[4]{\kappa\lambda} = \sqrt{2},$$
$$\gamma = \frac{1}{\sqrt{\kappa\lambda}}+\sqrt{\kappa\lambda} = 4,$$
$$\beta = \frac{1}{\sqrt{\kappa\lambda}}-\sqrt{\kappa\lambda} = 2\sqrt{3};$$
$$\cosh 2\phi = \sqrt{3}, \quad \sinh 2\phi = \sqrt{2}, \quad w^2 = 2-\sqrt{3};$$
$$\alpha = -2^4\times 3^2,$$
agreeing with Hermite's result.

Solving this equation, we shall find
$$\kappa = (\sqrt{3}-\sqrt{2})(2-\sqrt{3}), \quad \text{for} \quad K'/K = \sqrt{6};$$
$$\lambda = (\sqrt{3}+\sqrt{2})(2-\sqrt{3}), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2\div 3)}$$
(Legendre, *Fonctions elliptiques*).

$\Delta = 10$. Put $\kappa'\lambda' = 2\sqrt{\kappa\lambda}$ in the modular equation of the 5th order,

$$\kappa\lambda + \kappa'\lambda' + 2\sqrt[3]{4\kappa\lambda\kappa'\lambda'} = 1;$$

then
$$\kappa\lambda + 2\sqrt{\kappa\lambda} + 4\sqrt{\kappa\lambda} = 1;$$

or
$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 6 = 2\times 3;$$

$$\gamma = 2\sqrt{10},\quad w^3 = \sqrt{10}-3;$$

$$a = -2^4\times 3^4.$$

Solving these equations for κ and λ, we shall find

$$\kappa = (\sqrt{2}-1)^2(\sqrt{10}-3), \quad \text{for} \quad K'/K = \sqrt{10};$$

$$\lambda = (\sqrt{2}+1)^2(\sqrt{10}-3), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2\div 5)}.$$

$\Delta = 14$. From the modular equation of the 7th order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = 1,$$

we obtain
$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{2}\,\sqrt[8]{\kappa\lambda} = 1,$$

or
$$\frac{1}{\sqrt[8]{\kappa\lambda}} - \sqrt[8]{\kappa\lambda} = \sqrt[4]{2},$$

$$\frac{1}{\sqrt[4]{\kappa\lambda}} + \sqrt[4]{\kappa\lambda} = 2+\sqrt{2};$$

$$\gamma = \frac{1}{\sqrt{\kappa\lambda}} + \sqrt{\kappa\lambda} = 4(\sqrt{2}+1),$$

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 2\sqrt{(8\sqrt{2}+11)},$$

$$a = -2^4(8\sqrt{2}+11)^2.$$

Then
$$\sqrt{\kappa\lambda} = 2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)},$$

$$\cosh 2\phi = \sqrt{(8\sqrt{2}+11)}, \quad \sinh 2\phi = \sqrt{(8\sqrt{2}+10)},$$

$$\kappa = \left\{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\right\}\left\{\sqrt{(8\sqrt{2}+11)}-\sqrt{(8\sqrt{2}+10)}\right\},$$

for
$$K'/K = \sqrt{14};$$

$$\lambda = \left\{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\right\}\left\{\sqrt{(8\sqrt{2}+11)}+\sqrt{(8\sqrt{2}+10)}\right\},$$

for
$$\Lambda'/\Lambda = \sqrt{(2\div 7)}.$$

(G. H. Stuart, "Complex Multiplication of Elliptic Functions," *Quar. Jour. of Math.*, xx., p. 54).

$\Delta = 18$. Then $$a = -2^4 \times 7^4,$$

obtained from Hermite's *Équations Modulaires*, p. 51.

Then $$\cosh 2\phi = 7, \quad \sinh 2\phi = 4\sqrt{3}, \quad \cosh\phi = 2;$$

$$\sqrt{\kappa\lambda} = w^2 = 5\sqrt{2} - 7 = (\sqrt{2}-1)^3;$$

$$\kappa = (\sqrt{2}-1)^3 (2-\sqrt{3})^2, \qquad K'/K = 3\sqrt{2};$$

$$\lambda = (\sqrt{2}-1)^3 (2+\sqrt{3})^2, \qquad \Lambda'/\Lambda = \sqrt{2} \div 3.$$

$\Delta = 22$. From the equation of the 11^{th} order, we obtain

$$\sqrt{\kappa\lambda} + \sqrt{2}\sqrt[4]{\kappa\lambda} + 2\sqrt{2}\sqrt[4]{\kappa\lambda} = 1,$$

or $$\frac{1}{\sqrt[4]{\kappa\lambda}} - \sqrt[4]{\kappa\lambda} = 3\sqrt{2},$$

$$\gamma = \frac{1}{\sqrt{\kappa\lambda}} + \sqrt{\kappa\lambda} = 20;$$

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 6\sqrt{11};$$

$$\cosh\phi = 3\sqrt{11}, \quad \sinh\phi = 7\sqrt{2};$$

$$a = -\beta^4 = -2^4 \times 3^4 \times 11^2.$$

Then $$\sqrt{\kappa\lambda} = w^2 = 10 - 3\sqrt{11} = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2;$$

$$\kappa = (3\sqrt{11} - 7\sqrt{2})(10 - 3\sqrt{11}), \quad \text{for} \quad K'/K = \sqrt{22};$$

$$\lambda = (3\sqrt{11} + 7\sqrt{2})(10 - 3\sqrt{11}), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2 \div 11)}.$$

$\Delta = 26$. Taking Mr. Russell's form of the modular equation of the 13^{th} order,

$$P^7 - 2^5 R (105 P^4 - 2^7 \times 11 P^3 Q + 2^{12} Q^2) + 2^{16} P R^2 = 0,$$

where $$P = \kappa\lambda + \kappa'\lambda' - 1,$$

$$Q = \kappa\lambda\kappa'\lambda' - \kappa\lambda - \kappa'\lambda',$$

$$R = -\kappa\lambda\kappa'\lambda';$$

and putting $$\kappa\lambda = x^2, \quad \kappa'\lambda' = 2x,$$

then $$P = x^2 + 2x - 1 = -x(\beta - 2),$$

$$Q = 2x^3 - x^2 - 2x = -x^2(2\beta + 1),$$

$$R = -2x^3,$$

putting $$\beta = \frac{1}{x} - x;$$

and then substituting, and dividing out by x^7,

$$(\beta-2)^7 - 64\left\{105(\beta-2)^4 + 2^7\times 11(\beta-2)^2(2\beta+1) + 2^{12}(2\beta+1)^2\right\} + 2^{18}(\beta-2) = 0.$$

Putting $$\beta - 2 = 4t,$$

then $$t = \sinh^2\phi,$$

and $$t^7 - 105t^4 - 704t^3 - 1464t^2 - 1216t - 400 = 0,$$

or $$(t^2+t+4)(t^2+4t+25)(t^3-5t^2-8t-4) = 0.$$

Taking the cubic $$t^3 - 5t^2 - 8t - 4 = 0,$$

the other factors giving imaginary roots, then if $t = y^2$, the equation in y becomes

$$y^3 + 3y^2 + 2y + 2 = 0,$$

or $$(y+1)^3 = y - 1,$$

where $$y = \sinh\phi.$$

Putting $y+1 = v$, then $$y - 1 = v - 2,$$

and $$v^3 - v + 2 = 0;$$

which, compared with $$4v^3 - g_2 v - g_3 = 0,$$

has $$g_2 = 4,\quad g_3 = -8,$$

$$g_2^3 - 27g_3^2 = -64\times 26,$$

so that the cubic has only one real root.

The absolute invariant of this cubic

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} = -\frac{1}{26};$$

so that, putting $\operatorname{cosech} 3\alpha = \sqrt{26}$, then

$$v = \frac{2\cosh\alpha}{\sqrt{3}}$$

(*Proc. Lond. Math. Soc.*, Vol. XVII., p. 263).

$\Delta = 30$. Taking Fiedler's modular equation of the 15th order,

$$P^3 - 4PQ + 4R = 0,$$

where
$$P = \sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1,$$
$$Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'}+\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'},$$
$$R = \sqrt[4]{\kappa\lambda\kappa'\lambda'};$$
and putting
$$\kappa\lambda = x^8,\quad \kappa'\lambda' = 2\sqrt{\kappa\lambda} = 2x^4,$$
then
$$P = x^2+\sqrt[4]{2}\,x+1 = x\,(t+\sqrt[4]{2}),$$
$$Q = \sqrt[4]{2}\,x^3+x^2+\sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t+1),$$
$$R = \sqrt[4]{2}\,x^3,$$
putting
$$t = x+\frac{1}{x};$$
and then
$$(t+\sqrt[4]{2})^3-4\,(t+\sqrt[4]{2})(\sqrt[4]{2}\,t+1)+4\sqrt[4]{2} = 0,$$
$$t^3-\sqrt[4]{2}\,t^2-\sqrt{2}\,(2\sqrt{2}+1)\,t+\sqrt{2}\sqrt[4]{2} = 0,$$
factorizing into
$$\left\{t-\sqrt[4]{2}\,(\sqrt{2}+1)\right\}(t^2+\sqrt{2}\sqrt[4]{2}\,t-2+\sqrt{2}) = 0,$$
so that, if
$$t = \sqrt[4]{2}\,(\sqrt{2}+1),$$
$$\beta = \frac{1}{x^4}-x^4 = 2\sqrt{3}\,(4\sqrt{2}+5),$$
$$\cosh 2\phi = \sqrt{3}\,(4\sqrt{2}+5),$$
$$\sinh 2\phi = \sqrt{10}\,(3+2\sqrt{2}),$$
$$e^{2\phi} = \cosh 2\phi+\sinh 2\phi$$
$$= (\sqrt{6}+\sqrt{5})(4+\sqrt{15}),$$
$$e^{-2\phi} = (\sqrt{6}-\sqrt{5})(4-\sqrt{15});$$
$$\gamma = \frac{1}{x^4}+x^4 = 20+12\sqrt{2},$$
$$\sqrt{\kappa\lambda} = w^2 = \tfrac{1}{2}\,(\gamma-\beta)$$
$$= (2-\sqrt{3})(5-2\sqrt{6}).$$
Therefore
$$\kappa = w^2e^{-2\phi} = (2-\sqrt{3})(5-2\sqrt{6})(\sqrt{6}-\sqrt{\xi})(4-\sqrt{15}),$$
for
$$K'/K = \sqrt{30};$$
$$\lambda = w^2e^{2\phi} = (2-\sqrt{3})(5-2\sqrt{6})(\sqrt{6}+\sqrt{5})(4+\sqrt{15}),$$
for
$$\Lambda'/\Lambda = \sqrt{(2\div 15)}.$$
Similarly, $\kappa = (2+\sqrt{3})(5-2\sqrt{6})(\sqrt{6}-\sqrt{5})(4+\sqrt{15}),$

for
$$K'/K = \sqrt{(6 \div 5)};$$
$$\lambda = (2+\sqrt{3})(5-2\sqrt{6})(\sqrt{6}+\sqrt{5})(4-\sqrt{15}),$$
for
$$\Lambda'/\Lambda = \sqrt{(10 \div 3)}.$$

$\Delta = 34$. Taking Mr. Russell's form of the modular equation of the 17th order, and putting
$$\kappa\lambda = x^2, \qquad \kappa'\lambda' = 2x,$$
then
$$P = x^2+2x-1 \;= -x(\beta-2),$$
$$Q = 2x^3-x^2-2x = -x^2(2\beta+1),$$
$$R = -2x^3,$$
and the modular equation
$$P^9+2^6R(-287P^6+2^5\times 261P^4Q-2^{12}\times 15P^2Q^2+2^{17}Q^3)$$
$$+2^{10}R^2(7309P^3-2^8\times 117PQ)+2^{21}\times 3^3R^3 = 0$$
becomes, on putting $\beta-2 = 4t$, and dividing out x^9,
$$t^9-574t^6-8352t^5-35940t^4-63859t^3-58464t^2-29040t-6272 = 0,$$
which can be factorized into
$$(t+1)(t+4)^2(t^2-11t-8)\{(t^2+t-5)^2+24(2t+1)^2\}.$$
Taking the quadratic $\quad t^2-11t-8 = 0,$

then
$$t = \tfrac{1}{2}(11+3\sqrt{17});$$
and
$$\beta = 4t+2 = 6(\sqrt{17}+4).$$
If we had taken Sohncke's modular equation of the 17th order (*Crelle*, XVI.),
$$(v-u)^{18}-16uv(1-u^8)(1-v^8)\{17uv(v-u)^6-(v^4-u^4)^2+16(1-u^4v^4)^2\}$$
$$= 0,$$
connecting
$$u = \sqrt[4]{\kappa} \quad \text{and} \quad v = \sqrt[4]{\lambda};$$
and put
$$v/u = e^{\phi},$$
then, if
$$\kappa'\lambda' = 2\sqrt{\kappa\lambda} = 2u^2v^2,$$
$$(1-u^8)(1-v^8) = 4u^4v^4,$$
$$(1-u^4v^4)^2 = (v^4+u^4)^2 = 4u^4v^4\cosh^2 2\phi.$$
Putting
$$s = \cosh\phi - 1 = 2\sinh^2\tfrac{1}{2}\phi,$$
the equation becomes
$$2^9s^9-64(17\times 2^4s^3-4\sinh^2 2\phi+64\cosh^2 2\phi) = 0,$$

or, since $\cosh\phi = s+1, \quad \cosh 2\phi = 2s^2+4s+1,$

$$\sinh 2\phi = 2\,(s+1)\,\sqrt{(s^2+2s)},$$

$$s^6-17s^3+2\,(s+1)^2\,(s^2+2s)-8\,(2s^2+4s+1)^2 = 0,$$

or $$s^6-30s^4-137s^3-150s^2-60s-8 = 0.$$

But, since $\beta = 2\cosh 2\phi, \quad t = \sinh^2\phi,$

this equation in s will be found not in agreement with the previous equation for t.

There is consequently a misprint in Sohncke's equation; it should be

$$(v-u)^{18}-16uv\,(1-u^8)(1-v^8)\,\{17uv\,(v-u)^6-(v^4-u^4)^2+16\,(1+u^4v^4)^2\}$$
$$= 0,$$

and now the equation for s becomes

$$2^6s^6-64\,\{17\times 2^3s^3-4\sinh^2 2\phi+64\,(\cosh^2 2\phi+1)\} = 0,$$

or $$s^6-17s^3+2\,(s+1)^2\,(s^2+2s)-8\,(2s^2+4s+1)-8 = 0,$$

or $$s^6-30s^4-137s^3-150s^2-60s-16 = 0\,;$$

factorizing into

$$(s+1)(s^2-s-4)(s^2+2s+4)\,\{(s^2-s-1)^2+6s^2\} = 0,$$

the factor $$s^2-s-4 = 0,$$

giving $$s = \tfrac{1}{2}\,(\sqrt{17}+1),$$

the required solution; and thus

$$\cosh\phi = \tfrac{1}{2}\,(\sqrt{17}+3),$$
$$\cosh 2\phi = 3\,(\sqrt{17}+4).$$

$\Delta = 38$. Taking Fiedler's or Russell's form of the modular equation of the 19th order,

$$P^6-112P^3R+256QR = 0,$$

where $$P = \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1,$$
$$Q = \sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'},$$
$$R = -\sqrt{\kappa\lambda\kappa'\lambda'}\,;$$

and putting $$\kappa\lambda = x^4, \quad \kappa'\lambda' = 2x^2\,;$$

also $$x-\frac{1}{x} = t,$$

then $$P = x^2+\sqrt{2}\,x-1 \;= x\,(t+\sqrt{2}),$$
$$Q = \sqrt{2}\,x^3-x^2-\sqrt{2}\,x = x^2\,(\sqrt{2}\,t-1),$$
$$R = -\sqrt{2}\,x^3;$$
and therefore
$$(t+\sqrt{2})^5+16\sqrt{2}\,\{7\,(t+\sqrt{2})^2-16\,(\sqrt{2}\,t-1)\} = 0;$$
or, putting $$t+\sqrt{2} = \sqrt{2}\,y,$$
$$4\sqrt{2}\,y^5+224\sqrt{2}\,y^2-256\sqrt{2}\,(2y-3) = 0,$$
$$y^5+56y^2-64\,(2y-3) = 0;$$
or, putting $$y = 2v,$$
$$v^5+7v^2-8v+6 = 0,$$
a *Hauptgleichung* quintic for v, having only one real root between 2 and -3.

This equation can be factorized into
$$(v^2-v+3)(v^3+v^2-2v+2) = 0;$$
the factor $$v^2-v+3 = 0$$
giving $$v = \tfrac{1}{2}\,(1+i\sqrt{11});$$
while $$v^3+v^2-2v+2 = 0$$
has only one real root
$$v = -\tfrac{1}{3}\left[1+\sqrt[3]{\{37+3\sqrt{(114)}\}}+\sqrt[3]{\{37-3\sqrt{(114)}\}}\right].$$

Otherwise, the equation in t factorizes into
$$(t^2+22)(t^3+5\sqrt{2}\,t^2-2t+22\sqrt{2}) = 0;$$
in which equation $$t = -\beta.$$

$\Delta = 42$. Taking Fiedler's form of the modular equation of the 21st order,
$$Z_1''-2Z_0^{(0,2)} = 0,$$
where $$Z_1'' = \kappa\lambda+\kappa'\lambda'-1,$$
$$Z_0^{(0,2)} = -(\sqrt{\kappa\lambda'}+\sqrt{\kappa'\lambda})\,\sqrt[4]{\kappa\lambda\kappa'\lambda'}+(\sqrt{\kappa'}-\sqrt{\lambda'})\,\sqrt[4]{\kappa'\lambda'}$$
$$-(\sqrt{\kappa}-\sqrt{\lambda})\,\sqrt[4]{\kappa\lambda};$$
and putting $$\kappa\lambda = w^4,\quad \kappa'\lambda' = 2w^2,\quad \beta = \frac{1}{w^2}-w^2;$$
an equation can be found for the determination of w and β, and thence of κ and λ.

$\Delta = 46$. Taking Fiedler's or Russell's form of the modular equation of the 23rd order,

$$P^3 - 4R = 0,$$

where
$$P = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} - 1,$$
$$R = -\sqrt[4]{\kappa\lambda\kappa'\lambda'},$$

and putting
$$\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4,$$

then
$$P = x^2 + \sqrt[4]{2}\, x - 1 = -x\,(t - \sqrt[4]{2}),$$
$$R = -\sqrt[4]{2}\, x^3,$$

putting
$$t = \frac{1}{x} - x;$$

and then
$$(t - \sqrt[4]{2})^3 - 4\sqrt[4]{2} = 0,$$
$$t - \sqrt[4]{2} - \sqrt{2}\,\sqrt[4]{2} = 0;$$
$$\frac{1}{x} - x = \sqrt[4]{2}\,(\sqrt{2}+1),$$
$$\frac{1}{x^2} + x^2 = 3\sqrt{2}\,(\sqrt{2}+1),$$
$$\frac{1}{x^2} - x^2 = \sqrt{(50+36\sqrt{2})},$$
$$\beta = \frac{1}{x^4} - x^4 = (\sqrt{2}+1)\sqrt{(4\sqrt{2}+3)},$$
$$= 3\sqrt{2}\sqrt{(294+208\sqrt{2})}$$
$$= 6\sqrt{(147+104\sqrt{2})},$$
$$\gamma = 52 + 36\sqrt{2} = 4\,(13+9\sqrt{2}).$$

$\Delta = 58$. According to Hermite (*Équations Modulaires*, p. 51), this is a determinant Δ for which the number a is integral, and by approximate numerical calculation from the formula

$$16a \approx -e^{\pi\sqrt{\Delta}} + 104,$$

we find
$$a = -2^4 \times 3^8 \times 11^4, \quad \beta = 198, \quad \gamma = 26\sqrt{58}.$$

If we put
$$\kappa\lambda = x^4, \quad \kappa'\lambda' = 2x^2,$$

and then
$$t = \frac{1}{x} - x,$$

and substitute in Mr. Russell's modular equation of the 29th order, we shall obtain an equation of the 15th order for t, having a factor

corresponding to the value $\beta = 198$, and thus affording an independent verification of the numerical coefficients in this modular equation.

$\Delta = 62$. Taking Schröter's, Fiedler's, or Russell's modular equation of the 31st order,

$$(P^2-4Q)^2-4PR = 0,$$

and putting $\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4,$

then
$$P = \sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1 = x\,(t+\sqrt[4]{2}),$$
$$Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'}+\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = x^2\,(\sqrt[4]{2}\,t+1),$$
$$R = \sqrt[4]{\kappa\lambda\kappa'\lambda'} = \sqrt[4]{2}\,x^3,$$

where $t = \dfrac{1}{x}+x,$

then $\{(t+\sqrt[4]{2})^2-4\sqrt[4]{2}\,t-4\}^2-4\sqrt[4]{2}\,(t+\sqrt[4]{2}) = 0,$

factorizing into

$$\{t^2-(2-\sqrt{2})\sqrt[4]{2}\,t-6+3\sqrt{2}\}\{t^2-(2+\sqrt{2})\sqrt[4]{2}\,t-2+\sqrt{2}\} = 0,$$

of which the second factor will give the required numerical results.

$\Delta = 78$. Taking Fiedler's form of the modular equation of the 39th order, and putting

$$\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4, \quad t = x-\frac{1}{x},$$

then
$$Z_1 = x^2+\sqrt[4]{2}\,x-1 = x\,(t+\sqrt[4]{2}),$$
$$Z_2 = \sqrt[4]{2}\,x^3-x^2-\sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t-1),$$
$$Z_3 = -\sqrt[4]{2}\,x^3,$$
$$Z_{3'} = x^2\,\{t-\sqrt[4]{2})^2+4\};$$

and then the modular equation becomes, dividing out x^7,

$$(t+\sqrt[4]{2})^5\,\{(t-\sqrt[4]{2})^2+4\}+4\sqrt[4]{2}\,\{(t-\sqrt[4]{2})^2+4\}^2$$
$$+20\sqrt[4]{2}\,(t+\sqrt[4]{2})^2\,\{(t-\sqrt[4]{2})^2+4\}-8\sqrt[4]{2}\,(t+\sqrt[4]{2})^4$$
$$-144\sqrt{2}\,(t+\sqrt[4]{2}) = 0,$$

an equation of the 7th degree for t.

$\Delta = 94$. Taking Fiedler's or Russell's modular equation of the 47th order, and putting

$$\sqrt[4]{\kappa\lambda} = x^2, \quad \sqrt[4]{\kappa'\lambda'} = \sqrt[4]{2}\,x, \quad \frac{1}{x}+x = t,$$

then $$P = \quad x^2+\sqrt[4]{2}\,x+1 \quad = x\,(t+\sqrt[4]{2}),$$
$$Q = \sqrt[4]{2}\,x^3+x^2+\sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t+1),$$
$$R = \sqrt[4]{2}\,x^3,$$

and $$P^2-4Q = x^2\,(t^2-2\sqrt[4]{2}\,t+\sqrt{2}-4)$$
$$= x^2\left\{(t-\sqrt[4]{2})^2-4\right\}.$$

Then Russell's modular equation of the 47^{th} order (*Proc. Lond Math. Soc.*, Nov., 1887) becomes, dividing out x^6,

$$\left\{(t-\sqrt[4]{2})^2-4\right\}^3-28\sqrt[4]{2}\,(t+\sqrt[4]{2})^3-96\sqrt[4]{2}\,(t+\sqrt[4]{2})(\sqrt[4]{2}\,t+1)$$
$$-128\sqrt{2} = 0,$$

a sextic equation for t.

But, if we take Hurwitz's modular equation of the 47^{th} order (*Math. Ann.*, XIV.),

$$\left\{2\,(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1)-\sqrt[3]{4}\,\sqrt[12]{\kappa\lambda\kappa'\lambda'}\right\}^2$$
$$= 8\,(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1)-7\sqrt[3]{16}\,\sqrt[6]{\kappa\lambda\kappa'\lambda'},$$

and put $$\kappa\lambda = x^8, \qquad \kappa'\lambda' = 2x^4,$$

then $$(2x^2+2\sqrt[4]{2}\,x-2-\sqrt{2}\,\sqrt[4]{2}\,x)^2$$
$$= 8x^4+8\sqrt{2}\,x^2+8-14\sqrt{2}\,x^2$$
$$= 8x^4-6\sqrt{2}\,x^2+8,$$

or $$\left\{2x^2+\sqrt[4]{2}\,(2-\sqrt{2})\,x-2\right\}^2 = 8x^4-6\sqrt{2}\,x^2+8,$$
$$4x^4+4\sqrt[4]{2}\,(2-\sqrt{2})\,x^3+\sqrt{2}\,(6-4\sqrt{2})\,x^2-8x^2$$
$$-4\sqrt[4]{2}\,(2-\sqrt{2})\,x+4 = 8x^4-6\sqrt{2}\,x^2+8,$$

or $$4x^4-4\sqrt[4]{2}\,(2-\sqrt{2})\,x^3-4\sqrt{2}\,(3-2\sqrt{2})\,x^2+4\sqrt[4]{2}\,(2-\sqrt{2})\,x+y = 0,$$
$$x^4-\sqrt[4]{2}\,(2-\sqrt{2})\,x^3-\sqrt{2}\,(3-2\sqrt{2})\,x^2+\sqrt[4]{2}\,(2-\sqrt{2})\,x+1 = 0.$$

Putting $$\frac{1}{x}-x = v,$$

this equation becomes

$$v^2-\sqrt[4]{2}\,(2-\sqrt{2})\,v-3\sqrt{2}+6 = 0,$$
$$\left(v-\frac{\sqrt{2}-1}{\sqrt[4]{2}}\right)^2 = \frac{9-8\sqrt{2}}{\sqrt{2}}.$$

But putting $$\frac{1}{x}+x = t,$$

then $t^2-2+\sqrt[4]{2}\,(2-\sqrt{2})\sqrt{(t^2-4)}-\sqrt{2}\,(3-2\sqrt{2})=0,$

$$t^2-3\sqrt{2}+2+\sqrt[4]{2}\,(2-\sqrt{2})\sqrt{(t^2-4)}=0,$$

$$t^4-(6\sqrt{2}-4)\,t^2+22-12\sqrt{2}=(6\sqrt{2}-8)(t^2-4),$$

$$t^4-(12\sqrt{2}-12)\,t^2-10+12\sqrt{2}=0,$$

a quadratic for t^2, from which the equations for β and γ can be derived.

Appendix.

[The current number of the *Acta Mathematica*, XI. 4, contains an article by H. Weber, "Zur Theorie der elliptischen Functionen" (zweite Abhandlung), which gives a number of numerical results for the modular functions in Complex Multiplication, agreeing in many respects with the results given in this paper. By the aid of Weber's calculations, it is possible in some instances to add to and simplify some of the cases considered above, examples of which are given herewith, as well as developments of cases not treated completely before.

Class A.

This is the case of $\Delta\equiv 3$, mod. 8; it was convenient to put

$$a=\frac{(1-t^8)^3}{t^8}=\frac{(1-256s^{24})^3}{256s^{24}},$$

so that $$t^8=256s^{24}=16\kappa^2\kappa'^2;$$

and then $$s\approx q^{\frac{1}{24}}.$$

With this notation, then, for $\Delta=35$ (Weber, *Acta Math.*, XI., p. 388), $$2s^3-(\sqrt{5}+1)(s^2-s)-1=0.$$

$\Delta=51$: $$t^3+t^2+(\sqrt{17}+4)\,t-1=0 \quad [\text{p. } 385].$$

$\Delta=91$: $$2s^3+(\sqrt{13}+1)\,s^2+2s-1=0 \quad [\text{p. } 385].$$

$\Delta=99$: $t^3+(\sqrt{33}+4)\,t^2+(13+2\sqrt{33})\,t-1=0$ [p. 384],

leading to the value of a, namely

$$a=2^6\,(4591804316+799330532\sqrt{33}) \quad [\text{p. } 384].$$

Class C.

$\Delta=17$. Weber's equation gives

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\tfrac{1}{2}\,(\sqrt{17}+1).$$

$\Delta = 29$. Weber's equation for

$$x = \frac{1}{\sqrt[3]{(2\kappa\kappa')}}$$

is $$2x^3-9x^2-8x-5 = \sqrt{29}\,(x+1)^2;$$

which when rationalised becomes the reciprocal sextic

$$x^6-9x^5+5x^4-2x^3-5x^2-9x-1 = 0.$$

The corresponding equation in $z = \dfrac{1}{x^3}$ agrees with the one given in this paper, p. 328,

$$z^6+588z^5-979z^4+1960z^3+979z^2+588z-1 = 0,$$

which therefore can be reduced to the cubic

$$z^3+294z^2+155z+70 = \sqrt{29}\,(55z^2+28z+13).$$

$\Delta = 41$. The equation for

$$z = x+\frac{1}{x},$$

where $$\frac{1}{x} = \sqrt[6]{(2\kappa\kappa')}$$

is, according to Weber [*Acta Math.*, XI., p. 388],

$$z^2-\tfrac{1}{2}\,(\sqrt{41}+5)\,z+\tfrac{1}{2}\,(7+\sqrt{41}) = 0;$$

so that z is the root of the biquadratic

$$z^4-5z^3+3z^2+3z+2 = 0;$$

and then the equation for $\gamma = \dfrac{1}{2\kappa\kappa'}+2\kappa\kappa'$

is easily calculated, and also the equations for α and β.

$\Delta = 73$. Since $p = 2$ for this number, as well as for $\Delta = 17$, we can anticipate that α, β, γ will each be of the form $M\sqrt{73}+N$; and, in fact, by approximate numerical calculation, we shall find

$$\frac{1}{2\kappa\kappa'}+2\kappa\kappa' = \gamma = 4930\sqrt{73}+42120,$$

equivalent to $$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')} = \tfrac{1}{2}\,(\sqrt{73}+5).$$

$\Delta = 193$. This is another number for which $p = 2$, according to

Gauss (*Werke*, t. II., p. 288); and from the approximate values of $\dfrac{1}{\sqrt[6]{(2\kappa\kappa')}}$, namely, $\frac{1}{2}\sqrt{2}\,e^{\frac{1}{24}\pi\sqrt{\Delta}}$, we find

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\sqrt{193}+13,$$

whence α, β, γ can be inferred.

Similarly, for $\Delta = 97$, we shall find

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\tfrac{1}{2}\,(\sqrt{97}+9).$$

leading to $\qquad \gamma = 33210\sqrt{97}+327080,$

instead of the value given above, p. 338.

These values lead to the approximate equations

$$e^{\frac{1}{4}\pi\sqrt{97}} \approx 9\sqrt{97}+85,$$

$$e^{\frac{1}{4}\pi\sqrt{193}} \approx 52\sqrt{193}+720.$$

10*th Oct.*, 1888.]

Thursday, May 10*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

The following communications were made:—

Some Theorems on Parallel Straight Lines, together with some attempts to prove Euclid's Twelfth Axiom: J. Cook Wilson, M.A.

On Cyclicants or Ternary Reciprocants, and allied Functions E. B. Elliott, M.A.

On the Flexure and the Vibrations of a Curved Bar: Prof. H. Lamb, F.R.S.

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on Analogous Relations in Space of Three Dimensions: S. Roberts, F.R.S.

Lamé's Differential Equation; and Stability of Orbits: Prof. Greenhill, F.R.S.

The following presents were received:—

Cabinet Likeness of Dr. Glaisher, F.R.S., for the Society's Album.

"Proceedings of the Royal Society," Vol. XLIII., No. 264.

"Educational Times," for May.

"Proceedings of the Physical Society of London," Vol. IX., Part II., April, 1888.

"Mathematics from the 'Educational Times,'" Vol. XLVIII.

"A Treatise on Hydrodynamics," by A. B. Basset, M.A., Vol. I., 8vo; Cambridge and London, 1888.

"Bulletin de la Société Mathématique de France," Tome XVI., Nos. 2 and 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 4; Leipzig, 1888.

"Journal für die reine und angewandte Mathematik," Band CIII., Heft I.; Berlin, 1888.

"Acta Mathematica," XI., 2.

"Rendiconti del Circolo Matematico di Palermo," Fasc. 1 and 2, Tomo II.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VIII., No. 3; Coimbra, 1887.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXXII., Heft IV.; Zürich, 1887.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 1; Gennaio, 1888, Roma.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 55 and 56.

"The Earth's Polar Floods in Perihelion," by G. T. Carruthers (Subathu, India, March, 1888, 5 pp.).

Purchased: "Year-book of the Scientific and Learned Societies of Great Britain and Ireland" (Fifth Annual Issue; London, Charles Griffin & Co., 1888).

On the Flexure and the Vibrations of a Curved Bar.

By Professor HORACE LAMB, M.A., F.R.S.

[*Read May* 10*th*, 1888.]

The flexure of a curved bar has been treated in a general manner by Kirchhoff, Clebsch, and Thomson and Tait, but the special applications which have been made of the theory are very few. In this paper I propose to discuss the flexure in its own plane of a uniform bar whose axis forms in the unstrained state an arc of a circle. After establishing the general equations and the terminal conditions, some simple statical problems are solved, and I then proceed to discuss the vibrations of a "free-free" bar, with special reference to the case where the total curvature is slight. This latter problem is interesting as bearing on some observations by Chladni, referred to by Tyndall in his book on "Sound," Chap. IV.

Taking the centre of the circle as origin, and denoting the radius

by a, let the polar coordinates of any point of the bar be changed by the flexure from (a, θ) to $(a+R, \theta+\Theta)$, where R, Θ are small. If we neglect the extensibility of the bar, these quantities are not independent, but are connected by the relation

$$R = -a\frac{d\Theta}{d\theta}.^*$$

The rotation experienced by any element $ad\theta$ is easily found to be

$$\Theta+\Theta'',$$

the accents denoting differentiations with respect to θ, whence, for the change of curvature, we have

$$\Delta\rho^{-1} = (\Theta'+\Theta''')/a.\dagger$$

The formula for the potential energy is therefore

$$V = \tfrac{1}{2}B\int(\Delta\rho^{-1})^2\,ad\theta$$

$$= \tfrac{1}{2}\frac{B}{a}\int(\Theta'+\Theta''')^2\,d\theta.$$

The applied forces at any point of the bar may be specified by the radial component P and the tangential component Q, both estimated per unit length. We may also include the case where finite forces are concentrated in an infinitely short element of the length; these may be denoted by P_0, Q_0. The force on either end may be analysed into a radial component $\overline{P}$, a tangential component $\overline{Q}$, and a couple $\overline{N}$.

The variational equation of motion is then

$$\int(\ddot{R}\delta R+a^2\ddot{\Theta}\delta\Theta)\,\sigma a\,d\theta+\delta V = \int(P\delta R+Qa\delta\Theta)\,a\,d\theta$$

$$+\Sigma\left\{P_0\delta R+Q_0a\delta\Theta\right\}+\left[\overline{P}\delta R+\overline{Q}a\delta\Theta+\overline{N}\delta(\Theta+\Theta'')\right],$$

where σ is the mass per unit length, and the square brackets [ ] refer to the extremities.

If we substitute $R = -a\Theta'$, and integrate by parts in the usual

---

* Rayleigh, *Sound*, § 233.

† Or from the approximate formula, $\frac{1}{\rho} = \frac{1}{r} - \frac{1}{r^2}\frac{d^2r}{d\theta^2}$.

way, we find

$$\int \left\{ \sigma a^3 (\ddot{\Theta} - \ddot{\Theta}'') - \frac{B}{a} (\Theta'' + 2\Theta^{\text{iv}} + \Theta^{\text{vi}}) \right\} \delta\Theta \, d\theta$$

$$+ \Sigma \left[\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{\text{v}}) \right] \delta\Theta$$

$$- \Sigma \left[\Theta'' + \Theta^{\text{iv}} \right] \delta\Theta' + \Sigma \left[\Theta' + \Theta''' \right] \delta\Theta''$$

$$= \int \left(Q + \frac{dP}{d\theta} \right) a^3 \delta\Theta \, d\theta + \Sigma \, Q_0 a \, \delta\Theta - \Sigma P_0 a \delta\Theta'$$

$$+ \left[(-Pa^2 + \bar{Q}a + \bar{N}) \, \delta\Theta - \bar{P} a \, \delta\Theta' + \bar{N} \delta\Theta'' \right],$$

where the integrated terms on the left-hand side refer as well to the points of discontinuity (as regards the form of Θ) at which the forces P_0, Q_0 act, as to the extremities of the bar. The differential equation to be satisfied at each point of the bar is therefore

$$\sigma a^3 (\ddot{\Theta} - \ddot{\Theta}'') - \frac{B}{a} (\Theta'' + 2\Theta^{\text{iv}} + \Theta^{\text{vi}}) = a^3 \left(Q + \frac{dP}{d\theta} \right).$$

The terminal conditions are

$$\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{\text{v}}) = -Pa^2 + \bar{Q}a + \bar{N},$$

$$\frac{B}{a} (\Theta'' + \Theta^{\text{iv}}) = \bar{P}a,$$

$$\frac{B}{a} (\Theta' + \Theta''') = \bar{N};$$

whilst at a point of discontinuity we have

$$\left[\sigma a^3 \ddot{\Theta}' + \frac{B}{a} (\Theta' + 2\Theta''' + \Theta^{\text{v}}) \right] = Q_0 a,$$

$$\frac{B}{a} \left[\Theta'' + \Theta^{\text{iv}} \right] = P_0 a,$$

$$\left[\Theta' + \Theta''' \right] = 0,$$

the square brackets indicating that the differences of the values of the enclosed quantities on the two sides of the point in question are to be taken. These latter conditions may be simplified with the help of the obvious geometrical condition that the values of Θ, Θ', Θ' must be continuous.

As a first example, consider the equilibrium of the bar subject to applied force at its extremities only. The general equation becomes

$$\Theta'' + 2\Theta^{\text{iv}} + \Theta^{\text{vi}} = 0,$$

while the terminal conditions reduce to

$$\frac{B}{a^3}(\Theta'''+\Theta^{v}) = \bar{Q},$$

$$\frac{B}{a^2}(\Theta''+\Theta^{iv}) = \bar{P},$$

$$\frac{B}{a}(\Theta'+\Theta''') = \bar{N}.$$

The differential equation gives

$$\Theta = C+D\theta+(E+F\theta)\cos\theta+(G+H\theta)\sin\theta,$$

from which the terms in C, E, and G may, for the present purpose, be discarded as expressing a mere displacement of the bar as a whole. There remain three simple types of solution, from which the most general case can be derived by superposition. In the first place, if the applied forces reduce to two equal and opposite couples $\pm\bar{N}$ at the extremities, we find

$$\Theta = \frac{\bar{N}a}{B}\theta, \quad R = -\frac{\bar{N}a^2}{B},$$

i.e., the bar remains circular in form, but its radius is altered by the fraction $\bar{N}a/B$. Next, taking the origin of θ at the middle of the bar, consider the case where Θ is an odd function, *viz.*,

$$\Theta = D\theta+F\theta\cos\theta.$$

If there be no couples at the ends ($\theta = \pm\alpha$), we have

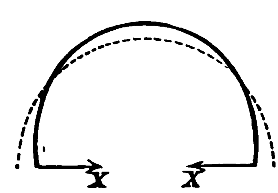
$$D = 2F\cos\alpha,$$

whence $$\pm\bar{Q} = 2F\frac{B}{a^3}\cos\alpha, \quad \bar{P} = 2F\frac{B}{a^2}\sin\alpha.$$

The resultant force at either extremity is along the chord. Denoting it by X, we have

$$\Theta = -\frac{a^2X}{B}(\cos\alpha+\tfrac{1}{2}\cos\theta)\,\theta,$$

$$R = \frac{a^3X}{B}(\cos\alpha+\tfrac{1}{2}\cos\theta-\tfrac{1}{2}\theta\sin\theta).$$



In particular, if $\alpha = \frac{1}{2}\pi$,

$$R_0 = -\tfrac{1}{4}\pi\, a^3X/B;$$

whilst, if $\alpha = \pi$, $$a\Theta_0 = \tfrac{3}{2}\pi a^3X/B.$$

Finally, we have the solution

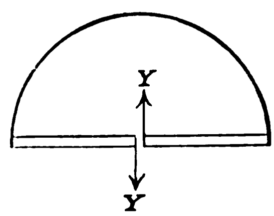
$$\Theta = H\theta \sin\theta,$$

which gives

$$\overline{Q} = 2H\frac{B}{a^2}\sin\alpha, \quad \pm\overline{P} = 2H\frac{B}{a^2}\cos\alpha, \quad \overline{N} = 2H\frac{B}{a}\sin\alpha.$$

If we write

$$Y = 2H.B/a^2,$$



we have the case of a bar bent by equal and opposite forces Y applied at the extremities of rigid pieces attached to the ends, in the manner shown in the figure. The case of a nearly complete circle ($\alpha \doteq \pi$) is worth notice.

As an example of points of discontinuity, take the case of a circular hoop deformed by a pair of equal and opposite forces at the extremities of a diameter. The differential equation is, as before,

$$\Theta'' + 2\Theta^{\text{iv}} + \Theta^{\text{vi}} = 0,$$

whilst the dynamical conditions to be satisfied at the points of discontinuity are

$$\left[\Theta' + 2\Theta''' + \Theta^{\text{v}}\right] = 0,$$

$$\frac{B}{a}\left[\Theta'' + \Theta^{\text{iv}}\right] = P_0 a,$$

$$\left[\Theta' + \Theta'''\right] = 0.$$

Combined with the geometrical conditions, these show that Θ, Θ', Θ'', Θ''', Θ^{v} are to be continuous, whilst

$$\frac{B}{a}\left[\Theta^{\text{iv}}\right] = P_0 a.$$

Taking the diameter in question as initial line, we may assume, from $\theta = 0$ to $\theta = \pi$,

$$\Theta = C + D\theta + (E + F\theta)\cos\theta + (G + H\theta\sin\theta),$$

and from $\theta = 0$ to $\theta = -\pi$,

$$\Theta = C_1 + D_1\theta + (E_1 + F_1\theta)\cos\theta + (G_1 + H_1\theta\sin\theta).$$

The foregoing conditions then lead to*

$$C_1 - C = 4H, \quad E_1 - E = -4H,$$

* [*Oct.* 1888.—A numerical error has been corrected here.]

$$D_1 = D = \frac{4}{\pi} H, \quad F = F_1 = 0, \quad G_1 = G,$$

where $$H = P_0 a^3/4B.$$

Hence we may write, from $\theta = 0$ to $\theta = \pi$,

$$\Theta = \frac{P_0 a^3}{2B} \left(-1 + \frac{2\theta}{\pi} + \cos\theta + \tfrac{1}{2}\theta \sin\theta \right),$$

whilst, from $\theta = 0$ to $\theta = -\pi$,

$$\Theta = \frac{P_0 a^3}{2B} \left(1 + \frac{2\theta}{\pi} - \cos\theta - \tfrac{1}{2}\theta \sin\theta \right).$$

We have here omitted certain terms of the form

$$K + L\cos\theta + M\sin\theta,$$

which are common to both expressions, and represent a mere displacement without deformation. The corresponding values of R are

$$R = -\frac{P_0 a^3}{2B} \left(\frac{2}{\pi} - \tfrac{1}{2}\sin\theta + \tfrac{1}{2}\theta\cos\theta \right),$$

and $$R = -\frac{P_0 a^3}{2B} \left(\frac{2}{\pi} + \tfrac{1}{2}\sin\theta - \tfrac{1}{2}\theta\cos\theta \right).$$

We thence ascertain that the diameter $\theta = 0$ is increased by the amount $(\pi^2 - 8)/4\pi \,.\, P_0 a^3/B$, whilst the perpendicular diameter is shortened by $(4-\pi)/2\pi \,.\, P_0 a^3/B$.*

As a final statical example, we may calculate the deformation, due to its own weight, of a hoop suspended from a point of the circumference. Taking the radius through this point as initial line, we have

$$P = -g\sigma\cos\theta, \quad Q = g\sigma\sin\theta,$$

so that the differential equation is

$$\Theta'' + 2\Theta^{\mathrm{iv}} + \Theta^{\mathrm{vi}} = -\frac{2g\sigma a^3}{B}\sin\theta.$$

We therefore write, omitting unnecessary terms,

$$\Theta = D\theta + F\theta\cos\theta + H\theta\sin\theta + K\theta^2\sin\theta,$$

where $$K = -\frac{g\sigma a^3}{4B}.$$

* [*Oct.* 1888.—This agrees with the result quoted by Pearson from some unpublished lectures of Saint Venant, *History of Elasticity*, § 1575.]

The dynamical conditions to be satisfied at the point of suspension are

$$\frac{B}{a}\Big[\Theta'+2\Theta'''+\Theta^{\mathrm{v}}\Big]_0^{2\pi}=0,$$

$$\frac{B}{a}\Big[\Theta''+\Theta^{\mathrm{iv}}\Big]_0^{2\pi}=P_0 a,$$

$$\Big[\Theta'+\Theta'''\Big]_0^{2\pi}=0.$$

To these we must add the geometrical conditions that Θ, Θ', Θ'' are to be continuous. We thence find

$$D=-4K,\quad F=4K,\quad H=-2\pi K,$$

$$P_0=-8\pi\frac{B}{a^3}K=2\pi a\sigma g.$$

In order that R', $(=-a\Theta'')$, may be zero at the point of suspension, we must add to Θ the terms $4\pi K(1-\cos\theta)$. We thus obtain

$$\Theta=K\left\{(\theta-\pi)^2\sin\theta+4(\theta-\pi)\cos\theta-4(\theta-\pi)-\pi^2\sin\theta\right\}.$$

It easily follows that the vertical diameter is increased by $(\pi^2-8)\,g\sigma a^4/4B$, whilst the horizontal diameter is shortened by $(4-\pi)\,g\sigma a^4/2B$.*

Let us next examine the flexural vibrations of the bar, supposed free from external force except at the extremities. If we assume that $\Theta \propto e^{ipt}$, and write

$$\sigma p^2/B=k^4,$$

the differential equation is

$$\Theta^{\mathrm{vi}}+2\Theta^{\mathrm{iv}}+(1-k^4a^4)\,\Theta''+k^4a^4\Theta=0.$$

At a *free* end we have the conditions

$$\Theta^{\mathrm{v}}+2\Theta'''+(1-k^4a^4)\,\Theta'=0,$$

$$\Theta^{\mathrm{iv}}+\Theta''=0,$$

$$\Theta'''+\Theta'=0,$$

* [*Oct.* 1888.—Comparing with the solution of the preceding problem, we learn that the elongation of the vertical, and the shortening of the horizontal, diameter are each half what they would have been if the weight of the hoop had been concentrated in its lowest point.]

the first of which may, in virtue of the differential equation, be replaced by

$$\int \Theta \, d\theta = 0.$$

The conditions to be satisfied at a *clamped* end are, of course,

$$\Theta = 0, \quad \Theta' = 0, \quad \Theta'' = 0.$$

Assuming that $\Theta = \Sigma A e^{\lambda\theta}$, we find

$$\lambda^6 + 2\lambda^4 + \lambda^2 + k^4a^4 (1 - \lambda^2) = 0.$$

The roots of this are functions of ka, and the elimination of the arbitrary constants from the six boundary conditions (three for each end) gives an equation to determine ka, and thence the frequency $p/2\pi$.

The interpretation of the solution in the general case would be difficult, but we may obtain some results of interest by proceeding to a first approximation in the case where the total curvature of the bar is slight. Fixing our attention more particularly on the case of a "free-free" bar, we see, by comparison with the known theory for the case where the bar is straight, that $\sigma p^2/B = m^4/l^4$ nearly, where l, $= 2a\alpha$, is the length, and m is a root of a certain transcendental equation. Hence $k^4a^4 = (m/2\alpha a)^4$, nearly, and is therefore large. One root of the cubic in λ^2 is therefore nearly equal to unity; and the remaining roots are given by

$$\lambda^2 + 1 = \pm k^2a^2 \text{ nearly.}$$

Continuing the approximations by ordinary methods, we find

$$\lambda^2 = \mu^2, \quad -\nu^2, \quad \varpi^2,$$

where

$$\mu^2 = k^2a^2 - \tfrac{3}{2} - \tfrac{7}{8}\frac{1}{k^2a^2} - \frac{2}{k^4a^4},$$

$$\nu^2 = k^2a^2 + \tfrac{3}{2} - \tfrac{7}{8}\frac{1}{k^2a^2} + \frac{2}{k^4a^4},$$

$$\varpi^2 = 1 + \frac{4}{k^4a^4}.$$

If we take the origin of θ at the middle of the bar, the fundamental modes will fall into two classes, according as Θ is an odd or an even function of θ. The former class is the more important as including the gravest mode, and is therefore taken first. We therefore assume

$$\Theta = F \sinh \mu\theta + G \sin \nu\theta + H \sinh \varpi\theta.$$

The conditions at the extremities ($\theta = \pm\alpha$) then give

$$\left.\begin{aligned} \frac{F}{\mu}\cosh\mu a - \frac{G}{\nu}\cos\nu a + \frac{H}{\varpi}\cosh\varpi a &= 0 \\ \mu^2(\mu^2+1)F\sinh\mu a + \nu^2(\nu^2-1)G\sin\nu a + \varpi^2(\varpi^2+1)H\sinh\varpi a &= 0 \\ \mu(\mu^2+1)F\cosh\mu a - \nu(\nu^2-1)G\cos\nu a + \varpi(\varpi^2+1)H\cosh\varpi a &= 0 \end{aligned}\right\}$$

whence, by elimination of F, G, H,

$$\begin{vmatrix} \frac{1}{\mu}\cosh\mu a, & -\frac{1}{\nu}\cos\nu a, & \frac{1}{\varpi}\cosh\varpi a \\ \mu^2(\mu^2+1)\sinh\mu a, & \nu^2(\nu^2-1)\sin\nu a, & \varpi^2(\varpi^2+1)\sinh\varpi a \\ \mu(\mu^2+1)\cosh\mu a, & -\nu(\nu^2-1)\cos\nu a, & \varpi(\varpi^2+1)\cosh\varpi a \end{vmatrix} = 0.$$

If we expand this according to the constituents of the last column, we find that the parts corresponding to the three constituents are of the orders k^7a^7, k^3a^3, k^3a^3, respectively. Hence, subject to an error of order $1/k^4a^4$, we may retain only the first of these, which gives

$$\mu a\tanh\mu a = -\nu a\tan\nu a.$$

To solve this by approximation, put

$$ka\alpha = \tfrac{1}{2}m + x,$$

where m is a root of $\quad \tanh\tfrac{1}{2}m = -\tan\tfrac{1}{2}m,$

and x is small. This makes

$$\mu a = ka\alpha - \tfrac{3}{4}\frac{\alpha^2}{ka\alpha}$$

$$= \tfrac{1}{2}m + x - \tfrac{3}{2}\frac{\alpha^2}{m},$$

and similarly $\quad \nu a = \tfrac{1}{2}m + x + \tfrac{3}{2}\frac{\alpha^2}{m}.$

Substituting in the equation, and retaining only the first powers of x and α^2, we find, after a little reduction,

$$x = -3\frac{\alpha^2}{m^2}\left(1 + \tfrac{1}{2}m\tan\tfrac{1}{2}m\right)\tan\tfrac{1}{2}m.$$

The values of m are approximately $3\pi/2$, $7\pi/2$, $11\pi/2$, &c., so that x is always negative. For the lowest root we have $m = 4{\cdot}7300$, whence

$$x = -{\cdot}17438\,\alpha^2.$$

To find the alteration of pitch due to the curvature, we have

$$\sigma p^2/B = k^4 = \left(\frac{m}{l}\right)^4 \left(1+\frac{2x}{m}\right)^4.$$

Hence, if $p_0/2\pi$ be the frequency of the corresponding mode for a straight bar

$$\frac{p}{p_0} = \left(1+\frac{2x}{m}\right)^2 = 1+\frac{4x}{m}, \text{ nearly,*}$$

so that the pitch is *lowered.* For the gravest mode

$$\frac{p}{p_0} = 1 - \cdot 14747\, a^2.$$

The position of the nodes ($R = 0$) is determined by

$$\begin{vmatrix} \mu \cosh \mu\theta, & \nu \cos \nu\theta, & \varpi \cosh \varpi\theta \\ \frac{1}{\mu} \cosh \mu a, & -\frac{1}{\nu} \cos \nu a, & \frac{1}{\varpi} \cosh \varpi a \\ \mu(\mu^2+1)\cosh \mu a, & -\nu(\nu^2-1)\cos \nu a, & \varpi(\varpi^2+1)\cosh \varpi a \end{vmatrix} = 0,$$

or $$(\nu^2+\varpi^2)(\nu^2-\varpi^2-1)\,\mu^2 \frac{\cosh \mu\theta}{\cosh \mu a} + (\mu^2-\varpi^2)(\mu^2+\varpi^2+1)\,\nu^2 \frac{\cos \nu\theta}{\cos \nu a}$$
$$+(\mu^2+\nu^2)(\mu^2-\nu^2+1)\frac{\cosh \varpi\theta}{\cosh \varpi a} = 0.$$

Recalling the approximate values of μ^2, ν^2, ϖ^2, we find that, subject to an error of the same order $(1/k^4a^4)$ as before, this reduces to

$$(\nu^2-1)\cos \nu a \cosh \mu\theta + (\mu^2+1)\cosh \mu a \cos \nu\theta = 0,$$

or $$\cos \nu a \cosh \mu\theta + \cosh \mu a \cos \nu\theta = \frac{1}{k^2a^2} \cos \nu\theta \cosh \mu a,$$

approximately. To solve this, write $\theta/a = z+\epsilon$, where z is a root of

$$\cos \tfrac{1}{2}m \cosh \tfrac{1}{2}mz + \cosh \tfrac{1}{2}m \cos \tfrac{1}{2}mz = 0.\dagger$$

* This calculation may be verified by the method explained in Lord Rayleigh's *Sound*, § 89; assuming as a hypothetical type that R has the same form as for a straight bar, viz.,

$$R \propto \cos \tfrac{1}{2}m \cosh \frac{m\theta}{2a} + \cosh \tfrac{1}{2}m \cos \frac{m\theta}{2a}.$$

† In comparing with the ordinary theory for a straight bar, it must be borne in mind that the origin is now at the middle point. *Cf.* Greenhill, *Messenger of Mathematics*, Dec. 1886.

We have already found

$$\mu a = \tfrac{1}{2}m + \left(x - \tfrac{3}{8}\,\frac{a^2}{m}\right),$$

$$\nu a = \tfrac{1}{2}m + \left(x + \tfrac{3}{8}\,\frac{a^2}{m}\right),$$

whence

$$\mu\theta = \mu a\,(z+\epsilon)$$

$$= \tfrac{1}{2}mz + \left(x - \tfrac{3}{8}\,\frac{a^2}{m}\right)z + \tfrac{1}{2}m\,\epsilon,$$

$$\nu\theta = \tfrac{1}{2}mz + \left(x + \tfrac{3}{8}\,\frac{a^2}{m}\right)z + \tfrac{1}{2}m\,\epsilon.$$

Substituting in the above equation, expanding, and retaining only the first powers of x, a^2, ϵ, we find, after effecting some reductions by means of the equation

$$\tanh \tfrac{1}{2}m = -\tan \tfrac{1}{2}m,$$

$$\tfrac{1}{2}m\,(\tanh \tfrac{1}{2}mz + \tan \tfrac{1}{2}mz)\,\epsilon$$

$$= -\left(x - \tfrac{3}{8}\,\frac{a^2}{m}\right)z \tanh \tfrac{1}{2}mz - \left(x + \tfrac{3}{8}\,\frac{a^2}{m}\right)z \tan \tfrac{1}{2}mz$$

$$+3\,\frac{a^2}{m}\tan \tfrac{1}{2}m - 4\,\frac{a^2}{m^2}$$

In the gravest mode

$$m = 4{\cdot}7300 = 271^\circ\,0'\,40'',$$

$$z = {\cdot}55164,$$

whence

$$\tfrac{1}{2}mz = 1{\cdot}3046 = 74^\circ\,45'\,20'',$$

$$\tan \tfrac{1}{2}mz = 3{\cdot}6694, \quad \tanh \tfrac{1}{2}mz = {\cdot}86295,$$

$$\tan \tfrac{1}{2}m = -{\cdot}98251.$$

Using the value already found for x, we obtain finally

$$\epsilon = -{\cdot}07959\,a^2,$$

so that the position of the nodes is given by

$$\pm\,\theta/a = {\cdot}55164 - {\cdot}07959\,a^2.$$

The conclusions that the effect of curvature is to lower the pitch, and at the same time to make the nodes approach the middle of the bar, are in agreement with the observations of Chladni.*

The asymmetrical fundamental modes may be treated more

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* *Akustik*, § 99.

briefly. Assuming

$$\Theta = F \cosh \mu\theta + G \cos \nu\theta + H \cosh \varpi\theta,$$

the terminal conditions lead to

$$\begin{vmatrix} \frac{1}{\mu} \sinh \mu a, & \frac{1}{\nu} \sin \nu a, & \frac{1}{\varpi} \sinh \varpi a \\ \mu^2 (\mu^2+1) \cosh \mu a, & \nu^2 (\nu^2-1) \cos \nu a, & \varpi^2 (\varpi^2+1) \cosh \varpi a \\ \mu (\mu^2+1) \sinh \mu a, & \nu (\nu^2-1) \sin \nu a, & \varpi (\varpi^2+1) \sinh \varpi a \end{vmatrix} = 0,$$

whence, to the same degree of approximation as before,

$$\mu a \coth \mu a = \nu a \cot \nu a.$$

Writing $\quad kaa = \tfrac{1}{2}m + y,$

where m is a root of $\quad \coth \tfrac{1}{2}m = \cot \tfrac{1}{2}m,$

and y is small, we find

$$y = -3 \frac{a^2}{m^2} (\tfrac{1}{2}m \cot \tfrac{1}{2}m - 1) \cot \tfrac{1}{2}m.$$

The effect of the curvature is to alter the frequency in the ratio

$$\frac{p}{p_0} = 1 + \frac{4y}{m}, \text{ nearly.}$$

It is easily seen that y is always negative, so that the pitch is in all cases lowered.

[*Note added Oct.* 1888.—It has been assumed throughout that the elongation of the elements of the bar may be neglected. It may be shown that the amount of elongation which actually occurs is of no importance in the problems above considered, except in one very special case. Take, for example, the problem discussed near the foot of p. 368. The stretching (or compressing) force will be greatest at the middle of the bar, where its value is X, and the energy per unit length due to the stretching will there be $X^2/2q\omega$, where q is Young's modulus, and ω the sectional area. Again, the bending moment is also greatest at the middle, where it is $X(1-\cos \alpha)\, a$, so that the energy due to the bending is, per unit length,

$$= \tfrac{1}{2} \frac{X^2 a^2}{B} (1-\cos \alpha)^2 = \tfrac{1}{2} \frac{X^2 a^2}{q\kappa^2\omega} (1-\cos \alpha)^2,$$

where κ is the proper radius of gyration of the section. The ratio of the former energy to the latter is $\kappa^2/a^2 (1-\cos \alpha)^2$, which in any practical case is small, unless indeed α be small. In the problem to which the figure near the top of p. 369 refers, the corresponding ratio is readily found to be κ^2/a^2, which is always small.]

On Cyclicants, or Ternary Reciprocants, and Allied Functions.

By E. B. Elliott, M.A.

[*Read May* 10*th*, 1888.]

CONTENTS.

1. It has become almost necessary to depart from the nomenclature which I have hitherto adopted in my papers on this subject (*Proceedings*, Vol. XVII., pp. 172—196; Vol. XVIII., pp. 142—164; Vol. XIX., pp. 6—23). The name *ternary reciprocant* was employed for reasons of analogy with Professor Sylvester's theory of reciprocants in two variables. As, however, the subject has grown, the advantages of this designation have become less marked and the danger of confusion in expression has been found to outweigh the convenience of keeping the analogy in prominence. I propose henceforward to use the name *cyclicant* in place of *ternary reciprocant*, and, in particular, *pure cyclicant* in place of *pure ternary reciprocant*. The leading idea of the cyclical interchange of three variables x, y, z is thus given the controlling influence in nomenclature which it probably should have had originally.

A *pure cyclicant* is then a function $R(z, x, y)$ of the second and higher partial differential coefficients of z with regard to x and y, which, if z_{rs} denote $\frac{1}{r!\,s!}\frac{d^{r+s}z}{dx^r dy^s}$, is homogeneous (of degree i) in the derivatives z_{rs} and isobaric in both first and second suffixes, the two partial weights being equal (each $\frac{1}{2}w$), and which persists in form, but for a first derivative factor, when the variables are cyclically

interchanged. The identities expressive of this persistence are (Vol. XVIII., pp. 157, 158)

$$R(z, xy) \equiv (-z_{10})^{i+\frac{1}{2}w} R(x, yz) \equiv (-z_{01})^{i+\frac{1}{2}w} R(y, zx) \ldots\ldots (1).$$

Pure cyclicants have, as was seen in the paper now referred to, four annihilators—

$$\Omega_1 \equiv \Sigma \left\{ (m+1)\, z_{m+1, n-1} \frac{d}{dz_{mn}} \right\} \equiv \eta \frac{d}{d\xi} (\zeta - z_{10}\xi - z_{01}\eta) \ldots\ldots (2),$$

$$\Omega_2 \equiv \Sigma \left\{ (n+1)\, z_{m-1, n+1} \frac{d}{dz_{mn}} \right\} \equiv \xi \frac{d}{d\eta} (\zeta - z_{10}\xi - z_{01}\eta) \ldots\ldots (3),$$

$$V_1 \equiv \Sigma \left\{ \Sigma \left(r z_{rs} z_{m+1-r, n-s} \right) \frac{d}{dz_{mn}} \right\} \equiv \frac{1}{2} \frac{d}{d\xi} \left\{ (\zeta - z_{10}\xi - z_{01}\eta)^2 \right\} \ldots (4),$$

$$V_2 \equiv \Sigma \left\{ \Sigma \left(s z_{rs} z_{m-r, n+1-s} \right) \frac{d}{dz_{mn}} \right\} \equiv \frac{1}{2} \frac{d}{d\eta} \left\{ (\zeta - z_{10}\xi - z_{01}\eta)^2 \right\} \ldots (5),$$

of which the first two express that it is a full invariant of the quantics (the emanants of z with regard to x and y),

$$\left.\begin{array}{l} (z_{20}, z_{11}, z_{02} \between (u, v)^2 \\ (z_{30}, z_{21}, z_{12}, z_{03} \between (u, v)^3 \\ \quad \&c. \qquad \&c. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

For the limits of the summations in Ω_1, Ω_2, V_1, V_2, see Vol. XIX., p. 6, and for the symbolical notation in the second expressions for those annihilators, see Vol. XVIII., pp. 150, &c.

The functions

$$\left.\begin{array}{l} E_2 \equiv (z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2 \\ E_3 \equiv (z_{30}, z_{21}, z_{12}, z_{03} \between (-z_{01}, z_{10})^3 \\ \quad \&c. \qquad \&c. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

obtained from the emanants (6) by giving u, v the values $-z_{01}$, z_{10}, I propose to call the quadratic cubic, &c. *cyclico-genitive forms*, for reasons partly indicated in my last paper and to be made more apparent presently.

A seminvariant of the cyclico-genitive forms which has the further property of being annihilated by V_1 I shall designate a *semicyclicant*, and the covariant of the cyclico-genitive forms which has for leading coefficient a semicyclicant I shall call a *cocyclicant*. The definition of a semicyclicant may be expressed without direct reference to the cyclico-genitive forms. It is a homogeneous and doubly isobaric

function of the derivatives z_{rs} which is annihilated by Ω_1 and by V_1. Call its degree i. Its two partial weights are different. Call them w_1, w_2, and let $w_1 - w_2 = m$.

One cyclical interchange of the variables in a semicyclicant produces from it, but for a first derivative factor, the result of interchanging first and second suffixes in its expression, and a second cyclical interchange produces the corresponding cocyclicant. If, in fact, S_0 be a semicyclicant, and $(S_0, S_1, \dots S_m)(-z_{01}, z_{10})^m$ the cocyclicant of which it is the leading coefficient, we have

$$\frac{S_0(x, yz)}{x_{01}^{i+w_1}} \equiv (-1)^m \frac{S_m(y, zx)}{y_{10}^{i+w_1}} \equiv (-1)^{i+w_2}(S_0, S_1, \dots S_m)(-z_{01}, z_{10})^m \quad \text{.....................(8)},$$

the notation (x, yz) denoting that x is taken as dependent and y and z as independent variables in order, and the absence of any explicit reference to the variables indicating that z is dependent.

The equivalences (8), which include (1) as particular cases, were proved in my last paper (Vol. XIX., p. 21),* where, however, only the restricted class of semicyclicants of which V_2, as well as V_1 and Ω_1, is an annihilator, were being considered. The proof in question will be found to have made no use of the supposed annihilation by V_2. It applied then equally to all semicyclicants, and need not be repeated. (It should be noticed that the same remark does not apply to the proofs of Props. IX. and X. on p. 15 of the paper in question. Those propositions distinctly depend on the annihilation by V_2 of the particular class of semicyclicants there studied. I see no reason for retaining the names *reciprocantive covariant* and *reciprocantive seminvariant.*)

It will be sometimes useful when speaking of cyclicants, semicyclicants and cocyclicants collectively, or without discrimination between them, to group them under the common designation *cyclic concomitants.*

2. The method of the last article of my last paper (Vol. XIX., pp. 22, 23) for the determination of all the linearly independent pure cyclicants of a given type i, $\frac{1}{2}w$, $\frac{1}{2}w$, is applicable equally for the determination of all the linearly independent semicyclicants of type i, w_1, w_2.

Of the cyclico-genitive forms (7) E_2 alone is a cocyclicant.

* There was there, however, an error in sign which is here corrected. The mistake was first made in the last line but eight of page 20, where P_{m-r} should be $(-1)^{w_2} P_{m-r}$, and repeated in the seventh line of page 21, where $(-1)^{i+m}$ should be $(-1)^{i+w_1}$.

3. To the important alternant equivalences given and used in my last paper [Vol. XIX., p. 9 (5) to (8)] may be added the following, the operation being on a homogeneous and doubly isobaric function of the derivatives z_{rs}.

$$V_1 \frac{d}{dx} - \frac{d}{dx} V_1 \equiv 2z_{20}(i+w_1) + z_{11}\Omega_1 \quad\text{.................}(9),$$

$$V_1 \frac{d}{dy} - \frac{d}{dy} V_1 \equiv z_{11}(i+w_1) + 2z_{02}\Omega_1 \quad\text{..............}(10),$$

$$V_2 \frac{d}{dx} - \frac{d}{dx} V_2 \equiv 2z_{20}\Omega_2 + (i+w_2) z_{11} \text{.................}(11),$$

$$V_2 \frac{d}{dy} - \frac{d}{dy} V_2 \equiv z_{11}\Omega_2 + 2z_{02}(i+w_2) \text{.................}(12),$$

$$\Omega_1 \frac{d}{dx} - \frac{d}{dx} \Omega_1 \equiv 0 \text{.......................................}(13),$$

$$\Omega_2 \frac{d}{dy} - \frac{d}{dy} \Omega_2 \equiv 0 \text{.......................................}(14).$$

The remaining alternants of the series,

$$\Omega_1 \frac{d}{dy} - \frac{d}{dy}\Omega_1 \quad\text{and}\quad \Omega_2 \frac{d}{dx} - \frac{d}{dx}\Omega_2,$$

appear to introduce new operators which I have not found time to study.* All are readily obtained, by means of (2) to (5), and

$$\frac{d}{dx} \equiv \Sigma_{r+s \nless 2} \left\{ (r+1)\, z_{r+1,s} \frac{d}{dz_{rs}} \right\} \equiv \frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta) - 2z_{20}\xi - z_{11}\eta \quad \text{.................}(15),$$

$$\frac{d}{dy} \equiv \Sigma_{r+s \nless 2} \left\{ (s+1)\, z_{r,s+1} \frac{d}{dz_{rs}} \right\} \equiv \frac{d}{d\eta}(\zeta - z_{10}\xi - z_{01}\eta) - z_{11}\xi - 2z_{02}\eta \quad \text{.................}(16),$$

$$i \equiv \Sigma_{r+s \nless 2} \left(z_{rs} \frac{d}{dz_{rs}} \right) \equiv \zeta - z_{10}\xi - z_{01}\eta \quad \text{.............................}(17),$$

$$w_1 \equiv \Sigma_{r+s \nless 2} \left(r z_{rs} \frac{d}{dz_{rs}} \right) \equiv \xi \frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta) \text{.......................}(18),$$

$$w_2 \equiv \Sigma_{r+s \nless 2} \left(s z_{rs} \frac{d}{dz_{rs}} \right) \equiv \eta \frac{d}{d\eta}(\zeta - z_{10}\xi - z_{01}\eta) \quad \text{.....................}(19),$$

either as in my last paper (Vol. XIX., pp. 9—12), or from the symbolical forms in the manner illustrated by Mr. Leudesdorf (Vol. XVIII., pp. 244, &c.).

* [*Oct.* 1888.—They are merely $\frac{d}{dx}$ and $\frac{d}{dy}$.]

For instance, the parts of $V_1\frac{d}{dx}$ and $\frac{d}{dx}V_1$ which involve symbols of second differentiation are identical; and the other parts are symbolically

$$\left(V_1\frac{d}{dx}\right) = \Sigma_{r+s<3}\left\{(\text{co. } \xi^r\eta^s \text{ in } V_1)\, r\xi^{r-1}\eta^s\right\} = \frac{dV_1}{d\xi},$$

and

$$\left(\frac{d}{dx}V_1\right) = \Sigma_{r+s<2}\left\{(r+1)\, z_{r+1,s}\,\xi^r\eta^s\right\}\frac{1}{2}\frac{d}{d\xi}\left\{(\zeta-z_{10}\xi-z_{01}\eta)^2\right\}$$

$$= \frac{d}{d\xi}\left\{(\zeta-z_{10}\xi-z_{01}\eta)\,\Sigma\left[(r+1)\, z_{r+1,s}\,\xi^r\eta^s\right]\right\}$$

$$= \frac{d}{d\xi}\left\{(\zeta-z_{10}\xi-z_{01}\eta)\left[\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta)-2z_{20}\xi-z_{11}\eta\right]\right\}$$

$$= \frac{dV_1}{d\xi}-2z_{20}(\zeta-z_{10}\xi-z_{01}\eta)-(2z_{20}\xi+z_{11}\eta)\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta)$$

$$= \frac{dV_1}{d\xi}-2z_{20}i-2z_{20}w_1-z_{11}\Omega_1.$$

Consequently,

$$V_1\frac{d}{dx}-\frac{d}{dx}V_1 = \left(V_1\frac{d}{dx}\right)-\left(\frac{d}{dx}V_1\right) = 2z_{20}(i+w_1)+z_{11}\Omega_1.$$

To save space I do not write out the other proofs. The first steps of all of them are included in

$$\left(\vartheta\frac{d}{dx}\right) = \Sigma_{r+s<3}\,(C_{rs}\, r\xi^{r-1}\eta^s) = \frac{d\vartheta}{d\xi}-2C_{20}\xi-C_{11}\eta,$$

and $$\left(\vartheta\frac{d}{dy}\right) = \Sigma_{r+s<3}\,(C_{rs}\, s\xi^r\eta^{s-1}) = \frac{d\vartheta}{d\eta}-C_{11}\xi-2C_{02}\eta,$$

where C_{rs} is the coefficient of $\xi^r\eta^s$ or of $\frac{d}{dz_{rs}}$ in ϑ.

4. From (13) alone, we draw the conclusion that, if Ω_1 annihilates a pure function I, it also annihilates $\frac{dI}{dx}$; in other words, that the operator $\frac{d}{dx}$ generates seminvariants of the system of quantics

$$(z_{20}, z_{11}, z_{02}\rangle(u, v)^2, \text{ \&c.},$$

from other seminvariants. This can hardly be new.

From (9) and (13) together, we derive a theorem of eduction of semicyclicants from semicyclicants. They tell us that, if a homogeneous doubly isobaric function S be annihilated by V_1 and by Ω_1, and if $i+w_1$, the sum of the degree and first partial weight of S, vanishes, then $\frac{dS}{dx}$ is also annihilated both by V_1 and by Ω_1.

Now, if S_0 be any pure cyclicant or semicyclicant of type i, w_1, w_2, $\frac{S_0}{z_{20}^{\frac{1}{2}(i+w_1)}}$ is such a function S, for z_{20} is another semicyclicant, its type being 1, 2, 0. Consequently, if S_0 is a pure cyclicant or semicyclicant,

$$z_{20}^{\frac{1}{2}(i+w_1)+1}\frac{d}{dx}\left(\frac{S_0}{z_{20}^{\frac{1}{2}(i+w_1)}}\right),$$

i.e.,
$$z_{20}\frac{dS_0}{dx}-(i+w_1)\,z_{30}S_0 \quad\text{......................}(20),$$

is another semicyclicant. Its type is $i+1$, w_1+3, w_2.

This formula of eduction of semicyclicants from semicyclicants is the same in form as, and includes, the formula for educing one Sylvesterian pure reciprocant from another. Analogy might lead us to speak of $S_0 \div z_{20}^{\frac{1}{2}(i+w_1)}$ as an *absolute* pure semicyclicant. In the expression of the fundamental property of such semicyclicants by (8), the first derivative factors do not appear.

Undoubtedly the same theorem of eduction might have been otherwise developed by means of (8) and the equivalence of operators,

$$\frac{1}{x_{01}}\frac{d}{dy} \equiv -\frac{1}{y_{10}}\frac{d}{dx} \equiv -z_{01}\frac{d}{dx}+z_{10}\frac{d}{dy} \quad\text{..............}(21),$$

in the first, second, and third members of which y and z, z and x, and x and y, respectively, are regarded as independent variables.

5. The chief object of the present paper is to give an introduction to the study of the geometrical usefulness of pure cyclicants and semicyclicants. With this object in view, it is necessary first to establish theorems of persistence in form, in case of linear transformation of the variables x, y, z, in close analogy to that of Professor Sylvester's ninth lecture (*American Journal*, Vol. VIII., p. 248) with regard to pure reciprocants.

The two theorems to be proved are:—

I. *A pure cyclicant reproduces itself, but for a factor involving first derivatives and the constants of transformation only, when the variables*

x, y, z are transformed by any scheme of linear transformation

$$\left.\begin{aligned} x &= lX+mY+nZ+p \\ y &= l'X+m'Y+n'Z+p' \\ z &= l''X+m''Y+n''Z+p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

II. *A pure semicyclicant in x as dependent variable, or a cocyclicant in z dependent, reproduces itself, but for a factor involving first derivatives and the constants of transformation only, when the variables are subjected to a restricted transformation, such as*

$$\left.\begin{aligned} x &= lX+mY+nZ+p \\ y &= l'X+m'Y+n'Z+p' \\ z &= \qquad\qquad\quad n''Z+p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

6. To prove the first of these two propositions.

It is readily seen that first derivatives transform by (22) into functions of first derivatives and the constants of transformation. In fact, the formulæ are

$$\frac{Z_{10}}{lz_{10}+l'z_{01}-l''} = \frac{Z_{01}}{mz_{10}+m'z_{01}-m''} = \frac{-1}{nz_{10}+n'z_{01}-n''} \ldots\ldots\ldots (24),$$

which at once reverse into

$$\frac{z_{10}}{\begin{vmatrix} Z_{10} & Z_{01} & -1 \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}} = \frac{z_{01}}{\begin{vmatrix} l & m & n \\ Z_{10} & Z_{01} & -1 \\ l'' & m'' & n'' \end{vmatrix}} = \frac{-1}{\begin{vmatrix} l & m & n \\ l' & m' & n' \\ Z_{10} & Z_{01} & -1 \end{vmatrix}} \ldots\ldots\ldots (25).$$

It is important to ascertain at once whether there is any exception to the fact that the substitution (22) may be replaced by successive partial substitutions, each changing only one variable at a time, such as

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= \lambda''x+\mu''y+\nu''Z+q'' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

$$\left.\begin{aligned} x &= \lambda X+\mu y+\nu Z+q \\ y &= y \\ Z &= Z \end{aligned}\right\} \ldots\ldots\ldots\; \ldots\ldots\ldots\ldots\ldots (27),$$

$$\left.\begin{aligned} X &= X \\ y &= l'X+m'Y+n'Z+p' \\ Z &= Z \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(28).$$

The complete substitution effected by these successive substitutions is

$$\begin{aligned} x &= (\lambda+\mu l')\,X+\mu m'Y+(\nu+\mu n')\,Z+q+\mu p', \\ y &= l'X+m'Y+n'Z+p', \\ z &= (\lambda''\lambda+\lambda''\mu l'+\mu''l')\,X+(\lambda''\mu m'+\mu''m')\,Y \\ &\quad +(\nu''+\lambda''\nu+\lambda''\mu n'+\mu''n')\,Z+q''+\lambda''q+\lambda''\mu p'+\mu''p'. \end{aligned}$$

For this scheme to be identical with (22) eight linear equations in λ'', μ'', ν'', q'', λ, μ, ν, q have to be satisfied. These are readily solved, the results being

$$\left.\begin{aligned} &\frac{\mu}{m} = \frac{\lambda}{lm'-l'm} = \frac{\nu}{nm'-n'm} = \frac{q}{pm'-p'm} = \frac{1}{m'}, \\ &\frac{\lambda''}{l''m'-l'm''} = \frac{\mu''}{lm''-l''m} = \frac{\nu''}{\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}} = \frac{q''}{\begin{vmatrix} l & m & p \\ l' & m' & p' \\ l'' & m'' & p'' \end{vmatrix}} = \frac{1}{lm'-l'm} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(29).$$

Thus suitable values of the coefficients in the successive substitutions are uniquely determinate unless either

$$m' = 0 \quad \text{or} \quad lm'-l'm = 0 \quad \dots\dots\dots\dots\dots\dots(30).$$

Even in these excepted cases, however, it is still possible that the substitution (22) may be produced by a succession of three partial substitutions by adopting a different order from that chosen above. Calling that order *zxy*, there are five other possible orders—*zyx*, *xyz*, *xzy*, *yzx*, *yxz*. Each of these orders is applicable to all but classes of cases for which particular conditions hold analogous to (30). In fact we have, if

$$\begin{vmatrix} L, & M, & N \\ L', & M', & N' \\ L'', & M'', & N'' \end{vmatrix} \text{ is the determinant reciprocal to } \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix},$$

so that, for instance, N'' is $lm'-l'm$, that of the six orders of partial substitutions all can produce the resultant substitutions (22), except

that the first fails when $m'=0$ or $N''=0$,

" second " $l=0$ or $N''=0$,

" third " $n''=0$ or $L=0$,

" fourth " $m'=0$ or $L=0$,

" fifth " $l=0$ or $M'=0$,

" sixth " $n''=0$ or $M'=0$.

Thus all fail if, and only if, simultaneously

either (α) $l=0,\quad m'=0,\quad n''=0,$

or (β) $L=0,\quad M'=0,\quad N''=0,$

or (γ) $m'=0,\quad n''=0,\quad M'=0,\quad N''=0,$

or (δ) $n''=0,\quad l=0,\quad N''=0,\quad L=0,$

or (ε) $l=0,\quad m'=0,\quad L=0,\quad M'=0.$

Of these (β) is a state of things with regard to the inverse substitution from X, Y, Z to x, y, z, exactly corresponding to (α) with regard to the direct substitution. Again, (γ), (δ), (ε) are one class of conditions, each being obtained by cyclical interchange of symbols from the former. In supplement, then, to the general case of a substitution resulting from three successive partial substitutions, as in (26), (27), (28), the sets of exceptional conditions (α) and (γ) need alone be considered. Of these (α) is the case of the substitution

$$\left.\begin{aligned} x &= \qquad\quad mY+nZ+p \\ y &= l'X \qquad\quad +n'Z+p' \\ z &= l''X+m''Y \qquad\quad +p'' \end{aligned}\right\} \quad\text{.........................}(31),$$

and (γ), *i.e.*, the case of

$$m'=0,\quad n''=0,\quad nl''-n''l=0,\quad lm'-l'm=0,$$

i.e., of $\quad m'=0,\quad n''=0,\quad nl''=0,\quad l'm=0,$

subdivides into four cases, viz.,

(*a*) $m'=0,\quad n''=0,\quad n=0,\quad l'=0,$

(*b*) $m'=0,\quad n''=0,\quad n=0,\quad m=0,$

(*c*) $m'=0,\quad n''=0,\quad l''=0,\quad l'=0,$

(*d*) $m'=0,\quad n''=0,\quad l''=0,\quad m=0,$

the four classes of substitutions corresponding to which are

$$\left.\begin{array}{ll} x = lX+mY & +p \\ y = & n'Z+p' \\ z = l''X+m''Y & +p'' \end{array}\right\} \dots\dots (32),$$

$$\left.\begin{array}{ll} x = lX & +p \\ y = l'X & +n'Z+p' \\ z = l''X+m''Y & +p'' \end{array}\right\} \dots\dots (33)$$

$$\left.\begin{array}{ll} x = lX\ +mY\ +nZ & +p \\ y = & n'Z+p' \\ z = \quad m''Y & +p'' \end{array}\right\} \dots\dots (34).$$

$$\left.\begin{array}{ll} x = lX & +nZ+p \\ y = l'X & +n'Z+p' \\ z = \quad m''Y & +p'' \end{array}\right\} \dots\dots (35).$$

The number of exceptional classes of substitutions to be considered may be still further reduced. For the pair (32) and (35) are similar to one another; and the result of inverting (33) is of the form (34). Again, (31) may be replaced by a sequence of (34) with a different p, followed by

$$lX = mY' + nZ' - \left(\frac{ml''}{m''} + \frac{nl'}{n'}\right) X',$$
$$m''Y = l''X' + m''Y',$$
$$n'Z = l'X' + n'Z',$$

a transformation in which neither of the conditions (30) is satisfied. Once more, (35) may be replaced by a sequence of (34) followed by

$$lX = \left(l - \frac{nl'}{n'}\right) X' - mY',$$
$$Y = Y',$$
$$n'Z = l'X' + n'Z',$$

which again is not special.

It will suffice, then, to prove the prerogative of persistence, first, for the general sequence of transformations (26), (27), (28), and secondly, for the special excepted transformation (34).

7. Apply, then, the first substitution (26) of the general sequence to the first of the three equivalent expressions for a pure cyclicant R in (1). It becomes

$$\nu''^{i} R(Z, xy).$$

Thus

$$R \equiv \nu''^{i} R(Z, xy) \equiv \nu''^{i}\left(-\frac{dZ}{dx}\right)^{i+\frac{1}{2}w} R(x, yZ) \equiv \nu''^{i}\left(-\frac{dZ}{dy}\right)^{i+\frac{1}{2}w} R(y, Zx) \qquad \text{............(36).}$$

Again, apply the second substitution (27) to the second of these three forms of R. It becomes, since by (25)

$$\frac{\dfrac{dZ}{dx}}{\dfrac{dZ}{dX}} = \frac{-1}{-\lambda-\nu\dfrac{dZ}{dX}},$$

$$\nu''^{i}\lambda^{i}\left\{\frac{-\dfrac{dZ}{dX}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(X, yZ).$$

Hence, by the laws expressed in (1), we have three forms

$$R \equiv (\nu''\lambda)^{i}\left\{\frac{1}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(Z, Xy) \equiv (\nu''\lambda)^{i}\left\{\frac{-\dfrac{dZ}{dX}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(X, yZ)$$

$$\equiv (\nu''\lambda)^{i}\left\{\frac{-\dfrac{dZ}{dy}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(y, ZX) \qquad \text{............(37).}$$

Lastly, apply the third partial substitution (28). By (25) we see that $\dfrac{dZ}{dX}$ and $\dfrac{dZ}{dy}$ have to be replaced respectively by

$$\frac{m'\dfrac{dZ}{dX}-l'\dfrac{dZ}{dY}}{m'+n'\dfrac{dZ}{dY}} \quad \text{and} \quad \frac{\dfrac{dZ}{dY}}{m'+n'\dfrac{dZ}{dY}};$$

and consequently that the last form of R in (37) becomes

$$(\nu''\lambda m')^{i}\left\{\frac{-\dfrac{dZ}{dY}}{\lambda m'+\nu m'\dfrac{dZ}{dX}+(\lambda n'-\nu l')\dfrac{dZ}{dY}}\right\}^{i+\frac{1}{2}w} R(Y, ZX).$$

But, as in (1),

$$\left(-\frac{dZ}{dY}\right)^{i+\frac{1}{2}w} R(Y, ZX) \equiv R(Z, XY).$$

Thus we have $\qquad R$, *i.e.*, $R(z, xy)$,

$$\equiv (\nu''\lambda m')^i \left\{\nu m'\frac{dZ}{dX}+(\lambda n'-\nu l')\frac{dZ}{dY}+\lambda m'\right\}^{-i-\frac{1}{2}w} R(Z, XY);$$

i.e., by (29),

$$\equiv \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^i$$

$$\times \left\{lm'-l'm-(mn'-m'n)\frac{dZ}{dX}-(nl'-n'l)\frac{dZ}{dY}\right\}^{-i-\frac{1}{2}w} R(Z, XY) \ldots (38).$$

Thus for the general case, when the linear transformation may be replaced by a sequence of partial substitutions (26), (27), (28), the prerogative of persistence of a pure cyclicant is proved. Moreover, the form of the extraneous factor introduced is determined.

As a verification it may be noticed that, since

$$(n''-nz_{10}-n'z_{01})\{lm'-l'm-(mn'-m'n)Z_{10}-(nl'-n'l)Z_{01}= \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

the same result is obtained by applying the reversed transformation to $R(Z, XY)$.

8. It remains to ascertain that the persistence holds also in case of the excepted transformation (34). Now, by this transformation, any pure derivative of z $(r+s \nless 2)$

$$\frac{d^{r+s}x}{dy^r dz^s} \text{ becomes } \frac{l^{r+s}}{n'^r m''^s}\,\frac{d^{r+s}X}{dZ^r dY^s}.$$

Thus, R being doubly isobaric and symmetrical in first and second suffixes,

$$R(x, yz) \equiv \frac{l^w}{(n'm'')^{\frac{1}{2}w}} R(X, ZY)$$

$$\equiv \frac{l^w}{(n'm'')^{\frac{1}{2}w}} R(X, YZ),$$

i.e.,
$$\frac{R(z,\, xy)}{z_{10}^{i+\frac{1}{2}w}} \equiv \frac{l^w}{(n'm'')^{\frac{1}{2}w}}\, \frac{R(Z,\, XY)}{Z_{10}^{i+\frac{1}{2}w}}.$$

There is, then, no exception to Theorem I. of § 5, as to the persistence in form of a pure cyclicant.

9. The proof of Theorem II. of § 5 is similar, but less cumbrous. The transformation (23) may be replaced by the successive partial substitutions

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= n''Z+p'' \end{aligned}\right\} \quad \text{..............................(40)},$$

$$\left.\begin{aligned} x &= \lambda X+\mu y+\nu Z+\varpi \\ y &= \qquad\; y \\ Z &= \qquad\qquad\; Z \end{aligned}\right\} \quad \text{...................(41)},$$

$$\left.\begin{aligned} X &= X \\ y &= l'X+m'Y+n'Z+p' \\ Z &= \qquad\qquad\quad\; Z \end{aligned}\right\} \quad \text{.................(42)},$$

which together are equivalent to

$$\begin{aligned} x &= (\lambda+\mu l')\, X+\mu m'Y+(\nu+\mu n')\, Z+\varpi+\mu p', \\ y &= \qquad\; l'X+\; m'Y+ \qquad\quad n'Z+p', \\ z &= \qquad\qquad\qquad\qquad\qquad\quad n''Z+p'', \end{aligned}$$

upon taking

$$\lambda = \frac{lm'-l'm}{m'},\quad \mu = \frac{m}{m'},\quad \nu = \frac{nm'-n'm}{m'},\quad \varpi = \frac{pm'-p'm}{m'} \text{...(43)};$$

the only failing case being when $m' = 0$.

If $m' = 0$, we may instead proceed with successive partial substitutions in the order z, y, x, and produce the resultant transformation (23), except when $l = 0$.

We must then consider separately the general case of a sequence of substitutions and the special one when both $l = 0$ and $m' = 0$.

10. Take, first, the general case. Applying the first partial substitution (40) to the third of the identical expressions in (8), we obtain

the same form multiplied by n''^{i+m}. Thus three equivalents of $\dfrac{S_0(x, yz)}{x_{01}^{i+w_1}}$ are

$$n''^{i+m}\frac{S_0(x, yZ)}{x_{01}^{i+w_1}} \equiv (-1)^m n''^{i+m}\frac{S_m(y, Zx)}{y_{10}^{i+w_1}}$$

$$\equiv (-1)^{i+w_1} n''^{i+m}(S_0, S_1, \dots S_m)(-Z_{01}, Z_{10})^m \dots (44),$$

where $x_{01}, y_{10}, Z_{10}, Z_{01}$ now mean $\dfrac{dx}{dZ}, \dfrac{dy}{dZ}, \dfrac{dZ}{dx}, \dfrac{dZ}{dy}$.

Next, apply the second partial substitution (41) to the first member of (44). It becomes

$$\lambda^i n''^{i+m}\frac{S_0(X, yZ)}{(\lambda X_{01}+\nu)^{i+w_1}},$$

i.e.,

$$\frac{\lambda^i n''^{i+m} y_{10}^{i+w_1}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}}\,\frac{S_0(X, yZ)}{X_{01}^{i+w_1}},$$

since

$$\frac{X_{10}}{-1} = \frac{X_{01}}{y_{10}} = \frac{-1}{y_{01}},$$

$X_{10}, X_{01}, y_{10}, y_{01}$ now meaning $\dfrac{dX}{dy}, \dfrac{dX}{dZ}, \dfrac{dy}{dZ}, \dfrac{dy}{dX}$.

And from this form of the original $\dfrac{S_0(x, yz)}{x_{01}^{i+w_1}}$ we have, by identities like (8) in the present variables, the two other forms

$$(-1)^m\frac{\lambda^i \nu''^{i+m}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}} S_m(y, ZX)$$

$$\equiv (-1)^{i+w_1}\frac{\lambda^i n''^{i+m} y_{10}^{i+w_1}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}}(S_0, S_1, \dots S_m)(-Z_{01}, Z_{10})^m \dots (45).$$

The last partial substitution (42) may now be applied to the last but one of these equivalent forms. At once

$S_m(y, ZX)$	becomes	$m'^i S_m(Y, ZX)$,
y_{10}	,,	$m'Y_{10}+n'$,
y_{01}	,,	$m'Y_{01}+l'$,

and consequently we obtain as the new form required

$$(-1)\ \frac{(\lambda m')^i n''^{i+m}}{\{\lambda m' Y_{10}-\nu m' Y_{01}+\lambda n'-\nu l'\}^{i+w_1}} S_m(Y, ZX),$$

i.e., by use of (43),

$$\frac{(lm'-l'm)^i\, n''^{i+m}\, Y_{10}^{i+w_1}}{\{(lm'-l'm)\,Y_{10}+(mn'-m'n)\,Y_{01}-(nl'-n'l)\}^{i+w_1}}\,(-1)^m\,\frac{S_m(Y,\,ZX)}{Y_{10}^{i+w_1}} \quad \text{..................(46).}$$

The result of the sequence of transformations equivalent to (23) is, then, to reproduce from the equivalent forms in (8) the same forms in the new variables multiplied by the factor

$$\frac{(lm'-l'm)^i\, n''^{i+m}\, Y_{10}^{i+w_1}}{\{(lm'-l'm)\,Y_{10}+(mn'-m'n)\,Y_{01}-(nl'-n'l)\}^{i+w_1}},$$

or, which is the same thing, by the factor

$$\frac{(lm'-l'm)^i\, n''^{i+m}}{\{lm'-l'm-(mn'-m'n)\,Z_{10}-(nl'-n'l)\,Z_{01}\}^{i+w_1}} \quad \text{.........(47).}$$

11. The temporarily excepted case of the transformation

$$\left.\begin{aligned} x &= \quad\; mY+nZ+p \\ y &= l'X \quad +n'Z+p' \\ z &= \qquad\quad n''Z+p'' \end{aligned}\right\} \quad \text{......................(48)}$$

is readily seen to be not really exceptional. This transformation may be replaced by the sequence of

$$\left.\begin{aligned} x &= mX' \quad + nZ+p \\ y &= \quad\; l'Y' + nZ+p' \\ z &= \qquad\quad n''Z+p'' \end{aligned}\right\},$$

and
$$\left.\begin{aligned} X' &= Y \\ Y' &= X \end{aligned}\right\}.$$

Of these partial transformations, the first is not special, and the second produces $(-1)^{w_1} S_m(Y,\ ZX)$ from $S_0(X',\ Y'Z)$, and $(-1)^{w_1} S_0(Y, ZX)$ from $S_m(X', Y'Z)$. In other words, it produces the second of the equivalent forms in (8) from the first, which is the same thing as reproducing the first.

Thus Theorem II. of § 5 is also established for all cases.

12. It is proposed now to consider the integration of a number of cyclicant, semicyclicant, and cocyclicant equations, and the converse passage from proper equations in x, y, z involving arbitrary functions to cyclicant, semicyclicant, and cocyclicant equations by elimination

of the arbitrary functions, as also of the variables and first derivatives. In other words, regarding the matter geometrically, it is proposed to deal with some classes of families of surfaces whose differential equations are the results of equating to zero pure cyclicants or semicyclicants or cocyclicants. A family of surfaces whose criterion is a pure cyclicant will have for its functional equation, if such can be found at all, one that is unaltered in character by any linear transformation of the variables. A family whose criterion is a semicyclicant in x as dependent variable, or a cocyclicant in z dependent, will, by the lawfulness of the transformation (23), have no special respect to any planes except those parallel to $z = 0$. A family of surfaces having properties which a single cyclicant equation is insufficient to express, but which are independent of any particular coordinate planes, will often at least have for the full expression of those properties the vanishing of all the coefficients of a cocyclicant. Examples of this will be given.

The propositions of § 5 indicate that, in determining pure cyclicant and semicyclicant equations, much use may, with advantage, be made of canonical forms of functional equations. Thus, if

$$F(x, y, z) = 0$$

satisfy an equation, "pure cyclicant" = 0.

The same is also satisfied by

$$F(lx+my+nz+p,\ l'x+m'y+n'z+p',\ l''x+m''y+n''z+p'') = 0;$$

and, if $\qquad \phi(x, y, z) = 0$

satisfy an equation, "semicyclicant in x" = 0,

or $\qquad$ "cocyclicant in z" = 0,

so also does

$$\phi(lx+my+nz+p,\ l'x+m'y+n'z+p',\ n''z+p'') = 0.$$

13. Of pure cyclic concomitants the lowest is z_{20}, the semicyclicant which is the leading coefficient of the quadratic cyclico-genitive form E_2. We have, in fact,

$$\frac{x_{20}}{x_{01}^3} \equiv \frac{y_{02}}{y_{10}^3} \equiv -(z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2$$
$$\equiv -E_2.$$

Now the integral of $x_{20} = 0$, *i.e.*, $\dfrac{d^2x}{dy^2} = 0$, is at once

$$x = yf(z) + \phi(z) \quad \text{.........................(49)},$$

which is quite as general as its apparent transformation by (23),

$$lx+my+nz+p = (l'x+m'y+n'z+p')\,f\,(n''z+p'')+\phi\,(n''z+p'').$$

This, then, is the equation of the family of surfaces whose differential equation is either

$$x_{30}=0, \quad \text{or} \quad y_{03}=0, \quad \text{or} \quad E_3=0 \ldots\ldots\ldots\ldots\ldots\ldots(50).$$

It is the family of surfaces generated by straight lines always parallel to the plane $z=0$.

The differential equation of surfaces cutting planes parallel to any other plane $\lambda x+\mu y+z=0$ than $z=0$ in straight lines is the one which would take either of the forms (50) upon putting in it z for $\lambda x+\mu y+z=0$, keeping x and y unaltered. The third form is the one which gives at once the forms of equation of the family, viz.,

$$(z_{30},\, z_{11},\, z_{03}\between(-z_{01}-\mu,\; z_{10}+\lambda)^2=0,$$

or, say,

$$e^{\lambda\,(d/dz_{10})+\mu\,(d\,dz_{01})}\,E_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(51).$$

If all planes whatever are cut by the surfaces in straight lines, this equation must be satisfied for all values of λ and μ, and conversely. Now, this necessitates that separately

$$z_{30}=0, \quad z_{11}=0, \quad z_{03}=0,$$

which are the differential equations of planes.

The results of this article, as no doubt also some of those which follow, are very familiar. They are given, however, as a first and instructive example of the method under consideration.

14. The later results of the last article exemplify facts which may at once be stated generally.

(i.) If $\qquad (S_0,\, S_1,\, \ldots\, S_m)(-z_{01},\, z_{10})^m=0$

be an equation of the form "cocyclicant $=0$," obtained as the differential equation of a family of surfaces having an assigned property with regard to planes in the direction of $z=0$, then

$$e^{\lambda\,(d/dz_{10})+\mu\,(d/dz_{01})}\,(S_0,\, S_1,\, \ldots\, S_m)(-z_{01},\, z_{10})^m=0 \ldots\ldots\ldots(51),$$

or

$$(S_0,\, S_1,\, \ldots\, S_m)(-z_{01}-\mu,\; z_{10}+\lambda)^m=0 \quad \ldots\ldots\ldots\ldots(51a),$$

is that of the family having that property with regard to planes parallel to the plane $\lambda x+\mu y+z=0$; and

(ii.) If surfaces have the property with regard to planes in an infinity of independent directions, they satisfy simultaneously the

differential equations

$$S_0 = 0, \quad S_1 = 0, \; \ldots \; S_m = 0\,;$$

equations which are not to be expected to be all independent.

15. The next simplest cyclic concomitant to z_{20} is the pure cyclicant

$$z_{20}z_{02} - \tfrac{1}{4}z_{11}^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52),$$

which is known to be the criterion of developable surfaces, and need not be further dwelt upon.

16. V_1 and Ω_1 both annihilate

$$3z_{30}z_{11} - 2z_{20}z_{21} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

which is accordingly a semicyclicant. By (8), therefore,

$$\frac{3x_{30}x_{11} - 2x_{20}x_{21}}{x_{01}^6} \equiv -\frac{3y_{03}y_{11} - 2y_{02}y_{12}}{y_{10}^6}$$

$$\equiv -\left(3z_{30}z_{11} - 2z_{20}z_{21},\; 6z_{30}z_{03} + z_{21}z_{11} - 4z_{20}z_{12},\; 4z_{21}z_{02} - z_{12}z_{11} - 6z_{20}z_{03},\right.$$
$$\left. 2z_{12}z_{02} - 3z_{03}z_{11}\right\rangle(-z_{01}, z_{10})^3 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Now, the first of these three identical expressions equated to zero gives

$$\frac{3x_{30}}{x_{20}} = \frac{2x_{21}}{x_{11}}, \quad i.e., \; \frac{1}{x_{20}}\frac{d}{dy}x_{20} = \frac{1}{x_{11}}\frac{d}{dy}x_{11}.$$

Therefore $$x_{20} = x_{11} f(z),$$

i.e., $$x_{10} = x_{01} F(z) + \phi(z)\,;$$

which, integrated by means of the auxiliary system

$$\frac{dx}{\phi(z)} = \frac{dy}{1} = \frac{dz}{-F(z)},$$

gives $$x + \int \frac{\phi(z)}{F(z)}\,dz = \psi\left(y + \int \frac{dz}{F(z)}\right),$$

i.e., $$x + u = \psi(y+v) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55),$$

u and v being arbitrary functions of z, and ψ an arbitrary functional symbol.

Thus either member of (54) equated to zero is the partial differential equation of the family of surfaces

$$lx + my + f_1(z) = \psi\{l'x + m'y + f_2(z)\} \;\ldots\ldots\ldots\ldots (56),$$

the generalisation of (55) by the transformation (23).

The property of the family in question is that any given member of it cuts all planes parallel to the fixed plane $z=0$ in identical and similarly situated curves. In particular, cylindrical surfaces are of the family, the loci of corresponding points of the sections—in general, curves of the type $lx+my+f_1(z)=0$, $l'x+m'y+f_2(z)=0$—being in this case straight lines. Again, any paraboloid whose axis is parallel to the plane $z=0$ is of the family.

Surfaces which have the property with regard to the plane $\lambda x+\mu y+z=0$ instead of $z=0$, have their differential equation written down upon inserting $-z_{01}-\mu$, $z_{10}+\lambda$ for $-z_{01}$ and z_{10} in the third of the identical expressions in (54), and equating to zero.

Again, any surface which cuts every system of parallel planes in a system of identical and similarly situated curves—or which cuts an infinite number of parallel systems in such a manner—must satisfy separately the equations

$$\left.\begin{aligned} 3z_{30}z_{11}-2z_{20}z_{21} &= 0 \\ 6z_{30}z_{02}+z_{21}z_{11}-4z_{20}z_{12} &= 0 \\ 4z_{21}z_{02}-z_{12}z_{11}-6z_{03}z_{20} &= 0 \\ 2z_{02}z_{12}-3z_{03}z_{11} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(57).$$

This is the case with cylindrical surfaces.

17. An equation involving one more arbitrary function than (56) is

$$w(lx+my)+u=\psi\{w(l'x+m'y)+v\} \quad \ldots\ldots\ldots\ldots(58),$$

where u, v, w are arbitrary functions of z. This is the functional equation of the family of surfaces of which any one cuts all planes parallel to $z=0$ in similar and similarly situated curves. All surfaces of revolution belong to the family, the plane $z=0$ being in their case at right angles to the axis of revolution. Another very particular included family is that of quadric surfaces, which retain the property in question whatever be the plane $z=0$.

From the canonical form

$$wx+u=\psi(wy+v) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58a),$$

of the equation (58), it is easy to obtain, by actual differentiation and elimination, the differential equation

$$\begin{vmatrix} 2x_{20} & x_{11} & \\ 3x_{30} & x_{21} & x_{20} \\ 4x_{40} & x_{31} & 2x_{30} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(59),$$

of the family, with x for dependent variable. This equation may be written

$$\frac{d}{dy}\left\{\frac{3x_{30}x_{11}-2x_{20}x_{21}}{x_{20}^2}\right\}=0 \quad \text{.......................(59a)},$$

so that, by (20) and (53), or from the fact that Ω_1 and V_1 (in x dependent) annihilate the left-hand member of (59), that left-hand member is a semicyclicant (in x).

The converse passage from (59) to the functional equation may be performed as follows. We may write (59) in the form

$$\left(x_{11}\frac{d}{dy}-2x_{20}\frac{d}{dz}\right)\frac{x_{20}^2}{x_{30}}=0,$$

of which, by Lagrange's method, the first integral is

$$\frac{x_{20}^2}{x_{30}}=f(x_{10}),$$

i.e., $$\frac{x_{30}}{x_{20}}=\frac{x_{20}}{f(x_{10})};$$

whence $$\log x_{20}=F(x_{10})+\phi(z),$$

i.e., $$x_{20}=\phi_1(z)\,f_1(x_{10}),$$

which gives $$F_1(x_{10})=y\phi_1(z)+\phi_2(z),$$

i.e., $$x_{10}=\psi_1\{y\phi_1(z)+\phi_2(z)\};$$

and, again integrating,

$$x\phi_1(z)+\phi_2(z)=\psi\{y\phi_1(z)+\phi_2(z)\},$$

which is the canonical form (58*a*).

The equation in z dependent equivalent to (59) is, by (8),

$$\left(Q_0,\ \frac{1}{6}\Omega_2 Q_0,\ \frac{1}{6.5}\Omega_2^2 Q_0,\ \ldots\ \frac{1}{6!}\Omega_2^6 Q_0\right)(-z_{01},\ z_{10})^6=0 \quad \text{...(60)},$$

where Q_0 denotes $$\begin{vmatrix} 2z_{20} & z_{11} & \\ 3z_{30} & z_{21} & z_{20} \\ 4z_{40} & z_{31} & 2z_{30} \end{vmatrix},$$

and $$\Omega_2 \equiv \Sigma_{r+s \nless 2}\left\{(s+1)\,z_{r-1,s+1}\frac{d}{dz_{rs}}\right\}.$$

If an infinite number of different sets of parallel planes cut a surface in sets of similar and similarly situated curves, the equation of

that surface satisfies all the differential equations

$$Q_0 = 0, \quad \Omega_2 Q_0 = 0, \quad \Omega_2^2 Q_0 = 0, \; \ldots \; \Omega_2^6 Q_0 = 0 \;\ldots\ldots\ldots(61).$$

18. In accordance with the remark at the end of the first paragraph of the last article, these last equations (61) must be satisfied by all quadric surfaces. But (*Proceedings*, Vol. XIX., p. 15) we know already the conditions of lower weight which such surfaces must satisfy, viz., the four conditions (two independent)

$$\begin{Vmatrix} z_{30} & z_{20} & \\ z_{21} & z_{11} & z_{20} \\ z_{12} & z_{02} & z_{11} \\ z_{03} & & z_{02} \end{Vmatrix} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62).$$

Our attention is then directed to the family of surfaces represented by the semicyclicant in x,

$$\begin{vmatrix} x_{30} & x_{20} & \\ x_{21} & x_{11} & x_{20} \\ x_{12} & x_{02} & x_{11} \end{vmatrix} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

or, as is the same thing, by the cocyclicant in z,

$$\begin{vmatrix} z_{30} & z_{20} & & z_{10}^3 \\ z_{21} & z_{11} & z_{20} & 3z_{10}^2 z_{01} \\ z_{12} & z_{02} & z_{11} & 3z_{10} z_{01}^2 \\ z_{03} & & z_{02} & z_{01}^3 \end{vmatrix} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(64).$$

It does not appear that any single equation involving arbitrary functions can be found which is the complete primitive of (63), so as to be the functional equation of the entire family of surfaces. We may write (14), however,

$$\frac{d}{dy}\left\{\frac{x_{20}x_{02}-\tfrac{1}{4}x_{11}^2}{x_{20}^{\frac{4}{3}}}\right\} = 0;$$

so that a first integral is

$$x_{20}x_{02}-\tfrac{1}{4}x_{11}^2 = x_{20}^{\frac{4}{3}} f(z) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (65).$$

In particular, then, the family includes all developable surfaces, for (52) is a particular case of (65).

In accordance with the known satisfaction of (62) by all quadrics, we notice that the reason is, that a central quadric cuts all planes of any parallel system, and a paraboloid all of any of a triply infinite number of parallel systems, in similar and similarly situated

conics having their centres collinear. Now, it is readily seen that the canonical form

$$xy = f(z) \quad\quad (66)$$

represents a family of surfaces having that property with regard to planes parallel to $z = 0$, and also is included in the more general family satisfying (63).

19. Enough isolated examples have been taken in the last six articles to indicate the importance of the study of cyclic concomitants in connection with the theory of families of surfaces. The remainder of the present paper will be devoted to the study of a very important particular class of concomitants, viz., to the class of cocyclicants whose semicyclicant sources are of second partial weight zero. The first of these is the quadratic cyclicogenitive form E_2. These have the very closest connection with pure reciprocants. (Having discarded the term *ternary reciprocant*, I henceforth use the word *reciprocants* to denote always the functions of the derivatives of one variable with regard to another, studied under that name by Professor Sylvester.)

It is useful to have a notation companion to that of (7) for the functions obtained by writing in the cyclicogenitive forms E_2, E_3, ... $-z_{01}-\mu$ and $z_{10}+\lambda$ for $-z_{01}$ and z_{10}. Let us use F_2, F_3, ... to denote these altered forms, so that, for all values greater than unity of the number r,

$$F_r \equiv (z_{r0},\, z_{r-1,1},\, \dots\, z_{0r} \circlearrowleft (-z_{01}-\mu,\, z_{10}+\lambda)^r \equiv e^{\lambda(d/dz_{10})+\mu(d/dz_{01})} E_r \quad\quad (67).$$

20. Let

$$\phi(a, b, c, \dots),$$

or say, taking x for dependent variable instead of y,

$$\phi\left(\frac{1}{2!}\frac{d^2x}{dy^2},\ \frac{1}{3!}\frac{d^3x}{dy^3},\ \frac{1}{4!}\frac{d^4x}{dy^4},\ \dots\right),$$

be any Sylvesterian pure reciprocant. Let its degree be i and its weight w; $a, b, c, \dots$ being regarded as of weights 2, 3, 4. The same function of the partial differential coefficients of x with regard to y, x being now regarded as a function both of y and z, is in our notation

$$\phi(x_{20}, x_{30}, x_{40}, \dots),$$

and satisfies the definition of a semicyclicant in x dependent, being homogeneous (of degree i), doubly isobaric (of weights w, 0), annihilated by V_1 (the same fact as that the reciprocant ϕ is by V) and also by Ω_1 (having no constituent of second suffix different from

zero). Hence, by (3),

$$\frac{\phi\,(x_{20},\,x_{30},\,x_{40},\,\ldots)}{x_{01}^{i+w}} \equiv (-1)^w\,\frac{\phi\,(y_{02},\,y_{03},\,y_{04},\,\ldots)}{y_{10}^{i+w}}$$

$$\equiv (-1)^i\,\{\text{that covariant of } E_2,\,E_3,\,E_4,\,\ldots \text{ whose leading term is } \phi\,(z_{20},\,z_{30},\,z_{40},\,\ldots)\,(-z_{01})^w\}$$

$$\equiv (-1)^i\,\phi\,(E_2,\,E_3,\,E_4,\,\ldots) \quad\ldots\ldots\ldots\ldots (68).$$

It is clear, then, that the study of cocyclicants of this class amounts to little more than a careful adaptation of results obtained by Sylvester and others with regard to pure reciprocants. They are the same functions of the cyclicogenitive forms E_2, E_3, ... as pure reciprocants are of the prepared derivatives a, b, ... Thus they are homogeneous and isobaric functions of E_2, E_3, ... which have the annihilator

$$4\,\frac{E_2^2}{2}\,\frac{d}{dE_3}+5E_2E_3\,\frac{d}{dE_4}+6\,(E_2E_4+\tfrac{1}{2}E_3^2)\,\frac{d}{dE_5}+7\,(E_2E_5+E_3E_4)\,\frac{d}{dE_6}+\ldots \quad\ldots\ldots\ldots (69),$$

and one may be educed from any other by operation with the generator

$$4\,(E_2E_4-E_3^2)\,\frac{d}{dE_3}+5\,(E_2E_5-E_3E_4)\,\frac{d}{dE_4}+6\,(E_2E_6-E_3E_5)\,\frac{d}{dE_5}+\ldots \quad\ldots\ldots\ldots (70).$$

21. The semicyclicants and cocyclicants obtained as in the last article are of immediately obvious geometrical interest. In fact, any pure reciprocant $\phi\,(a,\,b,\,c,\,\ldots)$ is known to be the criterion, *i.e.*, $\phi\,(a,\,b,\,c,\,\ldots)=0$ to be the differential equation, of a class of plane curves whose equations are unaltered in form by any linear transformation of x and y; and moreover it is known, conversely, that the criterion of any such class of curves is a pure reciprocant. Now, the process of elimination, by aid of differentiation, of any number of constants from an equation in x and y, is exactly the same as that of elimination, by aid of partial differentiations treating z as constant, of the same number of arbitrary functions of z from an equation involving those functions, just as the first equation involved the constants which they replace. In other words,

$$\phi\,(x_{20},\,x_{30},\,x_{40},\,\ldots)=0 \quad\ldots\ldots\ldots\ldots (71),$$

or either of its equivalents, by (68),

$$\phi\,(y_{02},\,y_{03},\,y_{04},\,\ldots)=0 \quad\ldots\ldots\ldots\ldots (71a),$$

or

$$\phi\,(E_2,\,E_3,\,E_4,\,\ldots)=0 \quad\ldots\ldots\ldots\ldots (71b),$$

is the differential equation of the family of curves which cut all planes parallel to $z = 0$ in curves of the type of which $\phi(a, b, c, \ldots)$ is the reciprocant criterion.

The same three equations may, in our ordinary notation of semi-cyclicants and cocyclicants, be written

$$\phi_0(x, yz) = 0,$$

$$\phi_w(y, zx) = 0,$$

$$(\phi_0, \phi_1, \phi_2, \ldots \phi_w)(-z_{01}, z_{10})^w = 0 \ \ldots\ldots\ldots\ldots\ldots(71c).$$

By § 14, it follows that the family of surfaces of which any cuts all parallels to any other given plane $\lambda x+\mu y+z=0$ in curves of the family of which $\phi(a, b, c, \ldots)$ is the criterion, has for its differential equation

$$(\phi_0, \phi_1, \phi_2, \ldots \phi_w)(-z_{01}-\mu, z_{10}+\lambda)^w = 0 \ \ldots\ldots\ldots\ldots(72),$$

or $$e^{\lambda(d/dz_{10})+\mu(d/dz_{01})} \,.\, \phi(E_2, E_3, E_4, \ldots) = 0,$$

or again, in the notation of (67),

$$\phi(F_2, F_3, F_4, \ldots) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72a).$$

22. The first pure reciprocant a leads in this manner to the differential equation of surfaces generated by straight lines parallel to a fixed plane. It produces the cocyclicant, &c., discussed in § 13.

The second pure reciprocant

$$ac - \tfrac{5}{4}b^2 \equiv (M)$$

is the criterion of parabolas. Hence either

$$M_0(x, yz) \equiv x_{20}x_{40} - \tfrac{5}{4}x_{30}^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(73),$$

or $$M_6(y, zx) \equiv y_{02}y_{04} - \tfrac{5}{4}y_{03}^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(73a),$$

or $$\mu \equiv E_2E_4 - \tfrac{5}{4}E_3^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(73b),$$

is the differential equation of the family of surfaces all sections of which by planes parallel to $z = 0$ are parabolas.

The family of which all sections by parallels to $\lambda x+\mu y+\nu = 0$ are parabolas, is represented by

$$\nabla\mu \equiv e^{\lambda(d/dz_{10})+\mu(d/dz_{01})}\,\mu = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(74);$$

i.e., by $$F_2F_4 - \tfrac{5}{4}F_3^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(74a).$$

Again, if a surface cut all planes in parabolas, it must satisfy all the equations

$$\left.\begin{array}{l} M_0 \equiv z_{20}z_{40} - \tfrac{5}{4}z_{30}^2 = 0, \\ M_1 \equiv \tfrac{1}{6}\Omega_2 M_0 \equiv \tfrac{1}{6}(z_{11}z_{40} + z_{20}z_{31} - \tfrac{5}{2}z_{30}z_{21}) = 0, \\ M_2 \equiv \tfrac{1}{5}\Omega_2 M_1 \equiv \dfrac{1}{6\,.\,5}(2z_{11}z_{31} - \tfrac{5}{2}z_{21}^2 + 2z_{02}z_{40} - 5z_{30}z_{12} + 2z_{20}z_{22}) = 0, \\ M_3 = \tfrac{1}{4}\Omega_2 M_2 \equiv \dfrac{1}{6\,.\,5\,.\,4}(6z_{02}z_{31} - 15z_{21}z_{12} + 6z_{11}z_{22} - 15z_{30}z_{03} + 6z_{20}z_{13}) = 0, \\ \quad\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ M_6 = \dfrac{1}{6\,!}\Omega_2^6 M_0 \equiv z_{02}z_{04} - \tfrac{5}{4}z_{03}^2 = 0, \end{array}\right\} \qquad \ldots\ldots\ldots\ldots(75),$$

which last results are probably rather matters of curiosity than value.

23. Again, $A \equiv a^2d - 3abc + 2b^3$

is the criterion of a conic (*Monge*).

Thus either of the three equivalent differential equations

$$A_0(x) \equiv x_{20}^2 x_{50} - 3x_{20}x_{30}x_{40} + 2x_{30}^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76),$$

or

$$A_0(y) \equiv y_{02}^2 y_{05} - 3y_{02}y_{03}y_{04} + 2y_{03}^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76a),$$

or

$$\alpha \equiv E_2^2 E_5 - 3E_2E_3E_4 + 2E_3^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76b),$$

is the differential equation of the family of surfaces denoted by

$$u_1x^2 + v_1y^2 + w_1 + 2u_2y + 2v_2x + 2w_2xy = 0,$$

where $u_1, v_1, w_1, u_2, v_2, w_2$ are arbitrary functions of z.

Again, $\nabla\alpha \equiv F_2^2F_5 - 3F_2F_3F_4 + 2F_3^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(77)$

represents the family of surfaces which cut parallels to $\lambda x + \mu y + z = 0$ in conics; and, if a surface cut all planes in conics, it satisfies all the equations $A_0 = 0, \quad \Omega_2 A_0 = 0, \ldots \Omega_2^9 A_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(78)$.

24. Professor Sylvester (*American Journal*, IX., p. 16) has proved that $16(\lambda+1)^2 M^3 + 25(\lambda-2)(2\lambda-1)A^2 = 0$

is the differential equation of curves of the class

$$ax + by + c = (a'x + b'y + c')^\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79),$$

a, b, c, a', b', c' being arbitrary constants.

It follows that, in the notation of the last two articles,

$$16\,(\lambda+1)^2\,\mu^3+25\,(\lambda-2)(2\lambda-1)\,\alpha^2=0 \quad\ldots\ldots\ldots\ldots\ldots(80)$$

and
$$16\,(\lambda+1)^2\,(\nabla\mu)^3+25\,(\lambda-2)(2\lambda-1)(\nabla\alpha)^2=0 \quad\ldots\ldots\ldots\ldots(81)$$

represent respectively the families of surfaces cutting planes parallel to $z=0$ and $\lambda x+\mu y+z=0$ in curves of the same type.

The results of the last two articles are included, as also are that

$$4^4\,(\nabla\mu)^3+5^3\,(\nabla\alpha)^2=0 \quad\ldots\ldots\ldots\ldots\ldots(82),$$

given by $\lambda=3$ or $\frac{1}{3}$, and

$$4\,(\nabla\mu)^3-\alpha^2=0 \quad\ldots\ldots\ldots\ldots\ldots(83),$$

given by $\lambda=\frac{2}{3}$ or $\frac{3}{2}$, represent surfaces cutting parallels to $\lambda x+\mu y+z=0$ in cubical and semicubical parabolas respectively.

25. The interpretation of Sylvester's $B, C, D, \ldots$ (*American Journal*, IX., p. 318) leads in like manner to those of the cocyclicants

$$\beta\equiv E_2^3E_6-2E_2^2E_4^2-\tfrac{7}{2}E_2^2E_3E_5+\tfrac{17}{2}E_2E_3^2E_4-4E_3^4 \quad\ldots\ldots\ldots\ldots(84),$$

$$\gamma\equiv E_2^4E_7-5E_2^3E_4E_5-4E_2^3E_3E_6+13E_2^2E_3E_4^2$$
$$+\tfrac{45}{4}E_2^2E_3^2E_5-\tfrac{103}{4}E_2E_3^3E_4+\tfrac{19}{2}E_3^5\ldots\ldots(85),$$

$$\delta\equiv E_2^5E_8-\tfrac{25}{8}E_2^4E_5^2-6E_2^4E_4E_6+7E_2^3E_4^3+E_3\{\ldots\} \quad\ldots\ldots\ldots\ldots(86),$$

&c., &c.,

and more generally to the interpretations of $\nabla\beta$, $\nabla\gamma$, $\nabla\delta$, ..., the results of replacing every E in β, γ, δ by the corresponding F.

Halphen's Δ ("Thèse sur les Invariants differentiels," pp. 12, &c.), or Sylvester's $AC-B^2$ (*American Journal*, IX., pp. 332, &c.), is the criterion of

$$\log(ax+by+c)+\omega\log(a'x+b'y+c')+\omega^2\log(a''x+b''y+c'')=k,$$

where ω is an imaginary cube root of unity, and a, b, c, a', b', c', a'', b'', c'', k are arbitrary constants. The differential equation of which the complete integral is

$$\log(u_1x+v_1y+w_1)+\omega\log(u_2x+v_2y+w_2)+\omega^2\log(u_3x+v_3y+w_3)=U \quad\ldots\ldots\ldots\ldots(87),$$

in which u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3, U are arbitrary functions of Z, is then

$$A_0(x, yz)\,C_0(x, yz)-\{B_0(x, yz)\}^2=0 \quad\ldots\ldots\ldots\ldots(88)$$

or $$A_9(y, zx)\,C_{15}(y, zx) - \{B_{12}(y, zx)\}^2 = 0 \quad\text{............(88a)},$$

or, again,

$$\alpha\gamma - \beta^2 \equiv \begin{vmatrix} E_3 & E_4 & E_5 & E_6 & E_7 \\ E_2 & E_3 & E_4 & E_5 & E_6 \\ -E_2^2 & 0 & E_3^2 & 2E_3E_4 & 2E_3E_5 + E_4^2 \\ 0 & E_2^2 & 2E_2E_3 & 2E_2E_4 + E_3^2 & 2E_2E_5 + 2E_3E_4 \\ 0 & 0 & E_2^2 & 3E_2E_3 & 3E_3^2 + 2E_2E_4 \end{vmatrix} = 0$$

$$\text{..................(88b)}.$$

A result including this, and also the $a = 0$ of § 23, obtained in like manner from a result of Sylvester's (*American Journal*, IX., pp. 337 338), is that, if λ be any constant,

$$2^4 7^3 (\lambda+1)^2 (\lambda-2)^2 (2\lambda-1)^2 (\alpha\gamma-\beta^2)^3 = 3^3 . 5^3 (\lambda^2-\lambda+1)^3 a^8 \;\text{...(89)}$$

is the differential equation of surfaces whose equations are of the type

$$(u_1x+v_1y+w_1)(u_2x+v_2y+w_2)^{-\lambda}(u_3x+v_3y+w_3)^{\lambda-1} = W \;\text{...(90)}$$

with certain special cases corresponding to the values 0, ∞, and 1 of λ. For these the differential equation is

$$2^6 . 7^3 (\alpha\gamma-\beta^2)^3 = 3^3 . 5^2 . a^8 \;\text{....................(91)},$$

and alternative complete integrals are

$$u_1x+v_1y+w_1 = \log(u_2x+v_2y+w_2) \;\text{...............(92)}$$

and $$u_1x+v_1y+w_1 = (u_2x+v_2y+w_2)\log(u_2x+v_2y+w_2) \;\text{......(93)}.$$

In all these results $u_1, v_1, w_1, u_2, \ldots, u_3, \ldots, W$ denote arbitrary functions of z. A particular result of (89) is that

$$2^8 (\alpha\gamma-\beta^2)^3 = 3^3 a^8 \;\text{.........................(94)}$$

represents surfaces where sections by parallels to $z = 0$ are cuspidal cubics.

The generalisations of these results, obtained by inserting $\nabla\alpha$, $\nabla\beta$, $\nabla\gamma$ for α, β, γ, *i.e.* F_2, &c., for E_2, &c., need not be stated at length.

26. A few more results may be stated without development.

The result of replacing λ in (90) by an arbitrary function of z

leads to a cocyclicant derived from Halphen's T (*Thèse*, p. 42), or Sylvester's $A^2D-3ABC+2B^3$. The comprehensive conclusion derived from this is that

$$24F_2^{-4}\left\{(\nabla\alpha)^2(\nabla\delta)-3(\nabla\alpha)(\nabla\beta)(\nabla\gamma)+2(\nabla\beta)^3\right\}\text{.........(95)},$$

i.e.,
$$\begin{vmatrix} 3F_3 & 0 & F_2 & 0 & 0 & 0 \\ 4F_4 & F_3 & F_3 & F_2 & 2F_2^2 & 0 \\ 5F_5 & 2F_4 & F_4 & 2F_3 & 5F_2F_3 & F_2^2 \\ 6F_6 & 3F_5 & F_5 & 3F_4 & 6F_2F_4+3F_3^2 & 3F_2F_3 \\ 7F_7 & 4F_6 & F_6 & 4F_5 & 7F_2F_5+7F_3F_4 & 4F_2F_4+2F_3^2 \\ 8F_8 & 5F_7 & F_7 & 5F_6 & 8F_2F_6+8F_3F_5+4F_4^2 & 5F_2F_5+5F_3F_4 \end{vmatrix}$$

...............(95*a*),

is the criterion of surfaces cutting planes parallel to $\lambda x+\mu y+z=0$ in curves whose equations referred to axes in their own plane are of the form

$$(ax+by+c)^{-1}(a'x+b'y+c')^{\lambda}(a''x+b''y+c'')^{1-\lambda}=k\text{......(96)},$$

where λ as well as the other constants is arbitrary.

Again, Roberts's reciprocant expression for the criterion of a general cubic curve (*Educational Times Reprint*, x., p. 47) leads us to the conclusion that

$$\begin{vmatrix} F_2 & F_3 & F_4 & F_2^2 & 0 & 0 \\ F_3 & F_4 & F_5 & 2F_2F_3 & F_2^2 & 0 \\ F_4 & F_5 & F_6 & 2F_2F_4+F_3^2 & 2F_2F_3 & F_2^3 \\ F_5 & F_6 & F_7 & 2F_2F_5+2F_3F_4 & 2F_2F_4+F_3^2 & 3F_2^2F_3 \\ F_6 & F_7 & F_8 & 2F_2F_6+2F_3F_5+F_4^2 & 2F_2F_5+2F_3F_4 & 3F_2^2F_4+3F_2F_3^2 \\ F_7 & F_8 & F_9 & 2F_2F_7+2F_3F_6+2F_4F_5 & 2F_2F_6+2F_3F_5+F_4^2 & 3F_2^2F_5+6F_2F_3F_4 \end{vmatrix}$$

....

is the criterion of surfaces whose sections by parallels to $\lambda x+\mu y+z=0$ are cubic curves.

And lastly, from Sylvester's criterion (*American Journal*, IX., p. 349) of curves of the n^{th} order, we deduce the criterion of surfaces cutting

planes parallel to $\lambda x+\mu y+z=0$ in curves of that order, viz.,

$$\left|\begin{array}{ccccccccccc}
(2.1) & (3.1) & (4.1) & (5.1) & (6.1) & (7.1) & (8.1) & (9.1) & (10.1) & (11.1) & \dots \\
(3.1) & (4.1) & (5.1) & (6.1) & (7.1) & (8.1) & (9.1) & (10.1) & (11.1) & (12.1) & \dots \\
(3.2) & (4.2) & (5.2) & (6.2) & (7.2) & (8.2) & (9.2) & (10.2) & (11.2) & (12.2) & \dots \\
(4.1) & (5.1) & (6.1) & (7.1) & (8.1) & (9.1) & (10.1) & (11.1) & (12.1) & (13.1) & \dots \\
(4.2) & (5.2) & (6.2) & (7.2) & (8.2) & (9.2) & (10.2) & (11.2) & (12.2) & (13.2) & \dots \\
(4.3) & (5.3) & (6.3) & (7.3) & (8.3) & (9.3) & (10.3) & (11.3) & (12.3) & (13.3) & \dots \\
(5.1) & (6.1) & (7.1) & (8.1) & (9.1) & (10.1) & (11.1) & (12.1) & (13.1) & (14.1) & \dots \\
(5.2) & (6.2) & (7.2) & (8.2) & (9.2) & (10.2) & (11.2) & (12.2) & (13.2) & (14.2) & \dots \\
(5.3) & (6.3) & (7.3) & (8.3) & (9.3) & (10.3) & (11.3) & (12.3) & (13.3) & (14.3) & \dots \\
(5.4) & (6.4) & (7.4) & (8.4) & (9.4) & (10.4) & (11.4) & (12.4) & (13.4) & (14.4) & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right| \qquad \text{..................(98),}$$

to $\frac{1}{2}n(n+1)$ rows and columns, where $(m.\mu)$ denotes the multiplier of k^m in the expansion of

$$(F_2k^2+F_3k^3+F_4k^4+\dots)^\mu.$$

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on analogous relations in Space of Three Dimensions. By Samuel Roberts.

[*Read May 10th,* 1888.]

I. *Plane Space.*

1. In studying some questions relating to the closed branches of curves, I was led to consider the clear spaces enclosed by the finite segments determined by the intersections of straight lines in a plane. By "clear spaces" I mean those not cut by any of the lines, and it will be convenient to call them simply "spaces." I have since found that, long ago, Steiner treated of the subject, in consequence of his

finding formulated in certain geometrical text-books connected with the Pestalozzian system the following proposition, viz.:—"To determine how many parts of the plane can be marked off by means of a given number of straight lines and circles altogether finite." Accordingly in an early paper entitled, "Einige Gesetze über die Theilung der Ebene und des Raumes" (*Crelle's Journal*, B. I., § 349—364), Steiner determines the number of parts in various cases, taking systems of straight lines with parallel groups, and of circles with concentric groups, afterwards proceeding to the solution of similar questions relating to planes and spheres. He assumes that no more than two lines intersect in the same point finitely situate, and imposes similar conditions on the circles, planes, and spheres, so that the final formulæ exhibit the number of parts "at most."

In the present paper, I study in somewhat more detail the nature of these figures. The determination of the number of parts cut off is plainly only one of many problems which arise in connection with such systems. For the figures formed by a system of straight lines in a plane are not only finite in number, but definite in form. Thus three straight lines not meeting in the same point finitely situate form by their finite segments a triangle; four straight lines, of which no three meet in the same point, make by their finite segments two triangles and a quadrilateral, and, although for higher numbers the general configuration is variable, it is so within limits.

I shall confine myself in what follows to the consideration of systems of straight lines and planes.

2. Let n straight lines in one plane intersect in points finitely situate, no three of the lines meeting in the same point. Several numerical relations are matter of immediate inference.

The number of points of intersection is $\frac{n \,.\, n-1}{2}$; that of the finite segments (which form the sides of the finite spaces) is $n\,(n-2)$; that of the segments unlimited in one direction (which I shall call "prolongations") is $2n$, the sum of the two sets being n^2.

If now an additional transversal be applied to the system, $n-1$ new finite spaces will be added, and, corresponding to the numbers of lines $3, 4 \ldots n$, the numbers of the finite spaces are $1, 3, 6 \ldots \frac{n-1 \,.\, n-2}{2}$. The number of open spaces is $2n$. Relatively to the finite figure the intersections may be distributed in four classes—(1) apices, (2) neutral or level points, (3) reentrant points, (4) interior points, altogether surrounded by the external contour.

Let the numbers of each class in the same order be a_1, a_2, a_3, a_4, then

$$a_1+a_2+a_3+a_4 = \frac{n(n-1)}{2}.$$

If we take into account for a moment the prolongations, it appears that to an apex belong two prolongations, to a neutral point belongs one, the reentrant and interior points are not immediately connected with any prolongation. Hence

$$2a_1+a_2 = 2n.$$

Further an apex terminates two finite segments; a neutral point, three; a reentrant or interior point, four; therefore

$$\left.\begin{aligned} 2a_1+3a_2+4a_3+4a_4 &= 2n(n-2) \\ \text{or} \qquad a_2+2a_3+2a_4 &= n(n-3) \\ -2a_1+2a_3+2a_4 &= n(n-5) \end{aligned}\right\}.$$

If K, L, M denote respectively the numbers of interior segments, of finite contour segments, and the sum of the numbers of the sides bounding the finite spaces, then

$$K = \frac{n(n-3)}{2}+a_4, \quad L = \frac{n(n-1)}{2}-a_4, \quad M = \frac{3n^2-7n}{2}+a_4.$$

For the number of contour sides is

$$a_1+a_2+a_3, \text{ and } L+M = 2n(n-2).$$

The maximum and minimum values of a_4 determine therefore the maximum and minimum values of K, M, and the minimum and maximum values of L.

If N is the number of right angles which make up the sum of the angles of the finite spaces,

$$N = n(n-1)+2a_4-4.$$

Let A_p denote the number of p-agons contained among the finite spaces, then

$$\left.\begin{aligned} A_n+A_{n-1}+\ldots+A_3 &= \frac{n-1\,.\,n-2}{2} \\ nA_n+(n-1)A_{n-1}+\ldots+3A_3 &= M = \frac{3n^2-7n}{2}+a_4 \end{aligned}\right\} \ldots\ldots(\text{A}).$$

Taking account only of the sides of the open spaces, and denoting by

B_p the number of such spaces having p sides, we have

$$\left.\begin{aligned} B_n+B_{n-1}+\ldots+B_2 &= 2n \\ nB_n+(n-1)\,B_{n-1}+\ldots+2B_2 &= \frac{n\,(n+7)}{2}-a_4 \end{aligned}\right\} \quad \ldots\ldots\ldots\text{(B)}.$$

The value of B_2 is a_1. But the possible forms fall short of the integer and positive solutions of these equations except when $n = 3$ or 4.

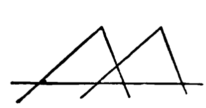
3. Still considering the finite figure, the maximum value of a_1 is n, if n is odd. For no line can contain more than two apices. If n be odd and the lines be numbered consecutively, we can arrange the cycle (1, 2), (2, 3) ... $(n-1, n)\,(n, 1)$, so that each line contains two apices.

When n is even, we cannot form a figure having n apices, since, if the lines be numbered as before and arranged in cycle, an evenly numbered line must, when we set out from an apex on it, cut all the oddly numbered lines previous to it in order, before the second apex is arrived at. Hence we cannot form the apex $(n, 1)$ in the cycle.

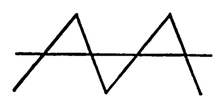
The maximum number of apices is consequently $n-1$. The minimum number of apices is in both cases 3, and any intermediate number can be given to the figure so that, for n odd, a_1 ranges from 3 to n, for n even, from 3 to $n-1$. It follows that a_2 (always even) ranges from 0 to $2n-6$ when n is odd, from 2 to $2n-6$ when n is even.

4. There must be at least one reentrant point between each pair of apices, except when a contour line contains no reentrant point. When $n = 3$, there are three such contour lines, and when $n = 4$ there are two; but when n is greater than 4 we can only have one such line, except in the case of $a_1 = 3$, when we may have two. For it will be observed that, given a figure for 4 lines, we can add as many transversals as we please, terminated at both ends by neutral points, that is to say, not containing apices. Hence, except in the case of $a_1 = 3$, we must have at least $a_3 = a_1$ or a_1-1.

We reduce the reentrant points between a pair of apices to a single one by aggregating them thus



On the other hand, by segregating them thus



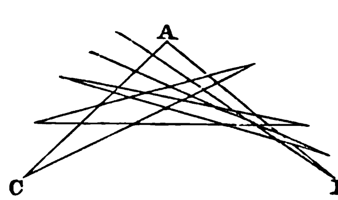
we get two reentrant points between the pair of apices; and we shall get the maximum value of a_3 by providing as many as possible of such pairs together, with as many as possible of reentrant points not immediately depending on apices.

We may set out with two lines, viz., the contour line without reentrants, and another beyond which we can, at most, place, if n is odd, $\frac{n-1}{2}$ aggregated apices. Add to these the two apices at the extremities of the contour line free from reentrants, and the number is $\frac{n+3}{2}$. If n is even, we can make at most, $\frac{n-2}{2}$ such apices, and, adding the two apices on the contour line free from reentrants, we have $\frac{n}{2}+1$. It follows that, up to and inclusive of $a_1 = \frac{n+3}{2}$ (n odd), and up to and inclusive of $a_1 = \frac{n}{2}+1$ (n even), we have (except in the special case of $a_1 = 3$) $a_3 = a_1 - 1$ for a minimum; for higher values the minimum value of a_3 is a_1.

5. In order to get the maximum value of a_3, we place, if a_1 is even, $\frac{a_1-2}{2}$ apices beyond one of two fundamental lines, say AB, and $\frac{a_1-4}{2}$ beyond AC, the other fundamental line. There are thus a_1-3 apices, each accompanied by two reentrant points, and we can get

$$n-3-\frac{a_1-2}{2}$$

other reentrant points at most. The following figure is a typical form for 8 apices and 9 lines:



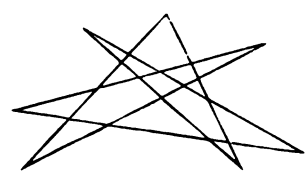
The maximum value of a_3 is for a_1 even,

$$n-3-\frac{a_1-2}{2}+2(a_1-3) = n+\frac{3a_1}{2}-8,$$

If a_1 is odd, the maximum of a_3 is

$$n-3-\frac{a_1-3}{2}+2(a_1-3) = n+\frac{3a_1}{2}-\frac{15}{2}.$$

The typical figure for 7 apices and 7 lines is



We can have any number of reentrant points between these li
and unity inclusive. Certain classes of figures (finite) can nov
indicated in a tabular form (Table I.). The numbers under
respective letters at the heads of the column denote their corresp(
ing values, the system in each row belonging to the same class.

To obtain the number of classes for each value of $n>4$, we obs(
that for n odd, and assuming in the first instance $a_3 = a_1$ for
lowest value of a_3, we have

$$2\left\{(n-5)+(n-4)+\ldots+\frac{3n-15}{2}\right\}+\frac{3n-13}{2},$$

and for n even

$$2\left\{(n-5)+(n-4)+\ldots+\frac{3n-16}{2}\right\}+\frac{3n-14}{2}.$$

But for the exceptional cases in which the minimum value
is $< a_1$, we must add in the first case $\frac{n+1}{2}$, and in the second
Therefore the number of classes is $\frac{1}{4}(5n^2-32n+51)$ for n odd,
$\frac{1}{4}(5n^2-38n+76)$ for n even.

The minimum value of a_4 (interior points) obtains when a_3-a_1
maximum, and is $\frac{n^2-8n+15}{2}$ for n odd, and $\frac{n^2-8n+16}{2}$ for n e
The maximum value is $\frac{n(n-5)}{2}+2$ $(n>3)$.

I.

n	a_1	a_2	a_3	a_4	M	N
3	3				3	2
4	3	2	1		10	8
5	3	4	1	2	22	20
	3	4	2	1	21	18
	4	2	3	1	21	18
	5		5		20	16
6	3	6	1	5	38	36
	3	6	2	4	37	34
	3	6	3	3	36	32
	4	4	3	4	37	34
	4	4	4	3	36	32
	5	2	5	3	36	32
	5	2	6	2	35	30
7	3	8	1	9	58	56
	3	8	2	8	57	54
	3	8	3	7	56	52
	3	8	4	6	55	50
	4	6	3	8	57	54
	4	6	4	7	56	52
	4	6	5	6	55	50
	5	4	4	8	57	54
	5	4	5	7	56	52
	5	4	6	6	55	50
	5	4	7	5	54	48
	6	2	6	7	56	52
	6	2	7	6	55	50
	6	2	8	5	54	48
	7		7	7	56	52
	7		8	6	55	50
	7		9	5	54	48
	7		10	4	52	46

n	a_1	a_2	a_3	a_4	M	N
8	3	10	1	14	82	80
	3	10	2	13	81	78
	3	10	3	12	80	76
	3	10	4	11	79	74
	3	10	5	10	78	72
	4	8	3	13	81	78
	4	8	4	12	80	76
	4	8	5	11	79	74
	4	8	6	10	78	72
	5	6	4	13	81	78
	5	6	5	12	80	76
	5	6	6	11	79	74
	5	6	7	10	78	72
	5	6	8	9	77	70
	6	4	6	12	80	76
	6	4	7	11	79	74
	6	4	8	10	78	72
	6	4	9	9	77	70
	7	2	7	12	80	76
	7	2	8	11	79	74
	7	2	9	10	78	72
	7	2	10	9	77	70
	7	2	11	8	76	68

6. As I have said, the equations (A) and (B) which must be satisfied give also inadmissible solutions. Some of the limitations on these general expressions can be immediately inferred. Thus, relative to the equations (A), the first number A_n must be unity or zero, since n lines can at most make one n-agon. It is moreover found, by actual inspection of the figure, when $n = 5$, that we cannot by an additional transversal create a hexagon and a pentagon. It follows that, for values of $n > 5$, $A_{n-1} = 0$ if $A_n = 1$.

Again, A_3 cannot be less than $n-2$. Suppose this is so up to $n-1$. In such a system, the removal of a line diminishes the number of triangles by one. Now take an n^{th} transversal not forming a divided triangle with at least one of the triangles of the $(n-1)$ system. That triangle is lost and not replaced by the removal of an original line. If the transversal makes divided triangles with all the triangles of the $(n-1)$ system, a triangle is still lost by the removal of an extreme line. The transversal must therefore make an additional triangle, and the (n) system has $(n-2)$ triangles, since three lines give one triangle, four lines give two triangles, &c. By "divided triangle" I mean a triangle divided by a line into a triangle and quadrilateral. The number of triangles cannot be diminished by adding transversely.

We can determine various general solutions. Thus a figure can be obtained $\frac{n-2 \,.\, n-3}{2}$ quadrilaterals and $n-2$ triangles. By adjusting the angle of intersection we can draw a line through a point on an interior segment so as to add two triangles, $n-3$ quadrilaterals, and two sides, one to each of two spaces, and one of them may be a triangle, in which case the transversal must make two triangles. Through a point on a contour segment we can draw a line adding two triangles, $n-3$ quadrilaterals, and one side to a space. Through a point on a prolongation we can draw a line adding one triangle and $n-2$ quadrilaterals. Any one of the numbers A_n, A_{n-1}, ... A_4 may vanish. Similarly other results applicable to the general number n can be obtained. But I have not succeeded in finding an exhaustive method of determining all the admissible solutions of the equations.

The accompanying scheme shows the admissible forms for $n = 5, 6$. I denote as before by P_q a q-agon (Table II.). The forms marked with an asterisk are inadmissible. All but six of these are excluded by the preceding considerations.

7. The general expressions of § 2 may be extended to the case in which the system contains groups of lines passing through one point. If p lines cointersect in one point, it has absorbed all the spaces, the finite edges, and all the points due to the intersection of p lines. If therefore, in a system of n lines, p pass through one point, q through another, r through another, and so on, the number of spaces is

$$\frac{n-1 \,.\, n-2}{2} - \frac{p-1 \,.\, p-2}{2} - \frac{q-1 \,.\, q-2}{2} - \frac{r-1 \,.\, r-2}{2} - \&c.,$$

and the number of finite edges is

$$n\,(n-2) - p\,(p-2) - q\,(q-2) - r\,(r-2) - \&c.$$

In the latter case we must take p, q, r, &c. = or > 2.

The points, on this general supposition, may be described as termi-

nating a certain number of finite segments and a certain number of prolongations. If a point terminates $2a_1$ segments in all, of which a_1 are prolongations, this is, in fact, an apex; if a_1-1 are prolongations, it is a level point; if there are no prolongations, it is an interior point. If there are α_1 prolongations where α_1 is $< a_1-1$, it is a re-entrant point. Including these in one class, let there be p points of the orders $a_1, a_2 \dots a_p$ terminating respectively $\alpha_1, \alpha_2 \dots \alpha_p$ prolongations; then the sum of the sides of the faces is

$$\Sigma_1^p a_p - \Sigma^p \alpha_p - C,$$

where C is the number of contour points, and $\Sigma_1^p a_p = 2n$.

II.

$n = 5$

	P_5	P_4	P_3
	1	2	3
	1	1	4
	1		5
		3	3
*		4	2
*		2	4

$n = 6$

	P_6	P_5	P_4	P_3
*	1	2	1	6
*	1	1	3	5
	1		5	4
*	1	1	2	6
*	1	2		7
	1		4	5
*	1	1	1	7
	1		3	6
*	1	1		8
*	1		2	7
*		1	6	3
	2		4	4
*	3		2	5
	1		5	4
	2		3	5
*	3		1	6
	1		4	5
	2		2	6
	3			7
	1		3	6
*	2		1	7
*			8	2
*			7	3
			6	4
*			5	5

8. If a group of p lines is a parallel one, we must further deduct p from the number of the edges, and $p-1$ from the number of spaces, and so for other groups of parallels. In his paper, Steiner does not consider intersections finitely situate of a higher order than 2, but only parallel groups. He gives the number of spaces in the form

$$1-U+A+\frac{a.a-1}{2},$$

where a is the number of single lines, U is the sum of the orders of the groups, and A is the sum of their products in pairs. This form gives a very symmetrical expression when circles also are involved. Putting aside for a moment the case of single lines, we may write U for n, and our expression becomes

$$\frac{(p+q+r+\&c.-1)(p+q+r+\&c.-2)}{2}$$
$$-\frac{p-1.p-2}{2}-\frac{q-1.q-2}{2}-\frac{r-1.r-2}{2}-\&c.,$$

or $\Sigma pq-k+1$, where k is the number of groups; but, since the groups are parallel, we must deduct

$$(p-1)+(q-1)+(r-1)+\&c. \quad \text{or} \quad p+q+r+\&c.-k,$$

giving

$$\Sigma pq-\Sigma p+1.$$

If now we suppose one of the groups, say the p group, to consist of single lines differently directed, we have deducted too much by

$$\frac{p-1.p-2}{2}+p-1 \quad \text{or} \quad \frac{p.p-1}{2},$$

so that Steiner's formula results.

9. If we take generally a system of points at which respectively $a_1, a_2 \dots a_p$ finite segments terminate, the number of segments is $\frac{a_1+\dots+a_p}{2}$, and the number of spaces is $\frac{a_1+a_2+\dots+a_p}{2}-p+1$. For, assuming the formula, if we add a point a_{p+1}, we increase the number of segments by a_{p+1}, and the number of spaces by $a_{p+1}-1$, and we have

$$\frac{a_1+a_2+\dots+a_p}{2}+a_{p+1}-p=\frac{a_1+a_2+\dots+a_{p+1}+a_{p+1}}{2}-(p+1)-1,$$

which is the same form, since the original system of points contains a_{p+1} points to which a segment has been added. The formula is true for $p=3, 4, \&c.$ Let μ be the number of contour points, then the

sum of the sides of the corresponding spaces is $a_1+a_2+\ldots+a_p-\mu$. The sum of the angles is equivalent to $2\mu-4+4(p-\mu)$ or $4p-2\mu-4$ right angles.

The formulæ of § 2 are, in fact, independent of the linear relations which reduce the number of admissible figures in the case of systems of lines. Disregarding linear relations, we can with 10 points and 15 segments, no more than 4 segments meeting in a point, construct 4 quadrilaterals and 2 triangles, or with 15 points and 24 segments, no more than 4 meeting in a point, we can construct 3 pentagons, 1 quadrilateral, and 6 triangles. These are inadmissible forms when the parts and segments are those due to a system of straight lines.

II. *Space of Three Dimensions.*

10. Let us now take a system of n planes, of which no more than three meet in one point, and no more than two have a common line, and no two are parallel. Moreover the points of intersection (triple points) are supposed to be finitely situate.

If we add one more plane to the system, it is cut in n lines which give $\frac{n-1\,.\,n-2}{2}$ new finite spaces, to each of which belongs an additional clear space or volume.

If u is the number of finite volumes of the system of n planes, we may write

$$\Delta u = \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2} - n+1,$$

whence

$$u = \frac{n(n-1)(n-2)}{2\,.\,3} - \frac{n(n-1)}{2} + n - 1 = \frac{(n-1)(n-2)(n-3)}{2\,.\,3}.$$

because u must vanish for $n=1$.

When we include the open volumes, and write v for the corresponding number, we have

$$\Delta v = \frac{n(n-1)}{2} + n + 1,$$

and $$v = \frac{n(n-1)(n-2)}{2\,.\,3} + \frac{n(n-1)}{2} + n + 1 = \frac{n^3+5n+6}{6}.*$$

because $n=3$ gives two spaces.

* This and some other particular cases will be found given as examples in the text-books, *e.g.*, in Mr. C. Smith's *Treatise on Algebra*, recently published, Examples XXIII.

The number of finite faces is $\frac{n(n-2)(n-3)}{2}$ since each plane is cut by $n-1$ planes, giving $\frac{n-2 \cdot n-3}{2}$ plane spaces. Including open spaces, the number is

$$\frac{n\left[(n-1)^2+(n-1)+2\right]}{2} \quad \text{or} \quad \frac{n(n^2-n+2)}{2}.$$

The number of finite edges is $\frac{n(n-1)(n-3)}{2}$, or, the prolongations being included, $\frac{n(n-1)^2}{2}$.

When p planes meet in one and the same point, but no more than two have a common line, the volumes, faces, and edges, due to a system of p planes, are absorbed in the common point. If, therefore, such groups of $p_1, p_2, \ldots p_m$ planes exist in the system of n planes, the number of finite volumes is

$$\frac{(n-1)(n-2)(n-3)}{2 \cdot 3} - \sum_{\iota=1}^{\iota=m} \frac{(p_\iota-1)(p_\iota-2)(p_\iota-3)}{2 \cdot 3},$$

that of the finite faces is

$$\frac{n(n-2)(n-3)}{2} - \sum_{\iota=1}^{\iota=m} \frac{p_\iota(p_\iota-2)(p_\iota-3)}{2},$$

and that of the finite edges is

$$\frac{n(n-1)(n-3)}{2} - \sum_{\iota=1}^{\iota=m} \frac{p_\iota(p_\iota-1)(p_\iota-3)}{2}.$$

11. We will next suppose that the system of n planes contains certain groups of planes having a common line, but that the several multiple lines do not intersect.

Let there be one such group of a_1 planes. If w is the number of finite volumes, we have

$$\Delta w = \frac{(n-1)(n-2)}{2} - \frac{(a_1-1)(a_1-2)}{2},$$

and

$$w = \frac{(n-1)(n-2)(n-3)}{2 \cdot 3} - (n-1)\frac{(a_1-1)(a_1-2)}{2} + \frac{a_1(a_1-1)(a_1-2)}{3},$$

for w must be zero, for $n = a_1$.

If now we cut the system by another group of a_2 planes having a common line, the increment of volumes is

$$\frac{(n-1)(n-2)}{2} - \frac{(a_1-1)(a_1-2)}{2} + (a_2-1)\left\{\frac{n(n-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}\right\},$$

and, changing n into $n-a_2$, we get

$$w=\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\frac{(a_2-1)(a_2-2)}{2}\right\}$$
$$+\frac{a_1(a_1-1)(a_1-2)}{3}+\frac{a_2(a_2-1)(a_2-2)}{3};$$

and the result will be of similar form when we include groups of a_3, $a_4 \dots a_q$ planes having common lines. In fact, assuming the general expression to be

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right\}$$
$$+\frac{a_1(a_1-1)(a_1-2)}{3}+\dots+\frac{a_q(a_q-1)(a_q-2)}{3}.$$

add another group of a_{q+1} planes having a common line. The increment of finite volumes is

$$\frac{(n-1)(n-2)}{2}-\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right\}$$
$$+(a_{q+1}-1)\left\{\frac{n(n-1)}{2}-\left[\frac{(a_1-1)(a_1-2)}{2}+\dots+\frac{(a_q-1)(a_q-2)}{2}\right]\right\}.$$

Writing now $n-a_{q+1}$ for n, and observing that

$$\frac{(n-a_{q+1}-1)(n-a_{q+1}-2)(n-a_{q+1}-3)}{2.3}+\frac{(n-a_{q+1}-1)(n-a_{q+1}-2)}{2}$$
$$+(a_{q+1}-1)\frac{(n-a_{q+1})(n-a_{q+1}-1)}{2},$$

is reducible to

$$\frac{1}{2.3}\left\{(n-1)(n-2)(n-3)-3a_{q+1}^2n+2a_{q+1}^3\right.$$
$$\left.+9a_{q+1}n-3a_{q+1}^2-6a_{q+1}-6n+6\right\},$$

we have finally,

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_1-2)}{2}+\dots\right.$$
$$\left.+\frac{(a_q-1)(a_q-2)}{2}+\frac{(a_{q+1}-1)(a_{q+1}-2)}{2}\right\}+\frac{a_1(a_1-1)(a_1-2)}{3}+\dots$$
$$+\frac{a_q(a_q-1)(a_q-2)}{3}+\frac{a_{q+1}(a_{q+1}-1)(a_{q+1}-2)}{3},$$

which verifies the form generally.

We can deal similarly with the open spaces. For if we suppose the finite figure constituted by n planes to be surrounded by a superficies, say, a sphere, the number of spaces will be the same as that of the parts into which the superficies is divided by the planes. Let groups of a_1, $a_2 \dots a_p$ planes have common lines respectively. The

effect of the multiple lines is to produce pairs of multiple points among the intersections of the arcs determined by the planes on the sphere, and each of these absorbs the same number of superficial spaces as if the arcs were straight lines. Hence the number of open spaces or regions is

$$n^2-n+2-(a_1-1)(a_1-2)-(a_2-1)(a_2-2)-(a_p-1)(a_p-2);$$

and, if $n=\Sigma a_1$, this result is

$$2\Sigma a_1 a_2+2\Sigma a_1-(p-1)\,2.$$

For brevity's sake, and because the finite figure possesses more interest, I concern myself chiefly with the finite volumes, &c.

12. If, however, some of the multiple lines intersect, the above determinations become incorrect. Suppose that a number of multiple lines of various orders meet in one point. The intersection has absorbed the volumes, faces, and edges due to a system made up of the same number of multiple lines of the same orders, and constituted by the same number of planes, but not co-intersecting. Hence, the foregoing expression gives us the form of the correction.

Let the system of n planes contain multiple lines of the orders $k_1, k_2 \ldots k_s$ meeting at a multiple point of the order m_1, of the orders $l_1, l_2 \ldots l_t$ meeting at a multiple point of the order m_2, of the orders $\kappa_1, \kappa_2 \ldots \kappa_v$, meeting at a multiple point of the order m_3, of the orders $\lambda_1, \lambda_2 \ldots \lambda_r$, meeting at a multiple point of the order m_4, and so on.

The expression for the number of finite volumes is

$$\begin{aligned}
&\frac{(n-1)(n-2)(n-3)}{2.3}-(n-1)\left\{\frac{(a_1-1)(a_2-2)}{2}+\ldots+\frac{(a_q-1)(a_q-2)}{2}\right\}\\
&\qquad+\frac{a_1(a_1-1)(a_1-2)}{2}+\ldots+\frac{a_q(a_q-1)(a_q-2)}{3}\\
&-\frac{(m_1-1)(m_1-2)(m_1-3)}{2.3}+(m_1-1)\left\{\frac{(k_1-1)(k_1-2)}{2}+\ldots\right.\\
&\qquad\left.+\frac{(k_s-1)(k_s-2)}{2}\right\}-\left[\frac{k_1(k_1-1)(k_1-2)}{3}+\ldots+\frac{k_s(k_s-1)(k_s-2)}{3}\right]\\
&-\frac{(m_2-1)(m_2-2)(m_2-3)}{2.3}+(m_2-1)\left\{\frac{l_1-1)(l_1-2)}{2}+\ldots\right.\\
&\qquad\left.+\frac{(l_t-1)(l_t-2)}{2}\right\}-\left\{\frac{l_1(l_1-1)(l_1-2)}{3}+\ldots+\frac{l_t(l_t-1)(l_t-2)}{3}\right\}\\
&-\&c.,
\end{aligned}$$

or as we may write it

$$F(n, a_1 \ldots a_q)-F(m_1, k_1 \ldots k_s)-F(m_2, l_1 \ldots l_t)$$
$$-F(m_3, \kappa_1 \ldots \kappa_v)-F(m_4, \lambda_1 \ldots \lambda_r)-\&c.,$$

where the symbol F denotes similar functions, except as to the number of the letters involved. It is to be remembered also that the letters k_1, &c., l_1, &c., and so on, really represent orders comprised in a_1, &c., and may be repeated in different functions. In this way we include cases in which the same multiple line intersects several others. If no multiple line meets another more than once (excepting at the usual triple points), the expression is simplified and becomes

$$\frac{(n-1)(n-2)(n-3)}{2.3}-(n-m_1)\left\{\frac{(k_1-1)(k_1-2)}{2}+\ldots+\frac{(k_s-1)(k_s-2)}{2}\right\}$$
$$-\frac{(m_1-1)(m_1-2)(m_1-3)}{2.3}$$
$$-(n-m_2)\left\{\frac{(l_1-1)(l_1-2)}{2}+\ldots+\frac{(l_t-1)(l_t-2)}{2}\right\}$$
$$-\frac{(m_2-1)(m_2-2)(m_2-3)}{2}-\&c.$$

The general formula includes the case of multiple points finitely situate, since we may consider these as constituted by double lines meeting together.

13. If we add to the system p parallel planes, the increment is

$$p\left[\frac{(n-1)(n-2)}{2}-\frac{(a_1-1)(a_1-2)}{2}-\frac{(a_2-1)(a_2-2)}{2}-\ldots\right.$$
$$\left.-\frac{(a_q-1)(a_q-2)}{2}\right].$$

A further addition of q parallel planes of different direction gives an increment

$$q\left[\frac{(n+p-1)(n+p-2)}{2}-\frac{(a_1-1)(a_1-2)}{2}-\frac{(a_2-1)(a_2-2)}{2}-\ldots\right.$$
$$\left.\ldots-\frac{(a_q-1)(a_q-2)}{2}\right]-\frac{p-1\,.\,p-2}{2}-p+1.$$

and so on.

If in the resulting expression we put $n=0$, and therefore dismiss $a_1, a_2 \ldots a_q$, we have

$$-1+p+q\left[\frac{(p-1)(p-2)}{2}-\frac{p-1\,.\,p-2}{2}-p+1\right]$$
$$-r\left[\frac{(p+q-1)(p+q-2)}{2}-\frac{(p-1)(p-2)}{2}-\frac{(q-1)(q-2)}{2}-p-q+2\right]$$
$$-\&c.$$

or $$-1+p-\Sigma pq+\Sigma pqr-\&c.,$$

which is the form in which Steiner gives the result. It includes the case in which single planes differently directed enter, for it is per-

missible to suppose the value of any of the letters p, q, r, &c. be unity.

14. Similar considerations enable us to determine the numbe faces and edges.

In the first instance, we take a system of n planes contain multiple lines not co-intersecting, and of the orders $a_1, a_2 \dots a_q$.

There are $n-a_1+a_2-\dots-a_q$ planes containing $n-1$ lines e a_1 planes containing $n-a_1+1$ lines, and so on; but it must be rem bered that in this way we count the multiple lines $a_1, a_2 \dots a_q$ ti respectively, whereas the other lines are counted only twice. He for edges, we have to make a final deduction of

$$(a_1-2)(n-a_1-1)+\dots+(a_q-2)(n-a_q-1).$$

Thus the formula for the number of edges will be

$$\frac{1}{2}\left\{\begin{array}{r} (n-a_1-a_2-\dots-a_q)\left\{(n-1)(n-3)-a_1(a_1-2)-\dots\right. \\ \left.\dots-a_q(a_q-2)\right\} \\ +a_1\left\{(n-a_1+1)(n-a_1-1)-a_2(a_2-2)-\dots\right. \\ \left.\dots-a_q(a_q-2)\right\} \\ +\dots+a_q\left\{(n-a_q+1)(n-a_q-1)-a_1(a_1-2)-\dots\right. \\ \left.\dots-a_{q-1}(a_{q-1}-2)\right\} \\ -(a_1-2)(n-a_1-1)-\dots-(a_q-2)(n-a_q-1) \end{array}\right\}$$
$$= F_1(n, a_1, a_2 \dots a_q).$$

And if a certain number of these multiple lines intersect in one po let these lines be of the orders $k_1, k_2 \dots k_s$. We must deduct for edges lost according to the number of planes constituting the in section. For m_1 planes the deduction will be $F_1(m_1, k_1, k_2 \dots k_s)$, so for any number of such intersections; *i.e.*, the general form will be

$$F_1(n, a_1, a_2 \dots a_q)-F_1(m_1, k_1, k_2 \dots k_s)-F_1(m_2, l_1, l_2 \dots l_t)-\&c.,$$

the symbol F_1 being interpreted in the same way as F. In manner, we get, for the number of faces when the multiple line not intersect,

$$\frac{1}{2}\left\{\begin{array}{r} (n-a_1-a_2\dots-a_q)\left[(n-2)(n-3)-(a_1-1)(a_1-2)-\dots\right. \\ \left.\dots-(a_q-1)(a_q-2)\right] \\ +a_1\left[(n-a_1)(n-a_1-1)-(a_2-1)(a_2-2)-\dots\right. \\ \left.\dots-(a_q-1)(a_q-2)\right] \\ +\dots+a_q\left[(n-a_q)(n-a_q-1)-(a_1-1)(a_1-2)-\dots\right. \\ \left.\dots-(a_{q-1}-1)(a_{q-1}-2)\right] \end{array}\right\}$$
$$= F_2(n\,a_1 a_2 \dots a_q),$$

and the correction for intersecting lines will be as before

$$-\left[F_2\,(m_1k_1k_2\dots k_s)+F_2\,(m_2l_1l_2\dots l_t)+\&c.\right],$$

a similar meaning being attached to the symbol F_2.

If certain groups of planes through a line intersect at infinity so that the multiple lines are parallel, the multiple point is the limit of an apex.

An apex subtends the same number of superficial spaces bounded by its lines as it would do if cut by a single plane; so that, when the apex is infinitely distant, we must deduct the number of volumes due to the planes through it, augmented by another plane. In addition, therefore, to the usual deduction of $F\,(m_1k_1k_2\dots k_s)$, where m_1 is the number of planes, and $k_1, k_2\dots k_s$ are the orders of the lines, we have to deduct

$$\frac{(m_1-1)(m_1-2)}{2}-\frac{(k_1-1)(k_1-2)}{2}-\dots-\frac{(k_s-1)(k_s-2)}{2}.$$

It will be observed that an apex may subtend more faces of the figure than the number mentioned, but the number of superficies subtended will be as stated, by the principles of perspective.

In the like case, we can determine the correction for the numbers of the faces and edges considered as finite, and the foregoing formulæ can be adapted to other cases which I do not treat of at length,—for example, to the case in which some of the parallel planes pass through a multiple point, and so forth. So that we may conclude that the numbers of the volumes, faces, and edges can be generally determined for systems of planes, consisting partly of single planes, of groups of planes having a common line finitely situate, or on the infinitely distant plane and multiple points formed by the intersection of multiple lines, and finitely or infinitely distant.

15. The numbers which occur relative to systems of a very moderate number of planes are large. Applying, for instance, the formula to the 45 real triple tangent planes of a cubic surface, we have to consider that they pass, five together, through 27 lines, and these intersect in 135 points in pairs, each point being constituted by 9 planes, since the two intersecting lines belong to a common plane of the system. The formulæ give therefore, for the numbers of the volumes or completely enclosed cells, the faces and the edges respectively

$$\frac{44.43.42}{2.3}-135\,\frac{8.7.6}{2.3}-27.44.\frac{4.3}{2}+27.\frac{5.4.3}{3}$$

$$+135.8.2.\frac{4.3}{2}-135.2.\frac{5.4.3}{3}=6656,$$

$$\tfrac{1}{2}\left\{\begin{array}{l}(45-5.27)(43.42-27.4.3)+5.27(40.39-26.4.3)\\ \qquad -135\,(9-10)(7.6-2.4.3)-2.5\,(4.3-4.3)\end{array}\right\} = 18765,$$

$$\tfrac{1}{2}\left\{\begin{array}{l}(45-5.27)(44.42-27.5.3)-5.27\,(41.39-26.5.3)\\ \qquad -27.3.39-135\,[(9-10)(8.6-2.5.3)\\ \qquad +5.2\,(5.3-5.3)-2.3.3]\end{array}\right\} = 17523,$$

and by the formula of § 11 there are 1658 open regions.

When we attempt to determine more particularly the forms of the volumes involved, the difficulty which we already encountered in the analogous plane problem is much intensified.

The edges of the finite figure made by n planes, no more than three meeting in one point, and no more than two having a common line, and no two being parallel, may be divided into four classes.

The edges may be (1) convex, (2) level, (3) re-entrant, (4) interior. Let a_1, a_2, a_3, a_4 be the numbers of the four kinds respectively. A convex edge belongs to two faces and one volume, a level to three faces and two volumes, a re-entrant to four faces and three volumes, an interior to four faces and four volumes. Hence, if we put F for the number of all the edges of all the faces taken separately, V for the number of all the edges of all the volumes taken separately, and E for the number of edges taken once only, we have

$$\left.\begin{array}{r}2a_1+3a_2+4a_3+4a_4 = F\\ a_1+2a_2+3a_3+4a_4 = V\\ a_1+\ a_2\ +\ a_3\ +\ a_4\ = E = \dfrac{n\,(n-1)(n-3)}{2}\end{array}\right\} \quad \ldots\ldots\ldots(\mathrm{C}),$$

and therefore $\qquad a_4 = E+V-F.$

These relations and others similarly obtainable are quite insufficient for a solution of the main question.

If we refer the letters a_1, a_2, a_3, a_4 back to the plane case, so that a_4 means the number of interior points, and represent by F, V, E, respectively, the number of the extremities of the finite edges taken separately, the sum of the number of the sides of the faces, and the number of intersections, the equations (C) hold in the same form, so that the value of a_4 is in the same form.

[redacted]

On Simplicissima in space of n-dimensions. (In continuation of a Paper in the "Proceedings of the London Mathematical Society," Vol. XVIII., pp. 325—359. *By* W. J. Curran Sharp, M.A.

[*Read April* 12*th*, 1888.]

Summary.

Addendum to previous paper—values of Σ', Σ''.

XVIII. Equations to a line, to parallel lines, linear loci through given lines, &c. Generalizations of theorems by Mr. J. J. Walker, Mr. McCay, Mr. Morley, and Professor Malet.

XIX. Properties when the connectors of a point with the vertices are similarly divided. Two theorems of Mr. Tucker's generalized.

XX. Analogue of anharmonic point in space of n dimensions and number of relations among the ratios. Homographic systems of lines and points. Anharmonic property of quadrics.

XXI. Relation among the common tangents to $(n+3)$ spherics, to $(n+2)$ spherics which all touch another.

XXII. Isogonally conjugate lines. If one system of lines meet in a point, the isogonally conjugate system does so. Isogonally conjugate loci.

XXIII. Concurrent linear loci through the intersections of faces. The harmonic conjugates of their intersections with the edges lie on a linear locus. What this is.

XXIV. Quadratic loci through pairs of points on the edges which are harmonically conjugate. If a quadric pass through the intersections with the edges of one set of concurrent loci, its other intersections lie on another set. Case of spherics. If $(r.s)^2 = m_r m_s$, certain related points lie on a line through centroid. Case in plane geometry.

XXV. Spherics through each vertex, and arbitrary points on the edges which meet there, all pass through a common point. Special cases. (Mr. Roberts.)

XXVI. Spherics with centre on a face, and sections by that common with a spheric, centre at a given point. centre of the orthogonal spheric is the isogonally jugate point, which has the same pedal spheric as given point. This, the original, and the orthog spheric are coaxal. (Mr. Roberts.) When one of isogonally conjugate points is at infinity, the p spheric is infinite.

XXVII. Rectangular simplicissimum, $(r.s)^2 = A_r + A_s$. Formu Central line. Spherics through the centroids of s sidiary simplicissima. Special cases. Right-ang simplicissima. Formulae. The pedal simplicissim

XXVIII. Mutually orthotomic spherics. $\left(\frac{\Sigma_1}{r_1}\right)^2 + \left(\frac{\Sigma_2}{r_2}\right)^2 + \dots \equiv$ The spherics on the common sections have the sa radical centre as the original one. (Prof. Clifford.)

XXIX. The spheric bisecting a system of spherics is the orthogo spheric of a system of imaginary spherics. Posit of centre. (Generalized from Mr. Roberts.)

XXX. Quadrics with indeterminate centres. Centre at infin Tangential equation. Intersections with straight li Rectangles under segments. Principal diametral and axes.

XXXI. Quadrics with two sections by linear loci common. Poi which have the same polars. Application to spher Limiting points.

XXXII. Foci of quadrics in space of n dimensions. Some ot foci.

XXXIII. Lines, planes, and other linear loci wholly on a locus order m in space of n dimensions.

XXXIV. Perpendiculars on parallel tangents to an m-ic lo Tangent loci through the intersection of two linear l

XXXV. The normal at point on a locus of order m, in space n-dimensions. How many normals can be drawn fr an arbitrary point. Kinds of contact and tangent lo

XXXVI. Alternative system of coordinates. Identical relati Equation to circum-spheric.

Addendum to Art. IX., Vol. XVIII., p. 343.

If
$$\Sigma \equiv S-(A_1\lambda+A_2\mu+A_3\nu\,...)(\lambda+\mu+\nu+...)$$
$$\equiv S-\frac{\lambda+\mu+\nu\,...}{V}\left(\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}+\nu\frac{dS'}{d\nu'}+...\right)$$
$$+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu+\nu\,...)^2=0$$

be the equation to a spheric radius ρ and centre $(\lambda', \mu', \nu', ...)$,

$$\Sigma' = V^2\rho^2,$$
$$\frac{d\Sigma'}{d\lambda'}=\frac{d\Sigma'}{d\mu'}=\frac{d\Sigma'}{d\nu'}\,... = 2V\rho^2;$$

and if $(\lambda'', \mu'', \nu''...)$ be any other point,

$$\Sigma''= V^2(\rho^2-d^2),$$

where d is the length of the connector of $(\lambda', \mu', \nu'...)$ and $(\lambda'', \mu'', \nu''...)$.

If $\Sigma = 0$ be the equation to the point spheric at $(\lambda', \mu', \nu'...)$

$$\Sigma'' = -V^2d^2.$$

The articles have been numbered in continuation of the former paper.

XVIII. The equations
$$\left\|\begin{matrix}\lambda, & \mu, & \nu, & ...\\ \lambda', & \mu', & \nu', & ...\\ \lambda'', & \mu'', & \nu'', & ...\end{matrix}\right\|=0$$

to the straight line which joins the points $(\lambda', \mu', \nu'...)$ and $(\lambda'', \mu'', \nu''...)$ may be put in the form

$$\frac{\lambda-\lambda'}{\lambda''-\lambda}=\frac{\mu-\mu'}{\mu''-\mu'}=\frac{\nu-\nu'}{\nu''-\nu'}=...=\frac{d}{d'},$$

where d and d' are the distances of $(\lambda, \mu, \nu\,...)$ and $(\lambda'', \mu'', \nu''...)$ from $(\lambda', \mu', \nu'...)$, which are only equivalent to $n-1$ independent equations.

These equations may be put into the form

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=\,...\,=\frac{Vd}{\sqrt{\{-S(a, b, c\,...)\}}},\text{ by IX.,}$$

where
$$a+b+c+...=0,$$

and the equations to the parallel line through $(\lambda_1, \mu_1, \nu_1 \ldots)$ are

$$\frac{\lambda-\lambda_1}{a}=\frac{\mu-\mu_1}{b}=\frac{\nu-\nu_1}{c}=\ldots$$

Similarly, the equations to the plane through the lines

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=$$

$$(a+b+c\ldots=0),$$

$$\frac{\lambda-\lambda'}{a'}=\frac{\mu-\mu'}{b'}=\frac{\nu-\nu'}{c'}=0$$

$$(a'+b'+c'+\ldots=0),$$

are

$$\left\|\begin{matrix}\lambda-\lambda', & \mu-\mu', & \nu-\nu' & \ldots \\ a, & b, & c & \ldots \\ a', & b', & c' & \ldots\end{matrix}\right\|=0$$

and

$$\left\|\begin{matrix}a, & b, & c & \ldots \\ a', & b', & c' & \ldots \\ a'', & b'', & c'' & \ldots\end{matrix}\right\|=0$$

gives the conditions that the line

$$\frac{\lambda-\lambda'}{a''}=\frac{\mu-\mu'}{b''}=\frac{\nu-\nu'}{c''}=\ldots,$$

$$(a''+b''+c''+\ldots=0),$$

should lie in this plane, and

$$\left\|\begin{matrix}\lambda-\lambda', & \mu-\mu', & \nu-\nu' & \ldots \\ a_1, & b_1, & c_1 & \ldots \\ a_2, & b_2, & c_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{p-1}, & b_{p-1}, & c_{p-1} & \ldots\end{matrix}\right\|=0$$

are the equations to the linear locus (in space of p dimensions) through the $p-1$ lines whose equations are obtained by giving to r all values from 1 to $p-1$ in

$$\frac{\lambda-\lambda'}{a_r}=\frac{\mu-\mu'}{b_r}=\frac{\nu-\nu'}{c_r}=\ldots,$$

where $$a_r+b_r+c_r \ldots = 0,$$

and
$$\left\|\begin{array}{cccc} a, & b, & c & \ldots \\ a_1, & b_1, & c_1 & \ldots \\ a_2, & b_2, & c_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{p-1}, & b_{p-1}, & c_{p-1} & \ldots \end{array}\right\| = 0$$

are the conditions that the additional line

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots,$$

$$(a+b+c \ldots = 0),$$

should lie in this locus.

Again, if $$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots = mp,$$

$$\frac{dS}{d\lambda} = \frac{dS'}{d\lambda'} + mp\frac{d}{da}S(a, b, c \ldots), \quad \frac{dS}{d\mu} = \frac{dS'}{d\mu'} + mp\frac{d}{db}S(a, b, c \ldots), \text{ \&c.},$$

and therefore the line represented by these equations when $a+b+c \ldots = 0$ will (XII.) be perpendicular to the linear locus

$$A_1\lambda + A_2\mu + A_3\nu + \ldots = 0,$$

if
$$\left\|\begin{array}{cccc} \frac{d}{da}S(a, b, c \ldots), & \frac{d}{db}S(a, b, c \ldots), & \frac{d}{dc}S(a, b, c \ldots) & \ldots \\ A_1, & A_2, & A_3 & \ldots \\ 1, & 1, & 1 & \ldots \end{array}\right\| = 0,$$

a condition which is easily transformed into

$$\frac{A_2-A_1}{\left(\frac{d}{db}-\frac{d}{da}\right)S(a, b, c \ldots)} = \frac{A_3-A_1}{\left(\frac{d}{dc}-\frac{d}{da}\right)S(a, b, c \ldots)} = \text{\&c.}$$

If $ABCDE$ be a simplicissimum in space of n dimensions, and B_1, C_1, D_1, &c. the mid-points of the connectors of P with B, C, D, &c., respectively (all the vertices except A), and if parallels to AB_1, AC_1, AD_1, &c., be drawn through the centroids of $CDE \ldots$, $BDE \ldots$, $BCE \ldots$, &c. respectively (the faces of $BCDE \ldots$); these will all meet in the same point Q, and AQ will pass through the centroid of $PBCDE \ldots$, where it will be divided in the ratio of $n-1 : 2$. (This is generalized from a Question of Mr. J. J. Walker's, *Educational Times*, Quest. 6391.) Take $ABCDE \ldots$ as the simplicissimum of

reference, and let P be $(\lambda', \mu', \nu' \ldots)$. Then B_1, C_1, &c. are

$$\left(\frac{\lambda'}{2}, \frac{\mu'+V}{2}, \frac{\nu'}{2} \ldots\right), \quad \left(\frac{\lambda'}{2}, \frac{\mu'}{2}, \frac{\nu'+V}{2} \ldots\right), \text{ \&c.,}$$

and the equations to AB_1, AC_1, &c. are

$$\frac{\lambda-V}{\lambda'-2V} = \frac{\mu}{\mu'+V} = \frac{\nu}{\nu'} = \frac{\pi}{\pi'} = \ldots,$$

$$\frac{\lambda-V}{\lambda'-2V} = \frac{\mu}{\mu'} = \frac{\nu}{\nu'+V} = \frac{\pi}{\pi'} = \ldots,$$

$$\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.}$$

and those to the parallels through the centroids of $(CDE \ldots)$, $(BDE \ldots)$, &c., *i.e.*, through the points

$$\left(0, 0, \frac{V}{n-1}, \frac{V}{n-1}, \ldots\right), \quad \left(0, \frac{V}{n-1}, 0, \frac{V}{n-1}, \ldots\right), \text{ \&c.,}$$

are
$$\frac{\lambda}{\lambda'-2V} = \frac{\mu}{\mu'+V} = \frac{\nu - \dfrac{V}{n-1}}{\nu'} = \frac{\pi - \dfrac{V}{n-1}}{\pi'} = \ldots,$$

$$\frac{\lambda}{\lambda'-2V} = \frac{\mu - \dfrac{V}{n-1}}{\mu'} = \frac{\nu}{\nu'+V} = \frac{\pi - \dfrac{V}{n-1}}{\pi'} = \ldots,$$

$$\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.,}$$

which all meet where

$$\frac{\lambda}{\lambda'-2V} = \frac{\mu - \dfrac{V}{n-1}}{\mu'} = \frac{\nu - \dfrac{V}{n-1}}{\nu'} = \frac{\pi - \dfrac{V}{n-1}}{\pi'} = \ldots \left(= \frac{1}{n-1}\right),$$

so that, if $(\lambda'', \mu'', \nu'' \ldots)$ denote the point Q,

$$\lambda'' = \frac{\lambda'}{n-1} - 2\frac{V}{n-1}, \quad \mu'' = \frac{\mu'}{n-1} + \frac{V}{n-1}, \quad \nu'' = \frac{\nu'}{n-1} + \frac{V}{n-1}, \text{ \&c.,}$$

and the equations to AQ are

$$\frac{\lambda}{\lambda''-V} = \frac{\mu}{\mu''} = \frac{\nu}{\nu''} = \frac{\pi}{\pi''} = \ldots,$$

or
$$\frac{\lambda-V}{\lambda'-(n+1)V} = \frac{\mu}{\mu'+V} = \frac{\nu}{\nu'+V} = \frac{\pi}{\pi'+V} = \ldots,$$

and this line passes through

$$\left(\frac{\lambda'}{n+1}, \frac{\mu'+V}{n+1}, \frac{\nu'+V}{n+1}, \frac{n'+V}{n+1}, \ldots\right), \text{ or } (\bar{\lambda}, \bar{\mu}, \bar{\nu} \ldots),$$

the centroid of *PBCDE* ..., and since

$$(n+1)\,\bar{\lambda} = (n-1)\,\lambda''+2V, \quad (n+1)\,\bar{\mu} = (n-1)\,\mu''+2\times 0, \text{ \&c.},$$

the centroid divides AQ in the ratio of $n-1:2$ (and therefore equally if $n=3$, as in Mr. Walker's Question).

Again, let the line

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots \; (\equiv p \text{ say}),$$

where $$a+b+c \ldots = 0$$

meet the faces of the simplicissimum of reference in the points

$$A'\,(0, \mu_1, \nu_1, \pi_1 \ldots), \quad B'\,(\lambda_2, 0, \nu_2, \pi_2 \ldots), \text{ \&c.},$$

and let the corresponding values of p be p_1, p_2, &c.; therefore

$$p_1 = -\frac{\lambda'}{a}, \quad \mu_1 = b\left(\frac{\mu'}{b}-\frac{\lambda'}{a}\right), \quad \nu_1 = c\left(\frac{\nu'}{c}-\frac{\lambda'}{a}\right), \text{ \&c.},$$

$$p_2 = -\frac{\mu'}{b}, \quad \lambda_2 = a\left(\frac{\lambda'}{a}-\frac{\mu'}{b}\right), \quad \nu_2 = c\left(\frac{\nu'}{c}-\frac{\mu'}{b}\right), \text{ \&c.},$$

$$\text{\&c.}, \qquad\qquad \text{\&c.};$$

and hence, if $(\bar{\lambda}, \bar{\mu}, \bar{\nu} \ldots)$ be the centroid of equal weights at A', B', &c.,

$$(n+1)\,\bar{\lambda} = -a\left(\frac{\lambda'}{a}+\frac{\mu'}{b}+\frac{\nu'}{c}+\ldots\right)+(n+1)\,\lambda',$$

$$(n+1)\,\bar{\mu} = -b\left(\frac{\lambda'}{a}+\frac{\mu'}{b}+\frac{\nu'}{c}+\ldots\right)+(n+1)\,\mu',$$

$$(n+1)\,\bar{\nu} = -c\left(\frac{\lambda'}{a}+\frac{\mu'}{b}+\frac{\nu'}{c}+\ldots\right)+(n+1)\,\nu',$$

$$\text{\&c.}, \qquad\qquad \text{\&c.},$$

therefore $$\frac{\bar{\lambda}}{a}+\frac{\bar{\mu}}{b}+\frac{\bar{\nu}}{c}+\ldots = 0$$

for all positions of $(\lambda', \mu', \nu' \ldots)$, and the centroid lies on a linear locus. The envelope of all such loci is

$$\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}}+\nu^{\frac{1}{2}}+\ldots = 0.$$

For, since
$$\frac{\lambda}{a}+\frac{\mu}{b}+\frac{\nu}{c}+\ldots=0,$$
and
$$a+b+c\ldots=0,$$
if $a+\delta a$, $b+\delta b$, $c+\delta c$, ... be the values of $a, b, c, \ldots$ for the adjacent locus,
$$\frac{\lambda}{a^2}\,\delta a+\frac{\mu}{b^2}\,\delta b+\frac{\nu}{c^2}\,\delta c+\ldots=0,$$
and
$$\delta a+\delta b+\delta c+\ldots=0,$$
therefore
$$\left(\frac{\lambda}{a^2}-\frac{\mu}{b^2}\right)\delta b+\left(\frac{\lambda}{a^2}-\frac{\nu}{c^2}\right)\delta c+\ldots=0,$$
and
$$\frac{\lambda}{a^2}=\frac{\mu}{b^2}=\frac{\nu}{c^2}=\ldots,$$
and therefore
$$\lambda^{\frac{1}{2}}+\mu^{\frac{1}{2}}+\nu^{\frac{1}{2}}+\ldots=0,$$
a locus of order 2^{n-1}, which passes through the middle points of the edges.

This is a proof and a generalization of a property of tetrahedron ennunciated by Mr. McCay, as Question 8840 in the *Educational Times*.

Also, if parallels be drawn through the vertices of the simplicissimum of reference to the line
$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=\ldots\;(\equiv p),$$
where
$$a+b+c+\ldots=0,$$
and these meet the opposite faces in the points $(0, \mu_1, \nu_1, \pi_1 \ldots)$, $(\lambda_2, 0, \nu_2, 0 \ldots)$, $(\lambda_3, \mu_3, 0, \pi_3 \ldots)$, &c., respectively; then, p_1, p_2, p_3, &c. being the corresponding values of p,
$$p_1=-\frac{V}{a},\quad p_2=-\frac{V}{b},\quad p_3=-\frac{V}{c},\ \&c.,$$
therefore
$$a=-\frac{V}{p_1},\quad b=-\frac{V}{p_2},\quad c=-\frac{V}{p_3},\ \&c.,$$
therefore
$$V\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\ldots\right)=0,$$
and
$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\ldots=0,$$
and p_1, p_2, p_3, &c. are proportional to the intercepts on the lines, between the vertices and the opposite faces.

This property was set, for the tetrahedron, by Mr. F. Morley (*Educational Times*, Quest. 6112).

Also $$\mu_1 = -\frac{Vb}{a}, \quad \nu_1 = -\frac{Vc}{a}, \quad \pi_1 = -\frac{Vd}{a}, \;\ldots,$$

$$\lambda_2 = -\frac{Va}{b}, \qquad \nu_2 = -\frac{Vc}{b}, \quad \pi_2 = -\frac{Vd}{b}, \;\ldots,$$

&c. &c. &c.,

and the content of the simplicissimum whose vertices are the feet of the parallels (VI.)

$$= \frac{1}{V^n} \begin{vmatrix} 0, & -V\frac{b}{a}, & -V\frac{c}{a}, & \ldots \\ -V\frac{a}{b}, & 0, & -V\frac{c}{b}, & \ldots \\ -V\frac{a}{c}, & -V\frac{b}{c}, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} .$$

$$= V \begin{vmatrix} 0, & 1, & 1, & \ldots \\ 1, & 0, & 1, & \ldots \\ 1, & 1, & 0, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} = nV \text{ (disregarding sign)},$$

a proposition set for the tetrahedron, by Professor Malet (*Educational Times*, Quest. 8047).

The same result would follow from the formula

$$nV \frac{A'P \, . \, B'P \ldots}{AP \, . \, BP \ldots} \quad \text{(VII.)},$$

by supposing P at an infinite distance, so that

$$\frac{A'P}{AP} = \frac{B'P}{BP} = \ldots = 1.$$

XIX. If the lines joining any point to the vertices of a simplicissimum be similarly divided, the lines joining the points of division to the centroids of the corresponding faces meet in a point. (A generalization of Quest. 7782, *Educational Times*.)

Take the simplicissimum as that of reference, and let $(\lambda', \mu', \nu' \ldots)$ be the point, then

$$\left(\frac{pV+q\lambda'}{p+q}, \frac{q\mu'}{p+q}, \frac{q\nu'}{p+q}, \ldots\right), \quad \left(\frac{q\lambda'}{p+q}, \frac{pV+q\mu'}{p+q}, \frac{q\nu'}{p+q}, \ldots\right), \text{ \&c.}$$

are the points of division; and the equations to the connector of $\left(0, \frac{V}{n}, \frac{V}{n}, \ldots\right)$, the centroid of the face upon $\lambda = 0$, and the first point of division, are

$$\begin{Vmatrix} \lambda, & \mu, & \nu, & \ldots \\ pV+q\lambda', & q\mu', & q\nu', & \ldots \\ 0, & 1, & 1, & \ldots \end{Vmatrix} = 0,$$

or

$$\begin{Vmatrix} \lambda, & \mu, & \nu, & \ldots \\ pV+q\lambda', & pV+q\mu', & pV+q\nu', & \ldots \\ 0, & 1, & 1, & \ldots \end{Vmatrix} = 0,$$

similarly those to the second connector are

$$\begin{Vmatrix} \lambda, & \mu, & \nu, & \ldots \\ pV+q\lambda', & pV+q\mu', & pV+q\nu', & \ldots \\ 1, & 0, & 1, & \ldots \end{Vmatrix} = 0,$$

and so on. Therefore all pass through the point

$$\left(\frac{pV+q\lambda'}{np+p+q}, \frac{pV+q\mu'}{np+p+q}, \ldots\right),$$

a point on the line joining $(\lambda', \mu', \nu', \ldots)$ to the centroid of the simplicissimum. The centroid of the derived simplicissimum, whose vertices are at the points of division, lies upon this line and divides it as the original lines are divided. The two simplicissima are similar and similarly situated, and the point $(\lambda', \mu', \nu', \ldots)$ is their centre of similitude, and hence that of the corresponding related spherics. Therefore, if P be the point $(\lambda', \mu', \nu', \ldots)$, O and O' the centres of the circumspherics, O' divides PO in the same ratio in which the connectors of P with the vertices are divided; and, if O be fixed, P and O' describe similar loci. If P lie upon the circumspheric of the simplicissimum of reference, the two circumspherics touch at that point. (This is generalized from Mr. Tucker's questions 7141 and 8169 in the *Educational Times*.) If O and O' be any other corresponding points on the two simplicissima, O' will divide OP in the same ratio, and O' and P will describe similar loci, and, if P lie on any other spheric connected with the simplicissimum, the corresponding spheric with respect to the derived simplicissimum will touch it at P.

XX. If a pencil of $2n$ coterminous lines meet a linear locus, in space

of n dimensions, in $2n$ points, the ratio of the products of any two pairs of $(n-1)$-ary simplicissima (each having its vertices at n of the points of intersection) is independent of the position of the transverse locus.

For, if $A_1, A_2, \dots A_{2n}$ be the points in which such a pencil meets a linear locus, and a_{rs} denote the angle between OA_r and OA_s, $(A_1 . A_2 \dots A_n)$ the content of the simplicissimum of which the vertices are at $A_1, A_2, \dots A_n$, p the perpendicular from O upon the linear locus upon which $A_1, A_2, \dots A_{2n}$ lie, and $D(1.2\dots n)$, the determinant

$$\begin{vmatrix} 1, & \cos a_{12}, & \cos a_{13}, & \dots & \cos a_{1n} \\ \cos a_{21}, & 1, & \cos a_{23}, & \dots & \cos a_{2n} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

by (IV.), $\quad (O.A_1.A_2 \dots A_n)^2 \times (O.A_{n+1}.A_{n+2} \dots A_{2n})^2$

$$= \frac{p^4}{n^4} (A_1.A_2 \dots A_n)^2 (A_{n+1}.A_{n+2} \dots A_{2n})^2$$

$$= \frac{OA_1^2 . OA_2^2 \dots OA_n^2}{(n!)^2} D(1.2\dots n) \frac{OA_{n+1}^2 . OA_{n+2}^2 \dots OA_{2n}^n}{(n!)^2} D(n+1.n+2\dots 2n)$$

$$= \frac{OA_1^2 . OA_2^2 \dots OA_{2n}^2}{(n!)^4} D(1.2\dots n) D(n+1.n+2\dots 2n),$$

therefore $\quad (A_1.A_2 \dots A_n)^2 (A_{n+1}.A_{n+2} \dots A_{2n})^2$

$$= \frac{OA_1^2 . OA_2^2 \dots OA_{2n}^2}{\{(n-1)!\}^4 p^4} D(1.2\dots n) D(n+1.n+2\dots 2n),$$

and therefore the ratio of the products of any two pairs of complementary $(n-1)$-ary simplicissima depends solely upon the values of the determinants, and is independent of the position of the transverse locus.

It also follows that two pencils which meet a linear locus in the same $2n$ points will have the ratios of the products of the corresponding pairs of complementary $(n-1)$-ary simplicissima upon any transverse linear loci the same.

All this is in analogy with the anharmonic ratios of four lines in a plane; and, as in that case, there will be equations among the ratios.

If the pencil consist of n rays coinciding with the edges of the simplicissimum of reference, which meet at $(V, 0, 0, \dots)$, and the

following n lines through the same point,

$$\mu = \frac{1}{k_1}\nu = \frac{1}{l_1}\pi = \dots,$$
$$\mu = \frac{1}{k_2}\nu = \frac{1}{l_2}\pi = \dots,$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\mu = \frac{1}{k_n}\nu = \frac{1}{l_n}\pi = \dots,$$

each of the $\frac{1}{2}\frac{(2n)!}{(n!)^2} = \frac{(2n-1)!}{n!\,(n-1)!}$ expressions for the products of pairs of complementary $(n-1)$-ary simplicissima, the vertices of which occupy all the intersections of the pencil with the linear locus $\lambda = 0$, will be expressible in terms of the $n(n-1)$ quantities, $k_1 \dots k_n$, $l_1 \dots l_n$, &c.; and there will therefore be $\frac{(2n-1)!}{n!\,(n-1)!} - n(n-1)$ relations among them; and hence, if $n=2$, one relation, as is well known.

It also follows that, if $M = 0$, $N = 0$, $P = 0$, &c.,

$$M = \frac{1}{k_1}N = \frac{1}{l_1}P = \dots,$$
$$M = \frac{1}{k_2}N = \frac{1}{l_2}P = \dots,$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$M = \frac{1}{k_n}N = \frac{1}{l_n}P = \dots,$$

be another system of $2n$ right lines meeting in a point, the ratios for these will be the same as for the system above, and the systems will be homographic.

Similarly, if there be two systems of $2n$ points, each lying on a linear locus, viz.,

$$(\lambda_1, \mu_1, \nu_1 \dots),\ (\lambda_2, \mu_2, \nu_2 \dots),\ \dots\dots\ (\lambda_n, \mu_n, \nu_n \dots),$$
$$\left(\frac{h_1\lambda_1 + k_1\lambda_2 + l_1\lambda_3 \dots}{h_1 + k_1 + l_1 \dots},\ \frac{h_1\mu_1 + k_1\mu_2 + l_1\mu_3 \dots}{h_1 + k_1 + l_1 \dots},\ \dots\right),$$
$$\left(\frac{h_2\lambda_1 + k_2\lambda_2 + l_2\lambda_3 \dots}{h_2 + k_2 + l_2 \dots},\ \frac{h_2\mu_1 + k_2\mu_2 + l_2\mu_3 \dots}{h_2 + k_2 + l_2 \dots},\ \dots\right),$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\left(\frac{h_n\lambda_1 + k_n\lambda_2 + l_n\lambda_3 \dots}{h_n + k_n + l_n \dots},\ \frac{h_n\mu_1 + k_n\mu_2 + l_n\mu_3 \dots}{h_n + k_n + l_n},\ \dots\right),$$

and $$(\lambda_1', \mu_1', \nu_1' \ldots),\ (\lambda_2', \mu_2', \nu_2' \ldots) \ldots (\lambda_n', \mu_n', \nu_n' \ldots),$$

$$\left(\frac{h_1\lambda_1'+k_1\lambda_2'+l_1\lambda_3' \ldots}{h_1+k_1+l_1 \ldots},\ \frac{h_1\mu_1'+k_1\mu_2'+l_1\mu_3' \ldots}{h_1+k_1+l_1 \ldots},\ \ldots\right),$$
$$\left(\frac{h_2\lambda_1'+k_2\lambda_2'+l_2\lambda_3' \ldots}{h_2+k_2+l_2 \ldots},\ \frac{h_2\mu_1'+k_2\mu_2'+l_2\mu_3' \ldots}{h_2+k_2+l_2 \ldots},\ \ldots\right),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\left(\frac{h_n\lambda_1'+k_n\lambda_2'+l_n\lambda_3' \ldots}{h_n+k_n+l_n \ldots},\ \frac{h_n\mu_1'+k_n\mu_2'+l_n\mu_3' \ldots}{h_n+k_n+l_n \ldots},\ \ldots\right),$$

these systems will also be homographic.

The anharmonic ratios of a pencil of the $2n$ coterminous lines represented by giving to r the values 1, 2 ... $2n$, in the equations

$$\frac{\lambda-\lambda'}{a_r}=\frac{\mu-\mu'}{b_r}=\frac{\nu-\nu'}{c_r}=\ldots\equiv\frac{V}{\sqrt{\{-S(a_r, b_r, c_r, \ldots)\}}}p_r\equiv m_r p_r, \text{ say,}$$

will be the ratios of the products of the corresponding complementary determinants of n rows and columns,

$$\begin{vmatrix} b_1, & c_1, & d_1 \ldots \\ b_2, & c_2, & d_2 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}.$$

For, if $(\lambda_1, \mu_1 \ldots)$, $(\lambda_2, \mu_2 \ldots)$, &c. be any points on the first n lines, and p_1, p_2, &c. their distances from $(\lambda', \mu' \ldots)$, the content of the simplicissimum of which these are the $(n+1)$ vertices is (V),

$$\frac{1}{V^n}\begin{vmatrix} \lambda', & \mu', & \nu', & \ldots \\ \lambda'+m_1p_1a_1, & \mu'+m_1p_1b_1, & \nu'+m_1p_1c_1, & \ldots \\ \lambda'+m_2p_2a_2, & \mu'+m_2p_2b_2, & \nu'+m_2p_2c_2, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \lambda'+m_np_na_n, & \mu'+m_np_nb_n, & \nu'+m_np_nc_n, & \ldots \end{vmatrix}$$

$$=\frac{m_1m_2\ldots m_n \cdot p_1p_2\ldots p_n}{V^n}\begin{vmatrix} \lambda', & \mu', & \nu' \ldots \\ a_1, & b_1, & c_1 \ldots \\ a_2, & b_2, & c_2 \ldots \\ \ldots & \ldots & \ldots \\ a_n, & b_n, & c_n \ldots \end{vmatrix}$$

$$=\frac{m_1m_2\ldots m_n \cdot p_1p_2\ldots p_n}{V^n}\begin{vmatrix} \lambda'+\mu'+\ldots, & \mu', & \nu', & \ldots \\ a_1+b_1+\ldots, & b_1, & c_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n+b_n+\ldots, & b_n, & c_n, & \ldots \end{vmatrix}$$

$$= \frac{m_1 m_2 \dots m_n \cdot p_1 p_2 \dots p_n}{V^{n-1}} \begin{vmatrix} b_1, & c_1, & \dots \\ b_2, & c_2, & \dots \\ \dots & & \dots \\ b_n, & c_n, & \dots \end{vmatrix},$$

from which the proposition above follows at once. This agrees the known results in Plane Geometry; for, the anharmonic rati pencils of parallel lines being equal, take $(V, 0, 0)$, as (λ', μ', ν') the lines

$$\mu = \frac{b_1}{c_1}\nu \equiv k_1\nu, \quad \mu = \frac{b_2}{c_2}\nu \equiv k_2\nu, \quad \mu = \frac{b_3}{c_3}\nu \equiv k_3\nu, \quad \mu = \frac{b_4}{c_4}\nu \equiv$$

and the value of the anharmonic ratio given above is

$$\frac{\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} \times \begin{vmatrix} b_3, & c_3 \\ b_4, & c_4 \end{vmatrix}}{\begin{vmatrix} b_1, & c_1 \\ b_3, & c_3 \end{vmatrix} \times \begin{vmatrix} b_2, & c_2 \\ b_4, & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} k_1, & 1 \\ k_2, & 1 \end{vmatrix} \times \begin{vmatrix} k_3, & 1 \\ k_4, & 1 \end{vmatrix}}{\begin{vmatrix} k_1, & 1 \\ k_3, & 1 \end{vmatrix} \times \begin{vmatrix} k_2, & 1 \\ k_4, & 1 \end{vmatrix}} = \frac{(k_1-k_2)(k_3-k_4)}{(k_1-k_3)(k_2-k_4)}$$

the known result. (Salmon's *Conics*, p. 56).

If $\alpha = 0$, $A = 0$, be the equations to two complementary l loci, each of which passes through n out of $2n$ given points, a s and S be the contents of the $(n-1)$-ary simplicissima, upon loci, whose vertices are at these points, and P_1, P_2 any two p the perpendiculars from which on $\alpha = 0$, $A = 0$ are p_1, q_1; respectively, by XII., therefore

$$\alpha_1 A_1 : \alpha_2 A_2 :: p_1 q_1 : p_2 q_2 :: p_1 s \cdot q_1 S : p_2 s \cdot q_2 S,$$

and αA is proportional to the product of the simplicissima of w the vertices are at $(\lambda, \mu, \nu \dots)$, and at the n points on each locus.

If then $\alpha = 0$, $A = 0$; $\beta = 0$, $B = 0$; $\gamma = 0$, $\Gamma = 0$, &c., be the tions to pairs of complementary linear loci through n of $2n$ g points, of which not more than n lie on a linear locus, the equ to any quadratic locus through the $2n$ points may be put in the

$$\mathrm{a}\alpha A + \mathrm{b}\beta B + \mathrm{c}\gamma\Gamma + \dots = 0,$$

where a, b, c, &c., are $\frac{n(n-1)}{2}+1$ in number; for $\frac{n(n+3)}{2}$ p determine the quadratic, and, since $2n$ of these are kn $\frac{n(n-1)}{2}$ will determine the locus.

This equation furnishes a relation between any $\frac{n(n-1)}{2}+1$ of the $\frac{(2n-1)!}{n!\,(n-1)!}$ products αA, βB, &c., which is satisfied when $(\lambda, \mu, \nu \ldots)$ lies on the locus: that is to say, an equation between $\frac{n(n-1)}{2}$ of the anharmonic ratios of the pencils drawn through any point on the quadric to the $2n$ given points upon it. And this generalizes and includes the anharmonic property of conics.

By the help of the identical relations which hold among the anharmonic ratios of the same pencil, the equivalent forms in terms of other ratios may be obtained.

XXI. If $(x_1, y_1, z_1 \ldots)$, $(x_2, y_2, z_2 \ldots) \ldots (x_{n+3}, y_{n+3}, z_{n+3} \ldots)$,

be the Cartesian co-ordinates of the centres of $n+3$ spherics, the radii of which are $r_1, r_2 \ldots r_{n+3}$, and if (1.2) be the length of an external common tangent to the first two spherics,

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+\ldots = (r_1-r_2)^2+(1.2)^2,$$

or
$$\begin{aligned}(1.2)^2 = x_1^2+y_1^2+z_1^2+\ldots-r_1^2+x_2^2+y_2^2+z_2^2+\ldots-r_2^2\\ -2\,(x_1x_2+y_1y_2+z_1z_2+\ldots-r_1r_2).\end{aligned}$$

[For an internal common tangent this becomes

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+\ldots = (r_1+r_2)^2+(1.2)^2,$$

or
$$\begin{aligned}(1.2)^2 = x_1^2+y_1^2+z_1^2+\ldots-r_1^2+x_2^2+y_2^2+z_2^2+\ldots-r_2^2\\ -2\,(x_1x_2+y_1y_2+z_1z_2+\ldots+r_1r_2)].\end{aligned}$$

therefore

$$\begin{vmatrix} 1, & 0, & 0, & 0, & 0, & 0 \\ x_1^2+y_1^2+z_1^2\ldots-r_1^2, & -2x_1, & -2y_1, & -2z_1, & \ldots\ 2r_1, & 1 \\ x_2^2+y_2^2+z_2^2\ldots-r_2^2, & -2x_2, & -2y_2, & -2z_2, & \ldots\ 2r_2, & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+3}^2+y_{n+3}^2+z_{n+3}^2\ldots-r_{n+3}^2, & -2x_{n+3}, & -2y_{n+3}, & -2z_{n+3}, & \ldots\ 2r_{n+3}, & 1 \end{vmatrix}$$

$$\times \begin{vmatrix} 0, & 0, & 0, & 0, & \ldots\ldots\ 0, & 1 \\ 1, & x_1, & y_1, & z_1, & \ldots\ldots\ r_1, & x_1^2+y_1^2+z_1^2+\ldots-r_1^2 \\ 1, & x_2, & y_2, & z_2, & \ldots\ldots\ r_2, & x_2^2+y_2^2+z_2^2+\ldots-r_2^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & x_{n+3}, & y_{n+3}, & z_{n+3}, & \ldots\ldots\ r_{n+3}, & x_{n+3}^2+y_{n+3}^2+z_{n+3}^2+\ldots-r_{n+3}^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots\ldots & 1 \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \ldots\ldots & (1.n+3)^2 \\ 1, & (2.1)^2, & 0, & (2.3)^2, & \ldots\ldots & (2.n+3)^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & (n+3.1)^2, & (n+3.2)^2, & (n+3.3)^2, & \ldots\ldots & 0 \end{vmatrix}$$

$$\equiv 0,$$

since each of the component determinants has $n+4$ rows and columns.

By suppressing the last row and the last column, a relation w obtained between the external common tangents to $n+2$ spl which all touch the same spheric.

This is adapted from a proof, by Dr. Casey, for the case o circles, given in Salmon's *Conic Sections*.

Similarly, by suppressing the two outer rows and columns, a tion will be obtained between the common tangents to $(n+1)$ sph each of which touches two spherics which also touch each other so on.

The identity of form, between the relation above and the con that the content of the simplicissimum (of $n+3$ vertices) w edges are equal to the common tangents should vanish, may some measure accounted for as follows. If all the spherics be e or if they be point spherics, (p, q) the common tangent to an of them is equal to the distance of their centres, and the conten be that of the simplicissimum whose vertices are at the centres, this vanishes as the $n+3$ vertices lie in the same space of n di sions.

In the same way, if $(n+2)$ equal spherics touch the same spl or $(n+2)$ point spherics lie on the same spheric, the outer row column of the determinant vanish, and the determinant (I.) bec $(-2)^{n+2}(n+1)!\,V^2R^2$, where V is the content of the simplicissi of $(n+2)$ vertices, whose vertices are at the centres of the equ point spherics, and which, as they all lie in the same space dimensions, vanishes. As those vertices all lie upon the same spl this is also the condition that $(n+2)$ points should lie on the spheric.

XXII. If there be two lines through a vertex (say λ : $\mu = \nu = \ldots = 0$) of the simplicissimum of reference, such tha perpendiculars from any point of the one to the faces which me that vertex are inversely proportional to those from any point o other upon the same faces, say two isogonally conjugate lines;

if the equations to the one be

$$k_2\mu = k_3\nu = k_4\pi = \ldots,$$

those to the other are

$$\frac{\mu}{k_2V_2^2} = \frac{\nu}{k_3V_3^2} = \frac{\pi}{k_4V_4^2} = \ldots .$$

Now, taking the system of concurrent lines from the vertices through the point $(\lambda', \mu', \nu' \ldots)$, the equations to those lines are

$$\begin{array}{llll} & \dfrac{\mu}{\mu'} = & \dfrac{\nu}{\nu'} = & \dfrac{\pi}{\pi'} = \ldots, \\[2ex] \dfrac{\lambda}{\lambda'} & = & \dfrac{\nu}{\nu'} = & \dfrac{\pi}{\pi'} = \ldots, \\[2ex] \dfrac{\lambda}{\lambda'} = & \dfrac{\mu}{\mu'} & = & \dfrac{\pi}{\pi'} = \ldots, \\[2ex] \&c. & & \&c., & \end{array}$$

and those to the isogonally conjugate lines are

$$\begin{array}{llll} & \dfrac{\mu\mu'}{V_2^2} = & \dfrac{\nu\nu'}{V_3^2} = & \dfrac{\pi\pi'}{V_4^2} = \ldots, \\[2ex] \dfrac{\lambda\lambda'}{V_1^2} & = & \dfrac{\nu\nu'}{V} = & \dfrac{\pi\pi'}{V_4^2} = \ldots, \\[2ex] \dfrac{\lambda\lambda'}{V_1^2} = & \dfrac{\mu\mu'}{V_2^2} & = & \dfrac{\pi\pi'}{V_4^2} = \ldots, \\[2ex] & \&c. & \&c., & \end{array}$$

which all meet at the point isogonally conjugate to $(\lambda', \mu', \nu' \ldots)$, *i.e.*, the point whose co-ordinates are proportional to

$$\frac{V_1^2}{\lambda'},\ \frac{V_2^2}{\mu'},\ \frac{V_3^2}{\nu'},\ \ldots\ldots .$$

Thus, if $(\lambda', \mu', \nu' \ldots)$ be the centroid, the lines isogonally conjugate to its connectors with the vertices, viz.,

$$\begin{array}{llll} & \dfrac{\mu}{V_2^2} = & \dfrac{\nu}{V_3^2} = & \dfrac{\pi}{V_4^2} = \ldots, \\[2ex] \dfrac{\lambda}{V_1^2} & = & \dfrac{\nu}{V_3^2} = & \dfrac{\pi}{V_4^2} = \ldots, \\[2ex] \dfrac{\lambda}{V_1^2} = & \dfrac{\mu}{V_2^2} & = & \dfrac{\pi}{V_4^2} = \ldots, \\[2ex] & \&c. & \&c., & \end{array}$$

all meet where $\lambda : \mu : \nu \ldots :: V_1^2 : V_2^2 : V_3^2 : \ldots$

(the centroid of weights proportional to V_1^2, V_2^2, V_3^2, &c., plac the opposite vertices), which may be called the **Symmedian p** Again, the equidistant lines (XIV., Note) are each of them its isogonal conjugate, and so are their intersections.

Other loci besides straight lines may be isogonally conjugate, such that to every point on the one a point on the other is isogo conjugate. If the one locus be of order m, the other will be at of order nm. If the highest powers of $\lambda, \mu, \nu \ldots$ in the equation t first locus be the p^{th}, q^{th}, $r^{\text{th}} \ldots$ respectively, the order of isogonally conjugate locus will be $p+q+r+\ldots-m$, and the c will be reduced by one for every time the original locus p through a vertex of the simplicissimum of reference.

To a linear locus there corresponds in general a locus of ord which passes through all the intersections of the faces. (In p geometry, a circumscribed conic.) An important example is isogonal conjugate of infinity noticed below (XXVI.) Another portant example, in plane geometry, is the conic which is the isog conjugate of a unicursal quartic (Salmon's *Higher Plane Cu* p. 244). Every such correspondence is a point to point corres dence.

As all spherics intersect in a common locus at infinity [or generally all loci whose equations are of the form

$$U-(\lambda+\mu+\ldots)(A\lambda+B\mu+\ldots)=0]$$

(see XXV. below), the isogonally conjugate loci intersect up common locus on

$$\frac{V_1^2}{\lambda}+\frac{V_2^2}{\mu}+\frac{V_3^2}{\nu}+\ldots=0,$$

the locus isogonally conjugate to infinity.

XXIII. Linear loci may be drawn through the intersections of two faces of a simplicissimum and a particular point; if the sir cissimum be taken as that of reference, and $(\lambda', \mu', \nu' \ldots)$ be point, the equations to these loci will be

$$\frac{\lambda}{\lambda'}=\frac{\mu}{\mu'},\quad \frac{\lambda}{\lambda'}=\frac{\nu}{\nu'},\quad \frac{\mu}{\mu'}=\frac{\nu}{\nu'},\ \text{\&c.},$$

or

$$\left\|\begin{matrix}\lambda, & \mu, & \nu, & \ldots\\ \lambda', & \mu', & \nu', & \ldots\end{matrix}\right\|=0,$$

and where $\dfrac{\lambda}{\lambda'} = \dfrac{\mu}{\mu'}$ meets the opposite edge (1.2),

i.e., $\nu = 0, \quad \pi = 0, \quad$ &c.,

$$\lambda = \frac{\lambda'}{\lambda'+\mu'}\,V, \text{ and } \mu = \frac{\mu'}{\lambda'+\mu'}\,V, \quad \nu = \pi = \ldots = 0,$$

and if this point be P_{12},

$$AP_{12} = \frac{\mu'}{\lambda'+\mu'}\,(1.2), \quad BP_{12} = \frac{\lambda'}{\lambda'+\mu'}\,(1.2),$$

and therefore

$$AP_{12}\,.\,\lambda' = BP_{12}\,.\,\mu', \quad BP_{23}\,.\,\mu' = CP_{23}\,.\,\nu', \quad CP_{13}\,.\,\nu' = AP_{13}\,.\,\lambda',$$

and therefore $\quad AP_{12}\,.\,BP_{23}\,.\,CP_{13} = BP_{12}\,.\,CP_{23}\,.\,AP_{13}$,

and the points P_{12}, P_{23}, P_{31} are the feet of concurrent lines from A, B, and C in their plane, and

$$AP_{12} : BP_{12} :: \frac{1}{\lambda'} : \frac{1}{\mu'}, \text{ and so on.}$$

If $(\lambda', \mu', \nu' \ldots)$ be the centroid, the edges are all bisected. If it be the centre of the in-spheric,

$$AP_{12} : BP_{12} :: \frac{1}{V_1} : \frac{1}{V_2},$$

and similarly for the centres of the ex-spherics (compare *Euclid* VI., 3 and A.).

If $\quad AP_{12} = AP_{13} = \ldots = d_1, \quad BP_{21} = BP_{23} = \ldots = d_2,$ &c.,

so that the simplicissimum is one in which $(r\,.\,s) = d_r + d_s$ (XVI.), the linear loci through the points of division of the edges (*i.e.*, the points of contact of the spheric which touches all these) and the opposite intersections of faces will all pass through the point

$$\lambda' : \mu' : \nu' : \ldots :: \frac{1}{d_1} : \frac{1}{d_2} : \frac{1}{d_3} : \ldots .$$

Again, if points P'_{rs} be taken on each edge $(r\,.\,s)$, such that P_{rs} and P'_{rs} are harmonically conjugate with respect to the vertices on that edge, and if $(\lambda'_{12}, \mu'_{12}, 0, 0, \ldots)$ be the co-ordinates of P'_{12},

$$AP'_{12} = \mu'_{12}\,\frac{(1.2)}{V}, \quad BP'_{12} = \lambda'_{12}\,\frac{(1.2)}{V},$$

and $$\frac{\lambda'_{12}}{\mu'_{12}} = -\frac{\lambda_{12}}{\mu_{12}} = -\frac{\lambda'}{\mu'},$$

and P'_{12} lies on the linear locus

$$\frac{\lambda}{\lambda'}+\frac{\mu}{\mu'}+\frac{\nu}{\nu'}+\ldots=0,$$

the locus of homology of the simplicissimum of reference and th whose vertices are the feet of the concurrent lines through $(\lambda', \mu',$ from the vertices (VIII.).

This locus is also the linear polar of the centroid with respe the quadric with regard to which the simplicissimum is self-co gate, and which has its centre at $(\lambda', \mu', \nu' \ldots)$. It is, too, the li polar of $(\lambda', \mu', \nu' \ldots)$ with respect to the $(n+1)$-ic locus compose the linear loci on which the faces lie.

XXIV. If a quadratic locus in space of n dimensions cut the e of a simplicissimum in points P_{12}, Q_{12}; P_{23}, Q_{23}; P_{31}, Q_{31}, &c. res tively, and points p_{12}, q_{12}; p_{23}, q_{23}; p_{31}, q_{31}, &c. be taken in the s edges harmonically conjugate to these points and the vertices the edges; these last points lie upon another quadratic locus.

For if $$A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu+\ldots=0$$

be the equation to the locus referred to the simplicissimum as th reference; the points P_{12}, Q_{12} are determined by the equation

$$A_{11}\lambda^2+2A_{12}\lambda\mu+A_{22}\mu^2=0,$$

and therefore p_{12}, q_{12} by

$$A_{11}\lambda^2-2A_{12}\lambda\mu+A_{22}\mu^2=0,$$

and therefore all the points p, q lie upon

$$A_{11}\lambda^2+A_{22}\mu^2+\ldots-2A_{12}\lambda\mu-\ldots=0.$$

If the points P_{12}, P_{23}, &c. be the intersections of the edges and current linear loci through the opposite intersections of faces; al equations—

$$\begin{aligned}A_{11}\lambda^2+2A_{12}\lambda\mu+A_{22}\mu^2&=0,\\ A_{22}\mu^2+2A_{23}\mu\nu+A_{33}\nu^2&=0,\\ A_{33}\nu^2+2A_{31}\nu\lambda+A_{11}\lambda^2&=0,\\ \&c. \qquad \&c.\end{aligned}$$

which can be formed, must hold simultaneously, and therefore al pressions of the form

$$A_{qq}A_{rr}A_{ss}-2A_{qr}A_{rs}A_{sq}-A_{qq}A_{rs}^2-A_{rr}A_{sq}^2-A_{ss}A_{qr}^2$$

must vanish* (*Educational Times*, Quest. 6813; *Reprint*, Vol. XXXVII., p. 31). When this is the case, the quadratic locus will meet the edges in another set of points upon concurrent linear loci, for

$$A_{11}\lambda^2+2A_{12}\lambda\mu+A_{22}\mu^2 \equiv K\left(\frac{\lambda}{\lambda'}-\frac{\mu}{\mu'}\right)\left(\frac{\lambda}{a}-\frac{\mu}{b}\right),$$

$$A_{22}\mu^2+2A_{23}\mu\nu+A_{33}\nu^2 \equiv K\left(\frac{\mu}{\mu'}-\frac{\nu}{\nu'}\right)\left(\frac{\mu}{b}-\frac{\nu}{c}\right),$$

&c. &c.

and the equation to the quadratic locus will (XVI.) be of the form

$$ll'\lambda^2+mm'\mu^2+\ldots-(lm'+l'm)\lambda\mu+\ldots=0;$$

or if $(\lambda', \mu', \nu'\ldots)$, $(\lambda'', \mu'', \nu''\ldots)$ be the points of concurrence of the linear loci through the points on the edges, and the opposite intersections of faces,

$$\frac{\lambda^2}{\lambda'\lambda''}+\frac{\mu^2}{\mu'\mu''}+\ldots-\left(\frac{1}{\lambda'\mu''}+\frac{1}{\lambda''\mu'}\right)\lambda\mu-\ldots=0,$$

and the locus of the harmonically conjugate points is the pair of linear loci

$$\left(\frac{\lambda}{\lambda'}+\frac{\mu}{\mu'}+\frac{\nu}{\nu'}+\ldots\right)\left(\frac{\lambda}{\lambda''}+\frac{\mu}{\mu''}+\frac{\nu}{\nu''}+\ldots\right)=0 \text{ (XXIII)}.$$

---

* If the quadric be the improper one composed of two linear loci, say

$$B_1\lambda+B_2\mu+B_3\nu+\ldots=0,$$

and

$$\frac{A_{11}}{B_1}\lambda+\frac{A_{22}}{B_2}\mu+\frac{A_{33}}{B_3}\nu+\ldots=0,$$

all the equations

$$A_{11}\frac{B_2}{B_1}+A_{22}\frac{B_1}{B_2}=2A_{12},$$

$$A_{22}\frac{B_3}{B_2}+A_{33}\frac{B_2}{B_3}=2A_{23},$$

$$A_{33}\frac{B_1}{B_3}+A_{11}\frac{B_3}{B_1}=2A_{13},$$

&c. &c.

which can be formed, must hold; and these only differ from the above in the sign of the terms which involve two different suffixes, and the conditions are therefore that all the expressions of the form

$$A_{qq}A_{rr}A_{ss}+2A_{qr}A_{rs}A_{sq}-A_{qq}A_{rs}^2-A_{rr}A_{sq}^2-A_{ss}A_{qr}^2$$

must vanish.

All this is in exact accordance with what was proved for Pl Geometry in Question 7585, *Educational Times* (*Reprint*, Vol. x p. 124).

If the quadratic locus be a spheric (X.),

$$\frac{\left(\frac{1}{\lambda'}+\frac{1}{\lambda''}\right)\left(\frac{1}{\mu'}+\frac{1}{\mu''}\right)}{(1\,.\,2)^2}=\frac{\left(\frac{1}{\mu'}+\frac{1}{\mu''}\right)\left(\frac{1}{\nu'}+\frac{1}{\nu''}\right)}{(2\,.\,3)^2},\ \&c.,$$

and if the simplicissimum be of the special class in which, $m_1, m_2 \ldots n$ being known linear magnitudes, $(r\,.\,s)^2 = m_r m_s$ [a class which inclu triangles and those tetrahedra for which

$$(1\,.\,2)\,(3\,.\,4) = (1\,.\,3)\,(2\,.\,4) = (1\,.\,4)\,(2\,.\,3)\big],$$

$$\frac{\frac{1}{\lambda'}+\frac{1}{\lambda''}}{m_1}=\frac{\frac{1}{\mu'}+\frac{1}{\mu''}}{m_2}=\frac{\frac{1}{\nu'}+\frac{1}{\nu''}}{m_3}=\ldots\ldots.$$

Now, if other points $(x', y', z'\ldots)$ $(x'', y'', z''\ldots)$ be taken such that

$$x' : y' : z' \ldots :: \frac{1}{m_1\lambda'} : \frac{1}{m_2\mu'} : \frac{1}{m_3\nu'} : \ldots$$

and
$$x'' : y'' : z'' \ldots :: \frac{1}{m_1\lambda''} : \frac{1}{m_2\mu''} : \frac{1}{m_3\nu''} \ldots$$

$$\therefore\quad px'+qx'' = py'+qy'' = pz'+qz'' = \ldots = \frac{(p+q)\,V}{n+1},$$

and the line joining $(x', y', z'\ldots)$ and $(x'', y'', z''\ldots)$ passes throu the centroid.

Now in such a simplicissimum

$$V^2R^2 = \frac{n}{2^{n-1}\,(n\,!)^2}\,m_1^2 m_2^2 \ldots m_{n+1}^2;$$

and
$$V_1^2R_1^2 = \frac{n-1}{2^n\,\{(n-1)\,!\}^2}\,m_2^2 \ldots m_{n+1}^2,\ \&c.,$$

therefore

$$m_1 = \sqrt{2n\,(n-1)}\,\frac{VR}{V_1R_1}\ \text{and}\ m_1 : m_2 : \ldots :: \frac{1}{V_1R_1} : \frac{1}{V_2R_2} : \ldots,$$

and therefore

$$x' : y' : z' : \ldots :: \frac{V_1R_1}{\lambda'} : \frac{V_2R_2}{\mu'} : \frac{V_3R_3}{\nu'} : \ldots;$$

so that, in space of two dimensions, $(x', y', z'\ldots)$, $(x'', y'', z''\ldots)$ are t

[redacted]

isogonal conjugates of

$$(\lambda', \mu', \nu' \ldots) \text{ and } (\lambda'', \mu'', \nu'' \ldots).$$

XXV. In March 1881, Mr. S. Roberts gave the theorem, analogous to a known fact in Plane Geometry, that "if an arbitrary point be taken on each edge of a tetrahedron, and spheres be described through each vertex and the assumed points on the edges which pass through it, these spheres intersect in a common point." This property is true if we read simplicissimum for tetrahedron, and spheric for sphere.

Let $(\lambda_{12}, \mu_{12}, 0, 0 \ldots)$, $(0, \mu_{23}, \nu_{23}, 0 \ldots)$, $(\lambda_{13}, 0, \nu_{13}, 0 \ldots)$, &c.

be the arbitrarily chosen points. Then (X.), if

$$S-(A_1\lambda+A_2\mu+A_3\nu \ldots)\ (\lambda+\mu+\nu \ldots) = 0$$

be the spheric through the vertex $\lambda = V$, $\mu = \nu = \ldots = 0$, and the points $(\lambda_{12}, \mu_{12}, 0 \ldots)$, $(\lambda_{13}, 0, \nu_{13} \ldots)$, &c.

It follows that

$$A_1 = 0, \quad A_2 = \frac{(1\,.\,2)^2}{V}\lambda_{12}, \quad A_3 = \frac{(1\,.\,3)^2}{V}\lambda_{13}, \text{ \&c.,}$$

and the equation to the spheric is

$$S-\frac{\lambda+\mu+\nu \ldots}{V}\left\{\lambda_{12}\mu\,(1\,.\,2)^2+\lambda_{13}\nu\,(1\,.\,3)^2+\ldots\right\} = 0.$$

Similarly, those to the other spherics are

$$S-\frac{\lambda+\mu+\nu \ldots}{V}\left\{\mu_{12}\lambda\,(1\,.\,2)^2+\mu_{23}\nu\,(2\,.\,3)^2+\ldots\right\} = 0, \text{ \&c.}$$

Multiplying these by $\lambda, \mu, \nu \ldots$ respectively, and adding the products, we have

$$\begin{aligned} VS-\big\{&(\lambda_{12}+\mu_{12})\,\lambda\mu\,(1\,.\,2)^2+(\lambda_{13}+\nu_{13})\,(1\,.\,3)^2 \\ &+(\mu_{23}+\nu_{23})\,\mu\nu\,(2\,.\,3)^2+\ldots\big\} \\ \equiv\ & VS-V\left\{\lambda\mu\,(1\,.\,2)^2+\mu\nu\,(2\,.\,3)^2+\nu\lambda\,(1\,.\,3)^2+\ldots\right\} \\ &(\text{since } \lambda_{12}+\mu_{12} = V = \mu_{23}+\nu_{23} = \text{\&c.}) \\ \equiv\ & VS-VS \equiv 0, \end{aligned}$$

and the spherics all pass through a common point.

In the special case when the arbitrary points are the intersections of the edges with the perpendiculars upon them from a fixed point, the common point of intersection of the spherics will be the fixed point, which will be the extremity of the diameter of each spheric

through the vertex upon it; and each spheric will pass through the feet of the perpendiculars from the fixed point upon all the faces of the simplicissimum (and of the subsidiary simplicissima) which pass through that vertex. An example will be found at the end of Art. XXVII. (p. 22.)

If the chosen points be the mid-points of the edges, the equations to the spherics become

$$S-\frac{\lambda+\mu+\nu+\dots}{2V}\cdot\frac{dS}{d\lambda}=0,$$

$$S-\frac{\lambda+\mu+\nu+\dots}{2V}\cdot\frac{dS}{d\mu}=0,$$

which are all satisfied at the centre of the circum-spheric, where $S = VR^2$, and

$$\frac{dS}{d\lambda}=\frac{dS}{d\mu}=\frac{dS}{d\nu}\dots=2VR^2.*$$

XXVI. If $(\lambda', \mu', \nu' \dots)$ be any point, $(0, \mu_1, \nu_1 \dots)$, $(\lambda_2, 0, \nu_2 \dots)$, & its orthographic projections on the faces of the simplicissimum of reference, and if ρ be the radius of a spheric whose centre is $(\lambda', \mu', \nu' \dots)$, and which cuts the faces in sections common with those of spherics, the centres of which are at the projections, and their radii $\rho_1, \rho_2, \dots$, &c., so that

$$\rho^2-\rho_1^2=\left(\frac{n\lambda'}{V_1}\right)^2,\quad \rho^2-\rho_2^2=\left(\frac{n\mu'}{V_2}\right)^2,\quad \&c.;$$

the equation to the spheric radius ρ is (IX.)

$$\Sigma \equiv S-\frac{\lambda+\mu+\nu+\dots}{V}\left(\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}+\dots\right)$$

$$+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu+\nu+\dots)^2=0,$$

and those to any spherics which cut the faces in the same sections are

$$\Sigma-k_1\lambda(\lambda+\mu+\nu\dots)=0,\quad \Sigma-k_2\mu(\lambda+\mu+\nu\dots)=0,\ \&c.\dots (A$$

* The more general proposition—that, "if arbitrary points be taken on each edge of a simplicissimum and quadratic loci $U_1 = 0$, $U_2 = 0$, &c., each of which intersects a given circumscribed locus

$$A_{12}\lambda\mu+A_{13}\lambda\nu+A_{23}\mu\nu+\dots=0$$

in two linear loci, one of which is infinity, be described to pass through each vertex and the assumed points upon the edges through it, the quadrics U_1, U_2, &c. meet in a common point,"—may be proved in exactly the same way.

Now, if the first of these be the spheric radius ρ_1 above,

$$\Sigma - k_1\lambda\,(\lambda+\mu+\nu+\dots)$$

$$\equiv S - \frac{\lambda+\mu+\nu+\dots}{V}\left\{\lambda\left(\frac{dS'}{d\lambda'}+k_1V\right)+\mu\frac{dS'}{d\mu'}+\nu\frac{dS'}{d\nu'}+\dots\right\}$$
$$+\frac{V^2\rho^2+S'}{V^2}\{\lambda+\mu+\nu+\dots\}^2$$

$$\equiv S - \frac{\lambda+\mu+\nu+\dots}{V}\left\{\lambda\frac{dS_1}{d\lambda_1}+\mu\frac{dS_1}{d\mu_1}+\dots\right\}$$
$$+\frac{V^2\rho_1^2+S_1}{V^2}(\lambda+\mu+\nu\dots)^2,$$

therefore
$$\left.\begin{aligned}
\frac{dS'}{d\lambda'}+k_1V-\frac{V^2\rho^2+S'}{V} &= \frac{dS_1}{d\lambda_1}-\frac{V^2\rho_1^2+S_1}{V}\\
\frac{dS'}{d\mu'}\qquad\quad-\frac{V^2\rho^2+S'}{V} &= \frac{dS_1}{d\mu_1}-\frac{V^2\rho_1^2+S_1}{V}\\
\frac{dS'}{d\nu'}\qquad\quad-\frac{V^2\rho^2+S'}{V} &= \frac{dS_1}{d\nu_1}-\frac{V^2\rho_1^2+S_1}{V}\\
\&c. \qquad\qquad &\qquad\qquad \&c.
\end{aligned}\right\}\dots\dots\dots\dots\text{(B)}.$$

Multiplying these equations by $\lambda', \mu', \nu'\dots$ respectively, and adding and arranging,

$$-S'-S_1+\lambda'\frac{dS_1}{d\lambda_1}+\mu'\frac{dS_1}{d\mu_1}+\nu'\frac{dS_1}{d\nu_1}+\dots = -V^2(\rho^2-\rho_1^2)+k_1\lambda'V.$$

But the sinister $= V^2(\rho^2-\rho_1^2)$, for it is V^2 into the square of the distance from $(\lambda', \mu', \nu'\dots)$ to $(0, \mu_1, \nu_1\dots)$ (IX.), therefore

$$k_1\lambda'V = 2V^2(\rho^2-\rho_1^2) = 2\frac{n^2\lambda'^2}{V_1^2}V^2, \quad\text{and}\quad k_1 = 2n^2\frac{\lambda'}{V_1^2}V.$$

Similarly
$$k_2 = 2n^2\frac{\mu'}{V_2}V, \;\&c.$$

Now at the radical centre of the spherics (A)

$$k_1\lambda = k_2\mu = k_3\nu = \dots$$

therefore
$$\frac{\lambda\lambda'}{V_1^2} = \frac{\mu\mu'}{V_2^2} = \frac{\nu\nu'}{V_3^2} = \dots,$$

and the radical centre is the point isogonally conjugate to $(\lambda', \mu', \nu'\dots)$,

say $(\lambda'', \mu'', \nu''...)$, and this is the centre of the spheric whic the system (A) orthogonally.

Let $(\lambda''', \mu''', \nu'''...)$ be the centre, and ρ''' the radius of the s through $(0, \mu_1, \nu_1 ...)$, $(\lambda_2, 0, \nu_2 ...)$, &c.

Now the equation to this, the pedal spheric of $(\lambda', \mu', \nu' ...)$, i

$$S - \frac{\lambda+\mu+\nu...}{V}\left(\lambda'''\frac{dS}{d\lambda}+\mu'''\frac{dS}{d\mu}+...\right)+\frac{V^2\rho'''^2+S'''}{V^2}(\lambda+\mu+\nu...$$

and since $(0, \mu_1, \nu_1, ...)$ lies upon it

$$S_1 - \left(\lambda'''\frac{dS_1}{d\lambda_1}+\mu'''\frac{dS_1}{d\mu_1}+...\right)+V^2\rho'''^2+S''' = 0;$$

but, from the equations (B),

$$\lambda'''\frac{dS'}{d\lambda'}+\mu'''\frac{dS'}{d\mu'}+ ... - \left(\lambda'''\frac{dS_1}{d\lambda_1}+\mu'''\frac{dS_1}{d\mu_1}+...\right)$$

$$= -k_1 V\lambda'''+S'-S_1+V^2(\rho^2-\rho_1) = S'-S_1+n^2\frac{V^2}{V_1^2}\lambda'^2-2n^2\frac{V^2}{1}$$

$$\therefore \quad \lambda'''\frac{dS'}{d\lambda'}+\mu'''\frac{dS'}{d\mu'}+...-S'''-S' = V^2\left\{\rho'''^2+n^2\frac{\lambda'^2}{V_1^2}-2n^2\frac{\lambda'}{V}\right.$$

and, if d be the distance from $(\lambda', \mu', \nu' ...)$ to $(\lambda''', \mu''', \nu'''...)$,

$$d^2 = \rho'''^2+n^2\frac{\lambda'^2}{V_1^2}-2n^2\frac{\lambda'\lambda'''}{V_1^2},$$

and $$k_1\left(\frac{\lambda'''}{V}-\frac{\lambda'}{2V}\right) = \rho'''^2-d^2;$$

therefore $k_1(2\lambda'''-\lambda') = 2V(\rho'''^2-d^2) = k_2(2\mu'''-\mu') =$ &c.,

therefore $$V = 2(\lambda'''+\mu'''+...)-(\lambda'+\mu'+...)$$

$$= 2V(\rho'''^2-d^2)\sum_{r=1}^{r=n+1}\frac{1}{k_r};$$

but $k_1\lambda'' = k_2\mu'' = k_3\nu'' = ... = q$ suppose,

therefore $$V = q\sum_{r=1}^{r=n+1}\frac{1}{k_r},$$

therefore $$q = 2V(\rho'''^2-d^2).$$

And $2\lambda'''-\lambda' = \lambda''$, $2\mu'''-\mu' = \mu''$, &c.

and $(\lambda''', \mu''', \nu'''...)$ is the mid-point of the connector of $(\lambda', \mu',$

and $(\lambda'', \mu'', \nu''\ldots)$, and the pedal spheric of $(\lambda', \mu', \nu'\ldots)$ is also that of the isogonally conjugate point $(\lambda'', \mu'', \nu''\ldots)$. And

$$\rho'''^2 = \frac{1}{V^2}\left\{\lambda'''\frac{dS'}{d\lambda'}+\mu'''\frac{dS'}{d\mu'}+\ldots-S'''-S'\right\}-n^2\frac{\lambda'^2}{V_1^2}+2n^2\frac{\lambda'\lambda'''}{V_1^2},$$

therefore

$$S'''+V^2\rho'''^2 = \tfrac{1}{2}(\lambda'+\lambda'')\frac{dS'}{d\lambda}+\tfrac{1}{2}(\mu'+\mu'')\frac{dS'}{d\mu'}+\ldots$$

$$-S'-n^2V^2\frac{\lambda'^2}{V_1}+n^2V^2\frac{\lambda'(\lambda'+\lambda'')}{V_1^2}$$

$$=\tfrac{1}{2}\left(\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+\ldots\right)+n^2V^2\frac{\lambda'\lambda''}{V_1^2}\quad\ldots\ldots\ldots\ldots\ldots\text{(C)},$$

and the equation to the pedal spheric is

$$\Sigma' \equiv S-\frac{\lambda+\mu+\nu\ldots}{2V}\left\{(\lambda'+\lambda'')\frac{dS}{d\lambda}+(\mu'+\mu'')\frac{dS}{d\mu}+\ldots\right\}$$

$$+\frac{(\lambda+\mu+\nu+\ldots)^2}{2V^2}\left\{\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+\ldots+2n^2V^2\frac{\lambda'\lambda''}{V_1^2}\right\}=0.$$

The pedal spheric becomes infinite when $(\lambda''', \mu''', \nu'''\ldots)$ lies on infinity, and this will be the case when either $(\lambda', \mu', \nu'\ldots)$ or $(\lambda'', \mu'', \nu''\ldots)$ does so. If $(\lambda'', \mu'', \nu''\ldots)$ lie on infinity, $(\lambda', \mu', \nu'\ldots)$ lies on

$$\frac{V_1^2}{\lambda}+\frac{V_2^2}{\mu}+\frac{V_3^2}{\nu}+\ldots=0,$$

the locus isogonally conjugate to infinity (a circumscribed locus of order $n-1$, which in space of two dimensions is the circumcircle). This locus contains all the edges and intersections of faces of the simplicissimum. Also, if the centre of a quadratic locus, with respect to which the simplicissimum is self-conjugate, lie upon this locus, the quadratic locus passes through the centres of all the spherics which touch the faces (XIV.), and those centres are the poles of infinity with respect to this locus.

When $(\lambda', \mu', \nu'\ldots)$ lies upon this locus, the pedal spheric degenerates into a linear locus upon which all the projections of $(\lambda', \mu', \nu'\ldots)$ lie. In two dimensions this is the case of the Simson lines.

For when
$$\frac{V_1^2}{\lambda'}+\frac{V_2^2}{\mu'}+\frac{V_3^2}{\nu'}+\ldots=0,$$

let $$\lambda'' = \frac{rV_1^2}{\lambda'}, \quad \mu'' = \frac{rV_2^2}{\mu'}, \quad \nu'' = \frac{rV_3^2}{\nu'}, \quad \text{\&c.},$$

therefore $$V = r\left\{\frac{V_1^2}{\lambda'} + \frac{V_2^2}{\mu'} + \frac{V_3^2}{\nu'} + \dots\right\}, \quad \text{and } r = \infty,$$

and the pedal spheric of $(\lambda', \mu', \nu' \dots)$ becomes

$$(\lambda+\mu+\nu \dots)\left[\frac{V_1^2}{\lambda'}\frac{dS}{d\lambda} + \frac{V_2^2}{\mu'}\frac{dS}{d\mu} + \frac{V_3^2}{\nu'}\frac{dS}{d\nu} + \dots\right.$$

$$\left.\dots - \frac{\lambda+\mu+\nu \dots}{V}\left(\frac{V_1^2}{\lambda'}\frac{dS'}{d\lambda'} + \frac{V_2^2}{\mu'}\frac{dS'}{d\mu'} + \frac{V_3^2}{\nu'}\frac{dS'}{d\nu'} + \dots + 2n^2V^2\right)\right] = 0,$$

and the Simson locus is

$$\frac{V_1^2}{\lambda'}\left(\frac{dS}{d\lambda} - \frac{dS'}{d\lambda'}\right) + \frac{V_2^2}{\mu'}\left(\frac{dS}{d\mu} - \frac{dS'}{d\mu'}\right) + \dots - 2n^2V^2 = 0,$$

the parallel to which through the circumcentre is

$$\frac{V_1^2}{\lambda'}\frac{dS}{d\lambda} + \frac{V_2^2}{\mu'}\frac{dS}{d\mu} + \frac{V_3^2}{\nu'}\frac{dS}{d\nu} + \dots = 0,$$

the polar of $(\lambda'', \mu'', \nu'' \dots)$ with respect to the circumspheric, and hence the equation to the perpendicular upon the Simson locus of $(\lambda', \mu', \nu' \dots)$ from $(\lambda_1, \mu_1, \nu_1 \dots)$ is (XVIII.)

$$\frac{\lambda'(\lambda-\lambda_1)}{V_1^2} = \frac{\mu'(\mu-\mu_1)}{V_2^2} = \frac{\nu'(\nu-\nu_1)}{V_3^2} = \text{\&c.}$$

Also, if $(\lambda_1, \mu_1, \nu_1 \dots)$ be a point such that the Simson locus divides the line from it to $(\lambda', \mu', \nu' \dots)$ as $p-1:1$,

$$p\frac{dS}{d\lambda} = (p-1)\frac{dS'}{d\lambda'} + \frac{dS_1}{d\lambda_1}, \quad p\frac{dS}{d\mu} = (p-1)\frac{dS'}{d\mu'} + \frac{dS_1}{d\mu_1}, \quad \text{\&c.},$$

and therefore $$p\left(\frac{dS}{d\lambda} - \frac{dS'}{d\lambda'}\right) = \frac{dS_1}{d\lambda_1} - \frac{dS'}{d\lambda'}, \quad \text{\&c.},$$

where $(\lambda, \mu, \nu \dots)$ lies on the Simson locus, and $(\lambda_1, \mu_1, \nu_1 \dots)$ may be any point upon

$$\frac{V_1^2}{\lambda'}\left(\frac{dS}{d\lambda} - \frac{dS'}{d\lambda'}\right) + \frac{V_2^2}{\mu'}\left(\frac{dS}{d\mu} - \frac{dS'}{d\mu'}\right) + \dots - 2pn^2V^2 = 0.$$

[This equation will be used further on (XXVII.).]

If ρ'' be the radius of the orthogonal spheric, its equation is

$$\sigma \equiv S - \frac{\lambda+\mu+\nu \dots}{V}\left(\lambda''\frac{dS}{d\lambda} + \mu''\frac{dS}{d\mu} + \dots\right) + \frac{V^2\rho''^2 + S''}{V^2}(\lambda+\mu \dots)^2 = 0,$$

and

$$\Sigma \equiv S - \frac{\lambda+\mu+\nu\ldots}{V}\left(\lambda'\frac{dS}{d\lambda}+\mu'\frac{dS}{d\mu}+\ldots\right)+\frac{V^2\rho^2+S'}{V^2}(\lambda+\mu\ldots)^2,$$

$$\Sigma+\sigma \equiv 2\left[S-\frac{\lambda+\mu+\nu\ldots}{2V}\left\{(\lambda'+\lambda'')\frac{dS}{d\lambda}+(\mu'+\mu'')\frac{dS}{d\mu}+\ldots\right\}\right]$$

$$+\frac{V^2(\rho''^2+\rho^2)+S''+S'}{2V^2}(\lambda+\mu+\nu\ldots)^2$$

$$\equiv 2\left[S-\frac{\lambda+\mu+\nu\ldots}{V}\left(\lambda'''\frac{dS}{d\lambda}+\mu'''\frac{dS}{d\mu}+\ldots\right)\right.$$

$$\left.+\frac{V^2\rho'''^2+S'''}{V^2}(\lambda+\mu+\nu\ldots)^2\right]$$

$$+V^2(\rho''^2+\rho^2-2\rho'''^2)+S'+S''-S''' \equiv 2\Sigma',$$

for $$V^2(\rho''^2+\rho_1^2)=\lambda''\frac{dS_1}{d\lambda_1}+\mu''\frac{dS_1}{d\mu_1}+\ldots-S''-S_1, \quad \text{(IX.)},$$

(because the spherics' radii ρ'' and ρ_1 are orthogonal)

$$=\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+\ldots-V^2\rho^2-S'+V^2\rho_1^2-S''+k_1V\lambda'', \ \text{[by (B)]},$$

therefore $$V^2(\rho''^2+\rho^2)=\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+\ldots-S'-S''+2n^2V^2\frac{\lambda'\lambda''}{V_1^2}.$$

Therefore, by (C),

$$V^2(\rho''^2+\rho^2-2\rho'''^2)+S'+S''-2S'''=0,$$

and Σ, σ, and Σ' are coaxal.

This article is in the main a generalization of Mr. S. Roberts' interesting Questions 9093 and 9170 in the *Educational Times* (*Reprint*, Vol. XLVIII., Appendix III.). It is also indebted to M. Emile Vigarié's Question 9013.

XXVII. The class of simplicissima in which $(r.s)^2 = A_r + A_s$ (XVI., Note) may be called rectangular simplicissima. For each edge is at right angles to the intersection of the faces opposite to the vertices upon it.

The equations to (1.2) are $\nu = \pi = \ldots = 0$, and those to the intersection of the opposite faces are $\lambda = 0$, $\mu = 0$, while those to the perpendiculars upon those faces from the opposite vertices are

$$A_2\mu = A_3\nu = A_4\pi = \ldots,$$

and $$A_1\lambda = A_3\nu = A_4\pi = \ldots,$$

and therefore those to the plane perpendicular to $\lambda = 0$, $\mu = 0$, are

$$A_3\nu = A_4\pi = \ldots,$$

and this contains the edge (1 . 2), which is therefore at right angle to the intersection.

In the case of such a simplicissimum (I.)

$$V^2 = \frac{-1}{(-2)^n(n!)^2}\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & A_1+A_2, & A_1+A_3, & \ldots \\ 1, & A_1+A_2, & 0, & A_2+A_3, & \ldots \\ 1, & A_1+A_3, & A_2+A_3, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= \frac{1}{(n!)^2} A_1A_2 \ldots A_{n+1}\left\{\frac{1}{A_1} + \frac{1}{A_2} + \ldots + \frac{1}{A_{n+1}}\right\},$$

and, since all subordinate simplicissima which have their vertices at some of the vertices of a simplicissimum of this class are evidently of the same kind,

$$V_1^2 = \frac{1}{\{(n-1)!\}^2} A_2A_3 \ldots A_{n+1}\left\{\frac{1}{A_2} + \frac{1}{A_3} + \ldots + \frac{1}{A_{n+1}}\right\},$$
&c. &c.,

therefore $$\frac{V_1^2}{n^2V^2} = \frac{1}{A_1}\,\frac{\Sigma\frac{1}{A_1} - \frac{1}{A_1}}{\Sigma\frac{1}{A_1}} = \frac{1}{p_1^2}, \text{ \&c.,}$$

therefore $$\frac{A_1}{p_1^2} + \frac{A_2}{p_2^2} + \frac{A_3}{p_3^2} + \ldots = n.$$

Again (I.),

$$V^2R^2 = \frac{-1}{(-2)^{n+1}(n!)^2}\begin{vmatrix} 0, & A_1+A_2, & A_1+A_3, & \ldots \\ A_1+A_2, & 0, & A_2+A_3, & \ldots \\ A_1+A_3, & A_2+A_3, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= \frac{1}{4(n!)^2} A_1A_2 \ldots A_{n+1}\left\{\Sigma A_1 . \Sigma\frac{1}{A_1} - (n-2)^2\right\},$$

therefore $$R^2 = \tfrac{1}{4}\,\frac{\Sigma A_1 . \Sigma\frac{1}{A_1} - (n-2)^2}{\Sigma\frac{1}{A_1}}.$$

It has been shown (XVI., Note) that a rectangular simplicissimum has, what no other has, a centre of perpendiculars, a polar spheric, the centre of which is at the orthocentre $(\lambda', \mu', \nu' \ldots)$, and a spheric which cuts each triangular face in its nine-point circle, and that these spherics and the circumspheric are coaxal.

The line of centres, the central line of the simplicissimum, passes through the centroid as well as the circumcentre $(\lambda'', \mu'', \nu'' \ldots)$ and the orthocentre $(\lambda', \mu', \nu' \ldots)$, for

$$\frac{dS''}{d\lambda''} = \frac{dS''}{d\mu''} = \frac{dS''}{d\nu''} = \ldots, \quad \text{by (X.)},$$

or $\quad A_1(V-2\lambda'') = A_2(V-2\mu'') = A_3(V-2\nu'') = \text{\&c.} = q''$, say,

therefore $$(n-1)\,V = q''\Sigma\frac{1}{A_1},$$

and $$A_1\lambda' = A_2\mu' = A_3\nu' = \text{\&c.} = q', \text{ say},$$

therefore $$V = q'\Sigma\frac{1}{A_1} \quad \text{and} \quad q'' = (n-1)\,q',$$

therefore $$V-2\lambda'' = (n-1)\,\lambda', \quad V-2\mu'' = (n-1)\,\mu',$$
$$V-2\nu'' = (n-1)\,\nu', \quad \text{\&c.},$$

and the centroid divides the line joining the orthocentre and the circumcentre in the ratio $n-1 : 2$ (*i.e.*, equally if $n = 3$); and, since the equations to this line are

$$\left\| \begin{matrix} \lambda, & \mu, & \nu, & \ldots \\ 1, & 1, & 1, & \ldots \\ \frac{1}{A_1}, & \frac{1}{A_2}, & \frac{1}{A_3}, & \ldots \end{matrix} \right\| = 0,$$

the point whose coordinates are as

$$\Sigma\frac{1}{A_1} - \frac{1}{A_1} : \Sigma\frac{1}{A_1} - \frac{1}{A_2} : \Sigma\frac{1}{A_1} - \frac{1}{A_3} : \text{\&c.},$$

i.e., as $$A_1V_1^2 : A_2V_2^2 : A_3V_3^2 : \text{\&c.},$$

that is, the isogonal conjugate of the orthocentre, lies upon it.

The common radical locus of these spherics is also that of any of them, and the spheric described through the centroids of all the p-ary simplicissima which can be formed so as to have each of their $p+1$ vertices coincident with a vertex of the simplicissimum. For at each such centroid $p+1$ of the coordinates will each be $\frac{V}{p+1}$, and the rest

zero, and if the equation to the spheric be

$$S-k\,(A_1\lambda+A_2\mu+\ldots+A_{p+1}\sigma+A_{p+2}\tau+\ldots)(\lambda+\mu+\nu\ldots)=0,$$

since it is satisfied, if

$$\lambda=\mu=\ldots=\sigma=\frac{V}{p+1},\quad \tau=\ldots=0,$$

$$p\,(A_1+A_2+\ldots+A_{p+1})\,\frac{V^2}{(p+1)^2}-k\,\frac{V^2}{p+1}\,(A_1+A_2+\ldots+A_{p+1})=0,$$

therefore $$k=\frac{p}{p+1},$$

and the equation is

$$S-\frac{p}{p+1}\,(A_1\lambda+A_2\mu+\ldots)(\lambda+\mu+\nu\ldots)=0.$$

This spheric also passes through the points where $p+1$ of the quan ties $A_1\lambda$, $A_2\mu$, &c. are equal and the rest zero, *i.e.*, through t centres of perpendiculars of the p-ary simplicissima.

The nine-point circle spheric is an example, when $p=1$. A when $p=n-1$, the spheric is that through the centroids of the fac which also passes through the feet of the perpendiculars upon th from the opposite vertices (that is, their orthocentres) and hence (XXVI.) through the projections of the point isogonally conjugate the orthocentre, and its centre is the mid-point of the connector these points. This spheric also divides the portions of the perp diculars intercepted between the vertices and the orthocentre in t ratio of $n-1:1$.

For let $$A_2\mu=A_3\nu=\ldots\left(=\frac{V-\lambda}{\Sigma\frac{1}{A_1}-\frac{1}{A_1}}\right).$$

(These are the equations to the perpendicular from $\lambda=\mu=\nu=\ldots=0$ upon $\lambda=0$.) Then, where this cuts the spheric,

$$S-\frac{p}{p+1}\,(A_1\lambda+A_2\mu+\ldots)(\lambda+\mu+\nu\ldots)=0,$$

$$A_1\lambda\left(\lambda-\frac{V}{p+1}\right)=\frac{V-\lambda}{\Sigma\frac{1}{A_1}-\frac{1}{A_1}}\left\{(n-1)\,V+\lambda-\frac{np}{p+1}\,V\right\},$$

and, if $p=n-1$, $$\lambda=0,$$

or $$\lambda=\frac{V}{n}+\frac{1}{A_1}\,\frac{V-\lambda}{\Sigma\frac{1}{A_1}-\frac{1}{A_1}}=\frac{V}{n}+(n-1)\frac{V}{A_1}\,\frac{1}{\Sigma\frac{1}{A_1}},$$

and
$$\mu = 0+(n-1)\frac{V}{A_2}\frac{1}{\Sigma\frac{1}{A_1}},$$

$$\nu = 0+(n-1)\frac{V}{A_3}\frac{1}{\Sigma\frac{1}{A_1}},$$

&c. &c.

and it is the circumspheric of the simplicissimum whose vertices are at the points of section of the perpendiculars, which is similar and similarly described to the original one.

The results of XIX. are all applicable.

The common radical locus of all these spherics is the locus of homology of the simplicissimum of reference, and the pedal simplicissimum (VIII.) For other interpretations see XXIII.

In space of two dimensions, this spheric, which is then identical with the nine-point circle spheric of this article, is the nine-point circle. In space of three dimensions, it is Professor Wolstenholme's second twelve-points sphere. See *Educational Times*, Quest. 3228 (*Reprint*, Vol. XLVIII., Appendix III.), to which I am indebted for the suggestion of much of this article.

At the centre of the spheric

$$S-\frac{p}{p+1}(A_1\lambda+A_2\mu\,...)(\lambda+\mu\,...)=0,$$

$$A_1\left\{\frac{V}{p+1}-2\lambda\right\}=A_2\left\{\frac{V}{p+1}-2\mu\right\}=A_3\left\{\frac{V}{p+1}-2\nu\right\}=\,...\,=k,$$

$$k=\frac{n-2p-1}{(p+1)\,\Sigma\frac{1}{A_1}}.$$

(And if $p=\frac{n-1}{2}$, and therefore n be odd, $k=0$, and the centre is the centroid; *e.g.*, if $n=3$ and $p=1$, the centre of the nine-points circle spheric, Professor Wolstenholme's first twelve-points sphere, is the centroid).

If these values of λ, μ, ν ... be substituted in the expression

$$S-\frac{p}{p+1}(A_1\lambda+A_2\mu+...)(\lambda+\mu+...),$$

the result will be $V^2\rho^2$, where ρ is the radius of the spheric.

In the case of a rectangular simplicissimum, the equation to the locus isogonally conjugate to infinity (XXVI.) becomes

$$\Sigma \frac{1}{A_1}\left(\frac{1}{A_1\lambda}+\frac{1}{A_2\mu}+\frac{1}{A_3\nu}+\dots\right)=\frac{1}{A_1^2\lambda}+\frac{1}{A_2^2\mu}+\frac{1}{A_3^2\nu}+\dots,$$

and if $(\lambda_1, \mu_1, \nu_1 \dots)$ be a point upon this, the equation to the linear locus, all lines from any point on which are divided as $p-1:1$ by the Simson locus of the point, becomes, since

$$\frac{dS}{d\lambda}=A_1(V-2\lambda)+A_1\lambda+A_2\mu+A_3\nu+\dots\equiv A_1(V-2\lambda)+P, \text{ say,}$$

$$\frac{dS}{d\mu}=A_2(V-2\mu)+A_1\lambda+A_2\mu+A_3\nu+\dots\equiv A_2(V-2\mu)+P, \text{ say,}$$

&c. &c.

$$\frac{1}{A_1\lambda_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_1}\right)\{2A_1(\lambda_1-\lambda)+P-P_1\}$$

$$+\frac{1}{A_2\mu_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_2}\right)\{2A_2(\mu_1-\mu)+P-P_1\}+\dots-2p\,\Sigma\frac{1}{A_1}=0,$$

or
$$2(n+1)\,\Sigma\frac{1}{A_1}-2\left(\frac{1}{A_1}+\frac{1}{A_2}+\dots\right)$$

$$-2\left\{\frac{\lambda}{\lambda_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_1}\right)+\frac{\mu}{\mu_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_2}\right)+\dots\right\}$$

$$+(P-P_1)\left\{\Sigma\frac{1}{A_1}\left(\frac{1}{A_1\lambda_1}+\frac{1}{A_2\mu_1}+\dots\right)-\left(\frac{1}{A^2\lambda_1}+\frac{1}{A_2^2\mu_1}+\dots\right)\right\}$$

$$-2p\,\Sigma\frac{1}{A_1}=0,$$

or, since $(\lambda_1, \mu_1, \nu_1 \dots)$ lies on the locus, isogonally conjugate to infinity, above,

$$(n-p)\,\Sigma\frac{1}{A_1}-\left\{\frac{\lambda}{\lambda_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_1}\right)+\frac{\mu}{\mu_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_2}\right)+\dots\right\}=0.$$

Now, if $(\lambda', \mu', \nu' \dots)$ be the orthocentre,

$$A_1\lambda'=A_2\mu'=\dots=\frac{V}{\Sigma\frac{1}{A_1}},$$

and $(n-p)\,\Sigma\,\frac{1}{A_1}-\left\{\frac{\lambda'}{\lambda_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_1}\right)+\frac{\mu'}{\mu_1}\left(\Sigma\frac{1}{A_1}-\frac{1}{A_2}\right)+\ldots\right\}$

$$\equiv (n-p)\,\Sigma\,\frac{1}{A_1}-\left\{\left(\frac{1}{A_1\lambda_1}+\frac{1}{A_2\mu_1}+\ldots\right)-\frac{1}{\Sigma\frac{1}{A_1}}\left(\frac{1}{A_1^2\lambda_1}+\frac{1}{A_2^2\mu_1}+\ldots\right)\right\}$$

$$\equiv (n-p)\,\Sigma\,\frac{1}{A_1},$$

and therefore vanishes if $n=p$. So that the Simson locus of $(\lambda_1, \mu_1 \ldots)$ divides the line joining $(\lambda_1, \mu_1 \ldots)$ to the orthocentre, as $n-1 : 1$, and therefore equally in plane geometry.

Of course the equation to the Simson locus is obtained by putting $p = 1$ in the equation to the dividing locus.

In order that the polar spheric

$$A_1\lambda^2+A_2\mu^2+A_3\nu^2+\ldots=0$$

may be real, it is necessary that one at least of the quantities

$$A_1, A_2, \ldots A_{n+1}$$

should be negative. Now,

$$A_1+A_2=(1.2)^2,\quad A_2+A_3=(2.3)^2,\quad A_3+A_1=(3.1)^2,\ \&c.,$$

and therefore $\quad 2A_1=(1.2)^2+(1.3)^2-(2.3)^2,$

or, with the notation of Art. IV.,

$$A_1=(1.2)(1.3)\cos a_{23},$$

and, generally, $\quad A_1=(1.r)(1.s)\cos a_{rs}.$

Therefore, if A_1 be negative, all the plane angles at the corresponding vertex are obtuse.

If $A_1=0$, each of these angles is a right angle.

In this case $\quad V^2=\frac{A_2.A_3\ldots A_{n+1}}{(n!)^2},$

$$V_1^2=\frac{A_2.A_3\ldots A_{n+1}}{\{(n-1)!\}^2}\left\{\frac{1}{A_2}+\frac{1}{A_3}+\ldots+\frac{1}{A_{n+1}}\right\}$$

$$=n^2V^2\left\{\frac{1}{A_2}+\frac{1}{A_3}+\ldots+\frac{1}{A_{n+1}}\right\},$$

$$V_2^2=\frac{A_3.A_4\ldots A_{n+1}}{\{(n-1)!\}^2}=n^2V^2\frac{1}{A_2},\ \&c.$$

therefore $\quad V_1^2=V_2^2+V_3^2+\ldots+V_{n+1}^2,$

a proposition which includes Euclid I. 47.

$$V^2 R^2 = \tfrac{1}{4} V^2 (A_1 + A_2 + \dots + A_{n+1}),$$

therefore $$R^2 = \tfrac{1}{4} (A_1 + A_2 + \dots + A_{n+1}).$$

In a rectangular simplicissimum, the equation to the spheric through the vertex opposite to $\lambda = 0$, and the feet of the perpendiculars from the opposite vertices on the faces which meet at that vertex, is $$S - (A_2\mu + A_3\nu + A_4\pi \dots)(\lambda + \mu + \nu \dots) = 0,$$
for this is satisfied at the vertex and at the points

$$\mu = 0, \quad A_1\lambda = A_3\nu = \dots; \qquad \nu = 0, \quad A_1\lambda = A_2\mu = \dots \text{ \&c.},$$

which are the feet of the perpendiculars.

The equations to the other spherics of this class are

$$S - (A_1\lambda + A_3\nu + A_4\pi \dots)(\lambda + \mu + \nu + \dots) = 0,$$
$$S - (A_1\lambda + A_2\mu + A_4\pi \dots)(\lambda + \mu + \nu + \dots) = 0,$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

and their radical centre is the orthocentre where

$$A_1\lambda = A_2\mu = A_3\nu = \dots;$$

and, as this lies upon them all, they meet in this common point, their centres being the mid-points* of its connectors with the vertices; and each of the spherics passes through the orthocentres of all the simplicissima, of any order less than n, which can be formed so as to have one vertex at the corresponding vertex, and the others at vertices of the original simplicissimum.

The radical loci of these spherics with the spheric through the feet of the perpendiculars on the faces

$$\left\{ S - \frac{n-1}{n} (A_1\lambda + A_2\mu + \dots)(\lambda + \mu + \dots) = 0 \right\},$$

are the faces of the polar simplicissimum, viz.,

[illegible]

[illegible]

&c. &c.

* See Proc., Vol. XXX [illegible] where there is a reference to these spherics, when regarded as those through a vertex and the feet of the perpendiculars upon the edges through it from the orthocentre.

XXVIII. If there be a system of $n+1$ mutually orthotomic spherics, their centres may be taken as the vertices of the simplicissimum of reference; then, if $r_1, r_2 \dots r_{n+1}$ be their radii,

$$r_p^2+r_q^2 = (p \cdot q)^2,$$

and the simplicissimum will be rectangular (XXVII.)

The equations to the spherics are

$$\left.\begin{aligned}\Sigma_1 &\equiv S-\frac{dS}{d\lambda}(\lambda+\mu+\nu+\dots)+r_1^2(\lambda+\mu+\nu\dots)^2=0\\ \Sigma_2 &\equiv S-\frac{dS}{d\mu}(\lambda+\mu+\nu+\dots)+r_2^2(\lambda+\mu+\nu\dots)^2=0\\ &\qquad\&c. \qquad\qquad\qquad \&c.\end{aligned}\right\} (A).$$

and if $(\lambda', \mu', \nu' \dots)$ be the centre, and r the radius of the spheric which cuts each of these orthogonally, its centre will be at the radical centre of the system (A);

i.e., $$\frac{dS'}{d\lambda'}-r_1^2(\lambda'+\mu'+\nu'\dots)=\frac{dS'}{d\mu'}-r_2^2(\lambda'+\mu'+\nu'\dots)=\&c.,$$

or, since $$S\equiv(r_1^2+r_2^2)\,\lambda\mu+(r_2^2+r_3^2)\,\mu\nu+(r_3^2+r_1^2)\,\nu\lambda+\&c.,$$

$$r_1^2\lambda' = r_2^2\mu' = r_3^2\nu' = \dots = k' \text{ say},$$

and this point is the orthocentre of the simplicissimum. Since the spheric centre $(\lambda', \mu', \nu' \dots)$ radius r cuts each of the spherics (A) orthogonally,

$$r^2+r_1^2=\frac{1}{V^2}\left(V\frac{dS'}{d\lambda'}-S'\right),\quad r^2+r_2^2=\frac{1}{V^2}\left(V\frac{dS'}{d\mu'}-S'\right),\ \&c.;$$

and $$\frac{dS'}{d\lambda}=r_1^2(\mu'+\nu'+\dots)+r_2^2\mu'+r_3^2\nu'+\dots=r_1^2V+\overline{n-1}\,k',$$

$$\frac{dS'}{d\mu'}=r_2^2V+\overline{n-1}\,k',\ \&c.,$$

and $$2S'=\lambda'\frac{dS'}{d\lambda'}+\mu'\frac{dS'}{d\mu'}+\dots=(n+1)\,Vk'+(n-1)\,Vk'=2n\,Vk',$$

and $$S'=n\,Vk';$$

also $$V=k'\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\dots+\frac{1}{r_{n+1}^2}\right),$$

therefore $$r^2+r_1^2=\frac{1}{V^2}\left\{r_1^2V^2+\overline{n-1}\,Vk'-n\,Vk'\right\}=r_1^2-\frac{k'}{V},$$

therefore $\frac{1}{r^2}+\frac{V}{k'}=0$, or $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots+\frac{1}{r_{n+1}^2}=0$;

and the equation to the orthotomic spheric is

$$\Sigma \equiv S-\left(\lambda\frac{dS'}{d\lambda'}+\mu\frac{dS'}{d\mu'}+\ldots\right)\frac{\lambda+\mu+\nu\ldots}{V}+\frac{V^2r^2+S'}{V^2}(\lambda+\mu+\nu\ldots)^2$$

$$\equiv S-(r_1^2\lambda+r_2^2\mu+r_3^2\nu+\ldots)(\lambda+\mu+\nu\ldots)$$

$$+\frac{V^2r^2+S'-\overline{n-1}k'V}{V^2}(\lambda+\mu+\ldots)^2$$

$$\equiv -(r_1^2\lambda^2+r_2^2\mu^2+r_3^2\nu^2\ldots)=0.$$

so that this spheric is the polar spheric of the simplicissimum (XVI. Note). It also appears that, in a system of $n+2$ mutually orthotomic spherics, one at least must be imaginary.

Putting Σ_1, Σ_2, ... &c. for the sinisters of the equations (A), so that

$$\Sigma_1 \equiv S-V\frac{dS}{d\lambda}+r_1^2V, \text{ \&c.},$$

$$\left(\frac{\Sigma_1}{r_1}\right)^2+\left(\frac{\Sigma_2}{r_2}\right)^2+\left(\frac{\Sigma_3}{r_3}\right)^2+\ldots+\left(\frac{\Sigma_{n+1}}{r_{n+1}}\right)^2$$

$$\equiv S^2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots+\frac{1}{r_{n+1}^2}\right)-2VS\left\{\frac{1}{r_1^2}\frac{dS}{d\lambda}+\frac{1}{r_2^2}\frac{dS}{d\mu}+\ldots\right\}$$

$$+V^2\left\{\frac{1}{r_1^2}\left(\frac{dS}{d\lambda}\right)^2+\frac{1}{r_2^2}\left(\frac{dS}{d\mu}\right)^2+\ldots\right\}+2V^2S\{1+1+\ldots\}$$

$$-2V^3\left\{\frac{dS}{d\lambda}+\frac{dS}{d\mu}+\ldots\right\}+V^4\{r_1^2+r_2^2+\ldots\},$$

and
$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots\equiv-\frac{1}{r^2},$$

$$\frac{1}{r_1^2}\frac{dS}{d\lambda}+\frac{1}{r_2^2}\frac{dS}{d\mu}+\ldots\equiv V-2\lambda+V-2\mu+\ldots$$

$$+\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots\right)(r_1^2\lambda+r_2^2\mu+\ldots)$$

$$\equiv(n-1)V-\frac{1}{r^2}(r_1^2\lambda+r_2^2\mu+\ldots),$$

$$\frac{1}{r_1^2}\left(\frac{dS}{d\lambda}\right)^2+\frac{1}{r_2^2}\left(\frac{dS}{d\mu}\right)^2+\ldots\equiv r_1^2(V-2\lambda)^2+r_2^2(V-2\mu)^2+\ldots$$

$$+2\left\{V-2\lambda+V-2\mu+\ldots\right\}\left\{r_1^2\lambda+r_2^2\mu+\ldots\right\}$$

$$+\left\{\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots\right\}\left\{r_1^2\lambda+r_2^2\mu+\ldots\right\}^2$$

$$\equiv\left(r_1^2+r_2^2+\ldots\right)V^2+2\left(n-3\right)V\left\{r_1^2\lambda+r_2^2\mu+\ldots\right\}$$

$$+4\left(r_1^2\lambda^2+r_2^2\mu^2+\ldots\right)-\frac{1}{r^2}\left\{r_1^2\lambda+r_2^2\mu+\ldots\right\}^2,$$

$$\frac{dS}{d\lambda}+\frac{dS}{d\mu}+\ldots\equiv\left(r_1^2+r_2^2+\ldots\right)V+\left(n-1\right)\left\{r_1^2\lambda+r_2^2\mu+\ldots\right\},$$

therefore $$\left(\frac{\Sigma_1}{r_1}\right)^2+\left(\frac{\Sigma_2}{r_2}\right)^2+\left(\frac{\Sigma_3}{r_3}\right)^2+\ldots+\left(\frac{\Sigma_{n+1}}{r_{n+1}}\right)^2$$

$$\equiv-\frac{S^2}{r^2}-2\left(n-1\right)V^2S+\frac{2}{r^2}VS\left(r_1^2\lambda+r_2^2\mu+\ldots\right)$$

$$+\left(r_1^2+r_2^2+\ldots\right)V^4+2\left(n-3\right)V^3\left(r_1^2\lambda+r_2^2\mu+\ldots\right)$$

$$+4V^2\left(r_1^2\lambda^2+r_2^2\mu^2+\ldots\right)-\frac{V^2}{r^2}\left(r_1^2\lambda+r_2^2\mu+\ldots\right)^2$$

$$+2\left(n+1\right)V^2S-2\left(r_1^2+r_2^2+\ldots\right)V^4$$

$$-2\left(n-1\right)V^3\left(r_1^2\lambda+r_2^2\mu+\ldots\right)+V^4\left(r_1^2+r_2^2+\ldots\right)$$

$$\equiv-\frac{S^2-2VS\left(r_1^2\lambda+r_2^2\mu+\ldots\right)+V^2\left(r_1^2\lambda+r_2^2\mu+\ldots\right)^2}{r^2}$$

$$+4V^2\left\{S-V\left(r_1^2\lambda+r_2^2\mu+\ldots\right)+\left(r_1^2\lambda^2+r_2^2\mu^2+\ldots\right)\right\}$$

$$\equiv-\left(\frac{\Sigma}{r}\right)^2,\text{ since }S-V\left(r_1^2\lambda+r_2^2\mu+\ldots\right)+\left(r_1^2\lambda^2+r_2^2\mu^2+\ldots\right)\equiv 0,$$

and $$\left(\frac{\Sigma}{r}\right)^2+\left(\frac{\Sigma_1}{r_1}\right)^2+\left(\frac{\Sigma_2}{r_2}\right)^2+\ldots+\left(\frac{\Sigma_{n+1}}{r_{n+1}}\right)^2\equiv 0.$$

The radical locus of the spherics $\Sigma_1=0$ and $\Sigma_2=0$ is $r_1^2\lambda=r_2^2\mu$, and the spheric to which their common section is diametral is

$$a\Sigma_1+b\Sigma_2=0,$$

when $a:b$ is determined by the condition that the centre lies on $r_1^2\lambda=r_2^2\mu$, when

$$a:b::\frac{1}{r_1^2}:\frac{1}{r_2^2},$$

and the spheric is

$$\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)S-\left(\frac{1}{r_1^2}\frac{dS}{d\lambda}+\frac{1}{r_2^2}\frac{dS}{d\mu}\right)\left(\lambda+\mu+\ldots\right)$$

$$+2\left(\lambda+\mu+\ldots\right)^2=0.$$

Similarly the equation to the spheric, which passes through the intersection of $\Sigma_2 = 0$ and $\Sigma_3 = 0$ and has its centre on their radical locus, is

$$\left(\frac{1}{r_2^2}+\frac{1}{r_3^2}\right) S-\left(\frac{1}{r_2^2}\frac{dS}{d\mu}+\frac{1}{r_3^2}\frac{dS}{d\nu}\right)(\lambda+\mu+\ldots)$$
$$+2(\lambda+\mu+\ldots)^2=0,$$

and so on. All these spherics have a common radical centre, which will be found to be at the point

$$\lambda:\mu:\nu:\ldots::\frac{1}{r_1^2}:\frac{1}{r_2^2}:\frac{1}{r_3^2}:\ldots,$$

the radical centre of the original spherics.

This article is a proof and generalisation of the late Professor Clifford's Questions 1748 and 1585, in the *Educational Times* (the latter as corrected by Mr. J. J. Walker, Question 7605).

XXIX. The centre of the spheric bisecting $n+1$ given spherics, whose radii are $r_1, r_2 \ldots$, is the radical centre of $n+1$ imaginary spherics, concentric with the given ones, but whose radii are $ir_1, ir_2 \ldots$ respectively, and is situated on the right line connecting the centre of the spheric through the given centres and the radical centre of the given spherics, the linear segment between the radical centres being bisected by the centre of the spheric through the given centres. This was enunciated, for circles and spheres, by Mr. S. Roberts, as Quest. 8572 in the *Educational Times*.

Let $(\lambda_1, \mu_1, \nu_1 \ldots)$, $(\lambda_2, \mu_2, \nu_2 \ldots)$, &c. be the centres of the given spherics; $(\lambda', \mu', \nu' \ldots)$ that of the bisecting spheric, r' its radius; $(\lambda'', \mu'', \nu'' \ldots)$ the centre, and r'' the radius of the spheric through the given centres; and $(\lambda''', \mu''', \nu''' \ldots)$ the radical centre of the given spherics.

The equations to these are (IX.)

$$S-\left(\lambda_1\frac{dS}{d\lambda}+\mu_1\frac{dS}{d\mu}\ldots\right)\frac{\lambda+\mu\ldots}{V}+\frac{V^2r_1^2+S_1}{V^2}(\lambda+\mu\ldots)^2=0,$$

$$S-\left(\lambda_2\frac{dS}{d\lambda}+\mu_2\frac{dS}{d\mu}\ldots\right)\frac{\lambda+\mu\ldots}{V}+\frac{V^2r_2^2+S_2}{V^2}(\lambda+\mu\ldots)^2=0,$$

&c., &c.,

and that to the bisecting spheric is

$$S-\left(\lambda'\frac{dS}{d\lambda}+\mu'\frac{dS}{d\mu}\ldots\right)\frac{\lambda+\mu+\ldots}{V}+\frac{V^2r'^2+S'}{V^2}(\lambda+\mu\ldots)^2=0.$$

Now, its radical locus with each of the given spherics must pass through the centre of that spheric, therefore

$$(\lambda'-\lambda_1)\frac{dS_1}{d\lambda_1}+(\mu'-\mu_1)\frac{dS_1}{d\mu_1}+\ldots-\left\{V^2(r'^2-r_1^2)+S'-S_1\right\}=0,$$

$$(\lambda'-\lambda_2)\frac{dS_2}{d\lambda_2}+(\mu'-\mu_2)\frac{dS_2}{d\mu_2}+\ldots-\left\{V^2(r'^2-r_2^2)+S'-S_2\right\}=0,$$

&c., &c.,

or

$$\left.\begin{aligned}\lambda'\frac{dS_1}{d\lambda_1}+\mu'\frac{dS_1}{d\mu_1}+\ldots-S_1-S'&=V^2(r'^2-r_1^2)\\ \lambda'\frac{dS_2}{d\lambda_2}+\mu'\frac{dS_2}{d\mu_2}+\ldots-S_2-S'&=V^2(r'^2-r_2^2)\\ \&c.,\qquad \&c.,&\end{aligned}\right\}\ldots\ldots\ldots\ldots(A)$$

and the spheric, centre $(\lambda', \mu', \nu'\ldots)$ and radius r', cuts the $n+1$ imaginary spherics, which have their centres at the given centres and their radii $ir_1, ir_2 \ldots$, orthogonally.

Again, the equation to the spheric through the centres is

$$S-\left(\lambda''\frac{dS}{d\lambda}+\mu''\frac{dS}{d\mu}+\ldots\right)\frac{\lambda+\mu+\ldots}{V}+\frac{V^2r''^2+S''}{V^2}(\lambda+\mu+\ldots)^2=0,$$

therefore

$$\left.\begin{aligned}S_1-\left(\lambda''\frac{dS_1}{d\lambda_1}+\mu''\frac{dS_1}{d\mu_1}+\ldots\right)+V^2r''^2+S''&=0\\ S_2-\left(\lambda''\frac{dS_2}{d\lambda_2}+\mu''\frac{dS_2}{d\mu_2}+\ldots\right)+V^2r''^2+S''&=0\\ \&c.,\qquad \&c.&\end{aligned}\right\}\ldots\ldots\ldots\ldots(B),$$

and at the radical centre of the given spherics

$$\left.\begin{aligned}&\lambda'''\frac{dS_1}{d\lambda_1}+\mu'''\frac{dS_1}{d\mu_1}+\ldots-V^2r_1^2-S_1\\ &=\lambda'''\frac{dS_2}{d\lambda_2}+\mu'''\frac{dS_2}{d\mu_2}+\ldots-V^2r_2^2-S_2\\ &=\&c.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(C);$$

therefore, from the equations (A) and (C),

$$\frac{\lambda'+\lambda'''}{2}\cdot\frac{dS_1}{d\lambda_1}+\frac{\mu'+\mu'''}{2}\cdot\frac{dS_1}{d\mu_1}+\ldots-S_1$$

$$=\frac{\lambda'+\lambda'''}{2}\cdot\frac{dS_2}{d\lambda_2}+\frac{\mu'+\mu'''}{2}\cdot\frac{dS_2}{d\mu_2}+\ldots-S_2=\&c.,$$

and comparing these with (B), since n equations determine a point, appears that

$$\frac{\lambda'+\lambda'''}{2}=\lambda'',\quad \frac{\mu'+\mu'''}{2}=\mu'', \text{ \&c.},$$

and $(\lambda'', \mu'', \nu''...)$ is the mid-point of the line joining $(\lambda''', \mu''', \nu'''...$ and $(\lambda', \mu', \nu'...)$.

If r''' be the radius of the orthogonal spheric,

$$V^2(r'''^2+r_1^2)=\lambda'''\frac{dS_1}{d\lambda_1}+\mu'''\frac{dS_1}{d\mu_1}+..-S'''-S_1,$$

and $S'''+V^2r'''^2$ equals each member of (C).

Now
$$S'''=4S''-2\left(\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+...\right)+S',$$

and

$$S'''+V^2r'''^2=2\left(\lambda''\frac{dS_1}{d\lambda_1}+\mu''\frac{dS_1}{d\mu_1}...\right)-\left(\lambda'\frac{dS_1}{d\lambda_1}+\mu'\frac{dS_1}{d\mu_1}...\right)-V^2r_1^2-$$

$$=V^2(2r''^2-r'^2)+2S''-S',$$

therefore

$$V^2\{r'''^2+r'^2-2r''^2\}=2\left(\lambda''\frac{dS'}{d\lambda'}+\mu''\frac{dS'}{d\mu'}+...\right)-2S''+2S'$$

$$=2V^2d^2,$$

if d denote the distance from $(\lambda', \mu', \nu'...)$ to $(\lambda'', \mu'', \nu''...)$.

XXX. If
$$A_{11}\lambda^2+A_{22}\mu^2+...+2A_{12}\lambda\mu+...=0$$

be a quadratic locus in space of n dimensions, the problem, to find th pole of the linear locus

$$B_1\lambda+B_2\mu+B_3\nu+...=0,$$

depends on the solution of the equations

$$\lambda+\mu+\nu+...-V=0,$$

$$A_{11}\lambda+A_{12}\mu+A_{13}\nu+...-B_1r=0,$$

$$A_{12}\lambda+A_{22}\mu+A_{23}\nu+...-B_2r=0,$$

$$\text{\&c.,}\qquad \text{\&c.};$$

so that r is determined by the equation

$$\begin{vmatrix} 1, & 1, & 1, & \ldots\ldots \dfrac{V}{r} \\ A_{11}, & A_{12}, & A_{13}, & \ldots\ldots B_1 \\ A_{21}, & A_{22}, & A_{23}, & \ldots\ldots B_2 \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

and, if $B_1\lambda + B_2\mu + B_3\nu + \ldots = 0$ be infinity,

$$\begin{vmatrix} \dfrac{V}{r}, & 1, & 1, & 1, \ldots\ldots \\ 1, & A_{11}, & A_{12}, & A_{13}, \ldots\ldots \\ 1, & A_{21}, & A_{22}, & A_{23}, \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

or
$$\frac{V}{r}\begin{vmatrix} A_{11}, & A_{12}, & A_{13}, \ldots\ldots \\ A_{21}, & A_{22}, & A_{23}, \ldots\ldots \\ A_{31}, & A_{32}, & A_{33}, \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} + \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots\ldots \\ 1, & A_{11}, & A_{12}, & A_{13}, & \ldots\ldots \\ 1, & A_{21}, & A_{22}, & A_{23}, & \ldots\ldots \\ 1, & A_{31}, & A_{32}, & A_{33}, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

and the centre will be indeterminate if both the determinants vanish.

If the centre be at infinity, V in the first equation must be omitted,

and
$$\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots\ldots \\ 1, & A_{11}, & A_{12}, & A_{13}, & \ldots\ldots \\ 1, & A_{21}, & A_{22}, & A_{23}, & \ldots\ldots \\ 1, & A_{31}, & A_{32}, & A_{33}, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

If the quadric touch a face, say $\lambda = 0$,

$$\begin{vmatrix} A_{22}, & A_{23}, & A_{24}, \ldots\ldots \\ A_{32}, & A_{33}, & A_{34}, \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

and, if it touch all the faces, all the principal minors of the discriminant are zero.

The condition that the locus should touch

$$B_1\lambda+B_2\mu+B_3\nu+\ldots=0,$$

is

$$\begin{vmatrix} 0, & B_1, & B_2, & B_3, & \ldots\ldots \\ B_1, & A_{11}, & A_{12}, & A_{13}, & \ldots\ldots \\ B_2, & A_{21}, & A_{22}, & A_{23}, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}=0,$$

and that it should touch infinity,

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots\ldots \\ 1, & A_{11}, & A_{12}, & A_{13}, & \ldots\ldots \\ 1, & A_{21}, & A_{22}, & A_{23}, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}=0,$$

the condition that the centre should be at infinity. The equations

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}\ldots=\frac{Vp}{\sqrt{\{-S\,(abc\ldots)\}}}=mp \text{ say,}$$

(where $a+b+c+\ldots=0$) p being the distance from $(\lambda, \mu, \nu\ldots)$ $(\lambda', \mu', \nu'\ldots)$, represent any line through $(\lambda', \mu', \nu'\ldots)$. (XVIII.)

Now, if $\lambda'+amp,\quad \mu'+bmp,\quad \nu'+cmp,$ &c.

be substituted for λ, μ, ν in

$$U\equiv A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu\ldots,$$

$$U\equiv U'+mp\left\{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+\ldots\right\}$$

$$+m^2p^2\left\{A_{11}a^2+A_{22}b^2+\ldots+2A_{12}ab+\ldots\right\},$$

and the points where this line cuts $U=0$ are determined by putti[ng] this quantity equal to zero.

There are, therefore, in general, two values of p, *i.e.*, the line cu[ts] the locus in two points;

if $$a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu}+\ldots=0,$$

the two values are equal in magnitude and opposite in sign. [But, [if] U' also vanish, they are both zero, and the line lies on the locus

$$\lambda\frac{dU'}{d\lambda}+\mu\frac{dU'}{d\mu'}+\ldots=0,$$

the polar locus of $(\lambda', \mu', \nu' \ldots)$, which is therefore the linear tangent locus at the point].* Therefore

$$a\frac{dU}{d\lambda}+b\frac{dU}{d\mu}+c\frac{dU}{d\nu}+\ldots=0$$

is a diametral locus bisecting all chords parallel to

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=\ldots;$$

and if $(\lambda'', \mu'', \nu'' \ldots)$ be the centre,

$$\frac{dU''}{d\lambda''}=\frac{dU''}{d\mu''}=\frac{dU''}{d\nu''}=\ldots,$$

and therefore, since $\quad a+b+c+\ldots=0,$

$$a\frac{dU''}{d\lambda''}+b\frac{dU''}{d\mu''}+c\frac{dU''}{d\nu''}+\ldots\equiv 0,$$

consequently all diametral loci pass through the centre, and all chords through the centre are bisected there. The same equation shows that the rectangles under the segments of parallel lines through $(\lambda', \mu', \nu' \ldots)$ and $(\lambda'', \mu'', \nu'' \ldots)$ are as $U' : U''$.

If $U=0$ be a spheric,

$$\frac{A_{11}+A_{22}-2A_{12}}{(1.2)^2}=\frac{A_{22}+A_{33}-2A_{23}}{(2.3)^2}=\ldots=h \text{ say},$$

---

* If $(\lambda', \mu', \nu' \ldots)$ be not upon the locus, the values of p will be equal, and the straight line a linear tangent to the locus, if

$$4U'\left\{A_{11}a^2+A_{22}b^2+\ldots+2A_{12}ab+\ldots\right\}=\left\{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+\ldots\right\}^2,$$

or $$4U'\left\{U-\lambda\frac{dU'}{d\lambda'}-\mu\frac{dU'}{d\mu'}\ldots+U'\right\}=\left\{\lambda\frac{dU'}{d\lambda'}+\mu\frac{dU'}{d\mu'}+\ldots-2U'\right\}^2,$$

i.e., if $$4UU'=\left\{\lambda\frac{dU'}{d\lambda'}+\mu\frac{dU'}{d\mu'}+\ldots\right\}^2,$$

which is therefore the equation to the group of straight lines which can be drawn from $(\lambda', \mu', \nu' \ldots)$ to touch the locus.

If the point be the centre $(\lambda'', \mu'' \ldots)$,

$$\frac{dU''}{d\lambda''}=\frac{dU''}{d\mu''}=\ldots=2r,$$

and therefore $\quad 2U''=2Vr,$

and the locus composed of enveloping straight lines becomes

$$U=\frac{U''}{V^2}(\lambda+\mu\ldots)^2,$$

the asymptotic locus which touches $U=0$ at infinity.

therefore

$$A_{11}a^2+A_{22}b^2+\ldots+2A_{12}ab+\ldots = A_{11}a\,(a+b+c\ldots)+A_{22}b\,(a+b+c\ldots)$$
$$+\&c.-h\left\{ab\,(1.2)^2+ac\,(1.3)^2+\ldots\right\} \equiv \frac{h}{m^2};$$

so that the equation in p becomes

$$U'+mp\left\{a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+\ldots\right\}+hp^2=0,$$

and the rectangle under the segments of a line through $(\lambda', \mu', \nu' \ldots)$ is independent of the direction of the line.

The equation above to the linear locus diametral to chords parallel to

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\&c.,$$

$$a\frac{dU}{d\lambda}+b\frac{dU}{d\mu}+c\frac{dU}{d\nu}+\ldots=0,$$

may be written

$$\left(\lambda\frac{d}{da}+\mu\frac{d}{db}+\nu\frac{d}{dc}+\ldots\right)U(a, b, c\ldots)=0;$$

and if this be at right angles to the chords it is a principal diametral locus, and the parallel to the chords through the centre is a principal axis; this requires (XVIII.) that

$$\frac{\left(\frac{d}{db}-\frac{d}{da}\right)U(a, b, c\ldots)}{\left(\frac{d}{db}-\frac{d}{da}\right)S(a, b, c\ldots)}=\frac{\left(\frac{d}{dc}-\frac{d}{da}\right)U(a, b, c\ldots)}{\left(\frac{d}{dc}-\frac{d}{da}\right)S(a, b, c\ldots)}=\&c. \quad \text{(I.)},$$

equations which are always satisfied when

$$U \equiv S-(\lambda+\mu+\ldots)(A_1\lambda+A_2\mu+\ldots),$$

i.e., when $U=0$ is a spheric, and satisfied for the same values of a, b, &c., for all quadrics of the system which have a common intersection with infinity, *i.e.*, $U=0$ and any quadric

$$U-(\lambda+\mu+\ldots)(A_1\lambda+A_2\mu+\ldots)=0;$$

have their axes parallel.

Also, any quadric will in general have n principal axes; for, putting each member of (I.) equal to p, and eliminating a, b, &c. between the n resulting equations and

$$a+b+c\ldots=0,$$

the resulting equation in p will be of order n, and each value of p gives a set of principal axes.

XXXI. If any number of quadratic loci have two sections by linear loci common (*i.e.*, be all of the form $U-pA\,.\,B=0$), the polars of any point with respect to them all pass through the intersection of two linear loci.

Let $U=0$, and

$$U-p\,(a\lambda+b\mu+\ldots)(A\lambda+B\mu+\ldots)=0,$$

be two of the loci, so that they have a common intersection with the linear loci

$$a\lambda+b\mu+c\nu+\ldots=0,$$

$$A\lambda+B\mu+C\nu+\ldots=0,$$

then the polars of any point $(\lambda', \mu', \nu' \ldots)$ with respect to them are

$$\lambda\frac{dU'}{d\lambda'}+\mu\frac{dU'}{d\mu'}+\nu\frac{dU'}{d\nu'}+\ldots=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\lambda\left\{\frac{dU'}{d\lambda'}-pa\,(A\lambda'+B\mu'+\ldots)-pA\,(a\lambda'+b\mu'+\ldots)\right\}$$

$$+\mu\left\{\frac{dU'}{d\mu'}-pb\,(A\lambda'+B\mu'+\ldots)-pB\,(a\lambda'+b\mu'+\ldots)\right\}$$

$$+\&c.=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and these will for all values of p pass through the intersection of (1), and

$$(a\lambda'+b\mu'+c\nu'\ldots)(A\lambda+B\mu+C\nu+\ldots)$$

$$+(A\lambda'+B\mu'+C\nu'+\ldots)(a\lambda+b\mu+c\nu\ldots)=0 \quad\ldots\ldots\ldots\ldots(3)$$

If the loci (1) and (3) be identical, the polar of $(\lambda', \mu', \nu' \ldots)$ is the same for all quadrics of the system. This involves

$$\frac{dU'}{d\lambda'}-ra\,(A\lambda'+B\mu'+\ldots)-rA\,(a\lambda'+b\mu'+\ldots)=0,$$

$$\frac{dU'}{d\mu'}-rb\,(A\lambda'+B\mu'+\ldots)-rB\,(a\lambda'+b\mu'+\ldots)=0,$$

&c. &c.

and if $$U\equiv A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu+\ldots,$$

the equation to determine r is

$$\begin{vmatrix} A_{11}-2rAa, & A_{12}-r(Ab+Ba), & A_{13}-r(Ac+Ca) & \dots \\ A_{21}-r(Ab+Ba), & A_{22}-2rBb, & A_{23}-r(Bc+Cb) & \dots \\ A_{31}-r(Ac+Ca), & A_{32}-r(Bc+Cb), & A_{33}-2rCc & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} =$$

which is, in fact, a quadratic in r, as the coefficients of all powers r above the second vanish identically, and there are therefore t such points.

If r' and r'' be the values of r, and $(\lambda', \mu', \nu' \dots)$, $(\lambda'', \mu'', \nu'' .$ the corresponding points,

$$\lambda' \frac{dU''}{d\lambda''} + \mu' \frac{dU''}{d\mu''} + \nu' \frac{dU''}{d\nu''} + \dots$$

$$= r' \{(a\lambda'+b\mu'+\dots)(A\lambda''+B\mu''+\dots)+(a\lambda''+b\mu''\dots)(A\lambda'+B\mu'\dots)$$

$$= \lambda'' \frac{dU'}{d\lambda'} + \mu'' \frac{dU'}{d\mu'} + \nu'' \frac{dU'}{d\nu'}$$

$$= r''\{a\lambda''+b\mu''+\dots)(A\lambda'+B\mu'+\dots)+(a\lambda'+b\mu'+\dots)(A\lambda''+B\mu''\dots$$

equations which involve

$$\lambda' \frac{dU''}{d\lambda''} + \mu' \frac{dU''}{d\mu''} + \dots = 0,$$

since r' is not in general $= r''$, and each of the points lies on the pol of the other.

Also, all points on the intersection of

$$a\lambda+b\mu+\dots = 0 \quad \text{and} \quad A\lambda+B\mu+\dots = 0$$

will evidently have the same polars with respect to all the quadrics.

If the quadratic loci be spherics, they will form a coaxal syste one of the linear loci being the common radical locus and the oth infinity. The middle point of the line joining the points, found above, will lie upon the radical locus, for in this case $a\lambda+b\mu+\dots =$ is the radical locus, and

$$A = B = \dots = 1,$$

therefore

$$(a\lambda'+b\mu'+\dots)(A\lambda''+B\mu''+\dots)+(a\lambda''+b\mu''+\dots)(A\lambda'+B\mu'+\dots) =$$

becomes $\qquad a(\lambda'+\lambda'')+b(\mu'+\mu'')+\dots = 0,$

and the two points so determined will be the limiting points (poi spherics) of the system, for the polar of every point with respect t

each of them will pass through it, and as in Plane Geometry all spherics which cut the system orthogonally will pass through the limiting points.

This is to a great extent a generalisation of articles *110—*112 in Salmon's conics.

XXXII. If

$$\Sigma \equiv S-\left(\lambda \frac{dS'}{d\lambda'}+\mu \frac{dS'}{d\mu'}+\dots\right) \frac{\lambda+\mu+\nu+\dots}{V}+S'\left(\frac{\lambda+\mu+\nu\dots}{V}\right)^2=0,$$

so that $\Sigma = 0$ is the point spheric at $(\lambda', \mu', \nu'\dots)$, the value of Σ will be $-V^2d^2$, where d is the distance between the points $(\lambda, \mu, \nu\dots)$ and $(\lambda', \mu', \nu'\dots)$. (Addendum at the beginning of this paper.)

Now, the equation to a quadratic locus $(U=0)$ in space of n dimensions involves $\frac{n(n+3)}{2}$ independent constants.

[The same number as is involved in the equation to the n-ic locus in space of two dimensions, and generally the number of independent constants in the equation to an m-ic locus in space of n dimensions, is equal to that in the equation to an n-ic locus in space of m dimensions, viz., $(n+m)!\div(n!\,m!)-1$.]

If, then, U be put equal to $\Sigma+A^2+B^2+$&c., where

$$A \equiv A_1\lambda+A_2\mu+A_3\nu+\dots,$$
$$B \equiv B_1\lambda+B_2\mu+B_3\nu+\dots,$$
$$\text{\&c.} \qquad \text{\&c.},$$

and the quantities A, B, &c. be $\frac{n}{2}$ or $\frac{n+1}{2}$ in number, according as n is even or odd, the proposed form will involve

$$n+\frac{n(n+1)}{2}=\frac{n(n+3)}{2} \quad \text{or} \quad n+\frac{(n+1)^2}{2}=\frac{n^2+4n+1}{2}$$

independent constants according as n is even or odd, and it will therefore be a form to which the equation to the quadric can in general be reduced.

In the first case, when n is even, the constants will all be determined, and if the point $(\lambda', \mu', \nu'\dots)$ be called a focus, there will be a determinate number of definite foci, such that the square of the radius vector from the focus to any point on the locus is a linear function of the squares of the perpendicular distances of that point from $\frac{n}{2}$ determinate linear loci (or directrices).

In the other case, when n is odd, the proposed form contai $\frac{n+1}{2}$ too many constants. Now, from the $\frac{n(n+3)}{2}$ independent equa tions which express the identity of U and $\Sigma+A^2+B^2+$&c., tl $\frac{(n+1)^2}{2}$ quantities $A_1, A_2 \ldots, B_1, B_2 \ldots$; &c., may be eliminated; an this will leave $\frac{n-1}{2}$ equations among the remaining constan $(\lambda', \mu', \nu' \ldots)$, the coordinates of the focus, which will therefore l upon a determinate locus; and for each position of the focus the will be determinate directrices, such that the squared distance of an point on $U=0$ from the focus is a linear function of the squares the perpendicular distances of that point from the correspondi directrices.

If a point $(\lambda'', \mu'', \nu'' \ldots)$ lie upon the intersection of the directr loci, so that $A''=0$, $B''=0$, &c., the polar of that point, which in general

$$\lambda \frac{d\Sigma''}{d\lambda''}+\mu \frac{d\Sigma''}{d\mu''}+\ldots+2AA''+2BB''+\ldots=0,$$

is identical with $$\lambda \frac{d\Sigma''}{d\lambda''}+\mu \frac{d\Sigma''}{d\mu''}+\ldots=0,$$

its polar with respect to the point spheric at the focus, and is ther fore at right angles to the line joining $(\lambda'', \mu'' \ldots)$ to the foc $(\lambda', \mu' \ldots)$, and the tangents from any point on the intersection of tl directrix loci subtend a right angle at the focus.

Also the equation to the group of straight lines drawn fro $(\lambda'', \mu'' \ldots)$ to touch the quadric is

$$(\Sigma+A^2+B^2+\ldots)(\Sigma''+A''^2+B''^2+\ldots)$$
$$=\left(\lambda \frac{d\Sigma''}{d\lambda''}+\mu \frac{d\Sigma''}{d\mu''}+\ldots+2AA''+2BB''+\ldots\right)^2,$$

and, since $\Sigma'=0$, and

$$\frac{d\Sigma'}{d\lambda'}=\frac{d\Sigma'}{d\mu'}=\ldots=0,$$

the equation to the group of straight lines drawn to touch the locu from $(\lambda', \mu' \ldots)$, the focus is

$$(A'^2+B'^2+\ldots)\,\Sigma+(AB'-A'B)^2+(BC'-B'C)^2+\ldots=0,$$

and the linear tangents from the intersections of

$$\frac{A}{A'}=\frac{B}{B'}=\ldots$$

with the spherical locus at infinity intersect at the focus. In space of three dimensions such an intersection only exists in the case of spheroids, but in higher space such intersections are normal; while in space of two dimensions the equation above reduces to $\Sigma = 0$.

There is a connection between the theory of foci and that of spherical sections (*i.e.*, sections by a linear locus which lie upon a spheric) (XI.). If the quadric

$$\Sigma + A^2 + B^2 + \ldots = 0$$

be cut spherically by a linear locus in its own space (of n dimensions)

$$A^2 + B^2 + \ldots \equiv PQ + T(\lambda + \mu + \ldots),$$

where $P = 0$, $Q = 0$, $T = 0$ are linear loci, and

$$A^2 + B^2 + \ldots = 0$$

meets infinity in a section identical with those by $P = 0$ and $Q = 0$, all loci parallel to which cut the quadric in spherical sections, while the intersection of the directrix loci meets infinity upon this same section.

If the quadric be cut spherically by a linear locus in space of $(n-1)$ dimensions,

$$A^2 + B^2 + \ldots \equiv PQ + QR + RP + T(\lambda + \mu \ldots),$$

where $P = 0$, $Q = 0$, $R = 0$, $T = 0$ are linear loci, and the intersection of

$$A^2 + B^2 + \ldots = 0$$

with infinity includes those of $P = 0$, $Q = 0$; $R = 0$, $Q = 0$; and $P = 0$, $R = 0$, and all linear loci parallel to these cut the quadric upon spherics in space of $(n-1)$ dimensions, while the intersection of the directrix loci meets infinity upon its intersections with the same loci.

And similarly with other spherical sections.

If a locus in space of n-dimensions be determined by a linear equation between the distances of any point upon it from any number of points, its equation may be put in the forn

$$a\sqrt{\Sigma_1} + b\sqrt{\Sigma_2} + \ldots = 0,$$

where $\Sigma_1 = 0$, $\Sigma_2 = 0$, &c. are the equations to the point spherics at the points.

If these be $n+1$ in number, to clear the equation of radicals it will be necessary to bring it to one of the 2^{n-1}-th power in Σ_1, Σ_2, &c., and therefore of the 2^n-th order in $\lambda, \mu, \nu \ldots$, and the locus will pass through the spherical locus at infinity at least twice.

If, however, $a \pm b \pm c \ldots = 0,$

the locus will be of one degree less and may only pass through spherical locus at infinity once.

XXXIII. Any point on the right line through $(\lambda', \mu', \nu' \ldots)$ $(\lambda'', \mu'', \nu'' \ldots)$ may be denoted by $(\lambda, \mu, \nu \ldots)$, where

$$\lambda = \frac{p\lambda' + q\lambda''}{p+q}, \quad \mu = \frac{p\mu' + q\mu''}{p+q}, \quad \&c.,$$

the line being divided in the ratio of $p : q$ at $(\lambda, \mu, \nu \ldots)$. Therel where this line cuts the quadratic locus

$$U \equiv A_{11}\lambda^2 + A_{22}\mu^2 + \ldots + 2A_{12}\lambda\mu + \ldots = 0,$$

$$\begin{aligned} p^2 \{A_{11}\lambda'^2 + A_{22}\mu'^2 + \ldots + 2A_{12}\lambda'\mu' + \ldots\} \\ + 2pq\{A_{11}\lambda'\lambda'' + A_{22}\mu'\mu'' + \ldots + A_{12}(\lambda'\mu'' + \lambda''\mu') + \ldots\} \\ + q^2\{A_{11}\lambda''^2 + A_{22}\mu''^2 + \ldots + 2A_{12}\lambda''\mu'' + \ldots\} = 0, \end{aligned}$$

which gives in general two definite values of $\frac{p}{q}$, which are equal

$$U'U'' = \left(\lambda' \frac{dU''}{d\lambda''} + \mu' \frac{dU''}{d\mu''} + \ldots\right)^2,$$

so that the enveloping locus formed by lines through $(\lambda', \mu', \nu'$ which meet $U = 0$ in two coincident points, is

$$UU' = \left(\lambda' \frac{dU}{d\lambda} + \mu' \frac{dU}{d\mu} + \ldots\right)^2.$$

Both values are 0, if $U'' = 0$, and

$$\lambda' \frac{dU''}{d\lambda''} + \mu' \frac{dU''}{d\mu''} + \ldots = 0,$$

so that
$$\lambda \frac{dU''}{d\lambda''} + \mu \frac{dU''}{d\mu''} + \ldots = 0$$

is the locus which meets $U = 0$ in two coincident points $(\lambda'', \mu'', \nu'' \ldots)$; (compare XXX., p. 28). Both indeterminate, $U' = 0$, $U'' = 0$, and

$$\lambda' \frac{dU''}{d\lambda''} + \mu' \frac{dU''}{d\mu''} + \ldots = 0;$$

when this is the case, the line lies wholly on $U = 0$, it also evident lies on the linear tangent loci at $(\lambda', \mu', \nu' \ldots)$ and $(\lambda'', \mu'', \nu'' \ldots$ Hence, taking any point $(\lambda', \mu', \nu' \ldots)$ on $U = 0$, as many straig lines, wholly on the locus, can be drawn through it as there a

solutions of the system of equations—

$$A_{11}\lambda''^2+A_{22}\mu''^2+\dots+2A_{12}\lambda''\mu''+\dots=0,$$

$$A_{11}\lambda'\lambda''+A_{22}\mu'\mu''+\dots+A_{12}(\lambda'\mu''+\lambda''\mu')+\dots=0.$$

In space of two dimensions these involve $\lambda'=\lambda''$, $\mu'=\mu''$, &c., and no such line can be drawn (unless the discriminant vanish).

In space of three dimensions $(\lambda'', \mu'', \nu'' \dots)$ may be any point in the section of the quadratic locus by the tangent plane at $(\lambda', \mu', \nu' \dots)$; and so in higher space. Hence, in space of three dimensions, there will be two such lines through each point, the rectilinear generators, as is well-known; while in higher space the number is unlimited.

A similar method is applicable, as shown in a paper on "Plane Sections of Surfaces, &c.," which I had the honour to read here in December, 1883, to find the planes which lie wholly on a locus.

If $$\lambda=\frac{p\lambda'+q\lambda''+r\lambda'''}{p+q+r},\quad \mu=\frac{p\mu'+q\mu''+r\mu'''}{p+q+r},\ \text{\&c.},$$

$(\lambda, \mu, \nu \dots)$ will denote any point on the plane through the three points $(\lambda', \mu', \nu' \dots)$, $(\lambda'', \mu'', \nu'' \dots)$, and $(\lambda''', \mu''', \nu''' \dots)$; and, where this plane meets $U=0$,

$$\begin{aligned}
&p^2\{A_{11}\lambda'^2+A_{22}\mu'^2+\dots+2A_{12}\lambda'\mu'+\dots\}\\
&+q^2\{A_{11}\lambda''^2+A_{22}\mu''^2+\dots+2A_{12}\lambda''\mu''+\dots\}\\
&+r^2\{A_{11}\lambda'''^2+A_{22}\mu'''^2+\dots+2A_{12}\lambda'''\mu'''+\dots\}\\
&+2pq\{A_{11}\lambda'\lambda''+A_{22}\mu'\mu''+\dots+A_{12}(\lambda'\mu''+\lambda''\mu')+\dots\}\\
&+2qr\{A_{11}\lambda''\lambda'''+A_{22}\mu''\mu'''+\dots+A_{12}(\lambda''\mu'''+\lambda'''\mu'')+\dots\}\\
&+2rp\{A_{11}\lambda'''\lambda'+A_{22}\mu'''\mu'+\dots+A_{12}(\lambda'''\mu'+\lambda'\mu''')+\dots\}=0,
\end{aligned}$$

and as many planes wholly on the locus can be drawn, through $(\lambda', \mu', \nu' \dots)$ a point on the surface, as there are solutions of

$$\begin{aligned}
A_{11}\lambda''^2+A_{22}\mu''^2+\dots+2A_{12}\lambda''\mu''+\dots&=0,\\
A_{11}\lambda'''^2+A_{22}\mu'''^2+\dots+2A_{12}\lambda'''\mu'''+\dots&=0,\\
A_{11}\lambda'\lambda''+A_{22}\mu'\mu''+\dots+A_{12}(\lambda'\mu''+\lambda''\mu')+\dots&=0,\\
A_{11}\lambda''\lambda'''+A_{22}\mu''\mu'''+\dots+A_{12}(\lambda''\mu'''+\lambda'''\mu'')+\dots&=0,\\
A_{11}\lambda'''\lambda'+A_{22}\mu'''\mu'+\dots+A_{12}(\lambda'''\mu'+\lambda'\mu''')+\dots&=0.
\end{aligned}$$

Such solutions are impossible in space of two dimensions, special in space of three, and normal in higher space.

In fact $(\lambda'', \mu'' \dots)$, $(\lambda''', \mu''' \dots)$ are any two points [not in a

straight line through $(\lambda', \mu' \ldots)$] on the section of the locus by linear tangent locus at $(\lambda', \mu' \ldots)$, which are so related that one l on the polar of the other. In the same way, by the substitution

$$\lambda = \frac{p\lambda' + q\lambda'' + r\lambda''' + s\lambda''''}{p+q+r+s}, \quad \mu = \frac{p\mu' + q\mu'' + r\mu''' + s\mu''''}{p+q+r+s},$$
$$\&c. \qquad \&c.,$$

it is easily shown that a hyper-plane through $(\lambda', \mu' \ldots)$ a point on quadratic locus will lie wholly on the locus, if it pass through th other points [not in the same plane with $(\lambda', \mu' \ldots)$] on the interse tion of the quadric with its linear tangent locus at $(\lambda', \mu' \ldots)$, a such that each lies on the polar of (*i.e.*, linear tangent locus at) t others.

And so with higher linear loci.

Hence, if the linear locus determined by $(p-1)$ points on a quad in space of n dimensions lie entirely on the quadratic locus, t higher linear locus determined by these and a p^{th} point will wholly on the locus, if this point be on the intersection of the qua ratic locus and the linear tangent loci at the $(p-1)$ first points.

If there be $n-2$ points $(p = n-1)$, $(\lambda_1, \mu_1 \ldots)$, $(\lambda_2, \mu_2 \ldots)$, & related as above, and $(\lambda, \mu \ldots)$ be the additional point required, t following equations hold :—

$$A_{11}\lambda^2 \quad + A_{22}\mu^2 \quad + \ldots + 2A_{12}\lambda\mu \qquad + \ldots = 0,$$
$$A_{11}\lambda_1\lambda \quad + A_{22}\mu_1\mu \quad + \ldots + A_{12}(\lambda_1\mu + \lambda\mu_1) \quad + \ldots = 0,$$
$$A_{11}\lambda_2\lambda \quad + A_{22}\mu_2\mu \quad + \ldots + A_{12}(\lambda_2\mu + \lambda\mu_2) \quad + \ldots = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$A_{11}\lambda_{n-2}\lambda + A_{22}\mu_{n-2}\mu + \ldots + A_{12}(\lambda_{n-2}\mu + \lambda\mu_{n-2}) + \ldots = 0,$$

involving $(\lambda, \mu \ldots)$, and

$$A_{11}\lambda_r\lambda_s + A_{22}\mu_r\mu_s + \ldots + A_{12}(\lambda_r\mu_s + \lambda_s\mu_r) + \ldots = 0,$$

where r and s may have all values from 1 to $n-2$. Hence, if the li joining the required point to $(\lambda_1, \mu_2 \ldots)$ be

$$\frac{\lambda - \lambda_1}{a} = \frac{\mu - \mu_1}{b} = \frac{\nu - \nu_1}{c} = \ldots,$$

where $$a + b + c + \ldots = 0,$$

then the ratios of the $n+1$ quantities a, b, c, &c, are determin

by the n equations

$$\begin{array}{lllll} A_{11}a^2 & +A_{22}b^2 & +\dots+2A_{12}ab & & +\dots=0, \\ A_{11}\lambda_1 a & +A_{22}\mu_1 b & +\dots+\ A_{12}(\lambda_1 b+\mu_1 a) & & +\dots=0, \\ A_{11}\lambda_2 a & +A_{22}\mu_2 b & +\dots+\ A_{12}(\lambda_2 b+\mu_2 a) & & +\dots=0, \\ & \&c. & & \&c., & \\ \multicolumn{5}{l}{A_{11}\lambda_{n-2}a+A_{22}\mu_{n-2}b+\dots+\ A_{12}(\lambda_{n-2}b+\mu_{n-2}a)+\dots=0,} \end{array}$$

and $$a+b+c+\dots=0,$$

and $(\lambda, \mu \dots)$ must lie on one of two definite straight lines through $(\lambda_1, \mu_1 \dots)$; and through a linear locus determined by $n-2$ points which lies wholly on a quadratic locus in space of n dimensions, two linear loci determined by $n-1$ points can in general be drawn so as to lie wholly on the locus. This is the case of the two rectilinear generators through a point on a quadric surface.

So, in space of 4 dimensions, two planes can be drawn through any straight line which lies wholly on the quadratic locus, so as to lie wholly upon it, and so on.

In the paper, on "Sections of Surfaces, &c.," already mentioned, and in the Solution of Quest. 8864 and 9004 in the *Educational Times* (*Reprint*, Vol. XLVIII.), I have shown that the intersection of a locus, in space of n dimensions, with its linear tangent locus is a locus in space of $(n-1)$ dimensions, having a node at the point of contact (9004), and that the intersection of a hyper-surface (in space of 4 dimensions) with its linear and quadratic polars is a curve of double curvature having a sextuple point at the point. (Professor Sylvester.)

XXXIV. If Q^2 be the determinant of the linear locus

$$\alpha\lambda+\beta\mu+\gamma\nu+\dots=0 \quad \text{(XII.)},$$

and $$\frac{\alpha}{Q}+k, \quad \frac{\beta}{Q}+k, \quad \frac{\gamma}{Q}+k, \ \&c.,$$

be substituted for α, β, γ, &c. in the tangential equation to an m-ic locus in space of n dimensions; the resulting equation in k will be of order $m(m-1)^{n-1}$, and its roots will be the perpendicular distances of the tangent loci parallel to

$$\alpha\lambda+\beta\mu+\gamma\nu+\dots=0$$

from that locus. This follows at once from the consideration that $\frac{\alpha}{Q}$, $\frac{\beta}{Q}$, $\frac{\gamma}{Q}$, &c. are the perpendiculars from the vertices of the

simplicissimum of reference upon

$$a\lambda+\beta\mu+\gamma\nu+\ldots=0 \qquad \text{(XII.)},$$

and that $\quad a\lambda+\beta\mu+\gamma\nu+\ldots+kQ\,(\lambda+\mu+\nu\ \ldots)=0$

represents a parallel locus at a perpendicular distance k from it.

If the resulting equation be

$$A_0 k^{m(m-1)^{n-1}}+A_1 k^{m(m-1)^{n-1}-1}+\&c.=0,$$

A_0, A_1, &c. will be of order 0, 1, &c. in $\alpha, \beta, \gamma\ldots$, and a variety of lations among the perpendiculars may be secured by equating fur tions of these coefficients to zero. Thus the locus $A_1=0$, which linear in $\alpha, \beta, \gamma\ldots$, represents a point such that its perpendicul distances from any system of parallel tangent loci have their al braical sum zero. This is a generalisation of Professor Male Question (6418) in the *Educational Times* (*Reprint*, Vol. XLII., p. 5 which extends a theorem of Chasles, for plane curves, to surfaces.

If the locus be a spheric, the difference of the two values of k w be $2r$, where r is the radius, and this might thus be found.

If $h\alpha_1+k\alpha_2$, $h\beta_1+k\beta_2$, $h\gamma_1+k\gamma_2$, &c. be substituted for α, β, γ, & respectively, in the tangential equation to an m-ic locus in space n-dimensions, the resulting $m(m-1)^{n-1}$-ic equation in $h:k$ will termine the $m(m-1)^{n-1}$-ic linear tangent loci which can be dra through the intersection of

$$\alpha_1\lambda+\beta_1\mu+\gamma_1\nu+\ldots=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

and $$\alpha_2\lambda+\beta_2\mu+\gamma_2\nu+\ldots=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2$$

what precedes is indeed a case of this, viz., when one of the line loci is infinity.

If instead of $h\alpha_1+k\alpha_2$, $h\beta_1+k\beta_1$, &c., $\frac{h}{Q_1}\alpha_1+\frac{k}{Q_2}\alpha_2$, $\frac{h}{Q_1}\beta_1+\frac{k}{Q_2}$, $\frac{h}{Q_1}\gamma_1+\frac{k}{Q_2}\gamma_2$, &c. be substituted, Q_1 and Q_2^2 being the determinants the linear loci above (XII.), the values of $h:k$ will be the ratios the perpendiculars from the linear tangent loci, through their int section, upon the two given linear loci, and the coefficients of t equation in $h:k$ will give the values of symmetric functions of the perpendiculars.

The discriminant of this binary equation will vanish if a dout tangent locus passes through the intersection of (1) and (2); and $\alpha, \beta, \gamma\ldots$ be written for $\alpha_2, \beta_2, \gamma_2\ldots$ in this, this equation must satisfied by the coefficients of the equations to all linear loci whi

intersect

$$\alpha_1\lambda+\beta_1\mu+\gamma_1\nu+\ldots=0$$

in sections common to doubly tangent loci.

XXXV. The equations to the normal to the locus $U=0$ (of the m^{th} order in space of n dimensions) at the point $(\lambda, \mu, \nu \ldots)$ are

$$\left\|\begin{matrix} \dfrac{dS'}{d\lambda'}-\dfrac{dS}{d\lambda}, & \dfrac{dS'}{d\mu'}-\dfrac{dS}{d\mu}, & \dfrac{dS'}{d\nu'}-\dfrac{dS}{d\nu}, & \ldots \\ \dfrac{dU}{d\lambda}, & \dfrac{dU}{d\mu}, & \dfrac{dU}{d\nu}, & \ldots \\ 1, & 1, & 1, & \ldots \end{matrix}\right\| = 0,$$

where $(\lambda', \mu', \nu' \ldots)$ are the current coordinates. This follows at once (XII.) from the equations to the perpendicular upon a given linear locus, the linear locus in this case being

$$\lambda'\frac{dU}{d\lambda}+\mu'\frac{dU}{d\mu}+\nu'\frac{dU}{d\nu}+\ldots=0,$$

the tangent linear locus to $U=0$ at the point $(\lambda, \mu, \nu \ldots)$ upon it.

From the $n-1$ equations given by the determinant, and the equation $U=0$, each of which is of order m in $\lambda, \mu \ldots$, &c., and

$$\lambda+\mu+\nu\ldots=V,$$

λ, μ, ν, &c. may be determined in terms of λ', μ', ν', &c., and these will be m^n solutions. Therefore m^n normals can be drawn from an arbitrary point to a given locus of the m^{th} order in space of n dimensions.

The linear tangent locus to any m-ic locus, in space of n dimensions, meets it in two coincident points along any line drawn in that locus. Hence an indefinitely near parallel locus meets it in a quadric locus in space of $(n-1)$ dimensions, and there are as many different kinds of contact at a point as there are kinds of quadrics in space of $(n-1)$ dimensions; while there are as many kinds of tangent loci as there are kinds of m-ic loci having a double point. (See XXXIII., *ad finem.*)

XXXVI. The quantities $\dfrac{dS}{d\lambda}, \dfrac{dS}{d\mu}, \dfrac{dS}{d\nu}\ldots$ are linear functions of $\lambda, \mu, \nu \ldots$, $n+1$ in number, and are connected by an identical relation.

For

$$\left.\begin{array}{l} V-\lambda-\mu \qquad -\nu \qquad \ldots \equiv 0 \\ \dfrac{dS}{d\lambda}-\mu\,(1.2)^2-\nu\,(1.3)^2 \ldots \equiv 0 \\ \dfrac{dS}{d\mu}-\lambda\,(2.1)^2-\nu\,(2.3)^2 \ldots \equiv 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots$$

therefore

$$\begin{vmatrix} V, & 1, & 1, & 1, & \ldots \\ \dfrac{dS}{d\lambda}, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\ \dfrac{dS}{d\mu}, & (2.1)^2, & 0, & (2.3)^2, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \equiv 0.$$

These may therefore be employed as a coordinate system in the of λ, μ .... The equation to the circumspheric in terms of the ordinates is easily obtained as follows.

Taking the identities (A) except the first, and

$$2S-\lambda\frac{dS}{d\lambda}-\mu\frac{dS}{d\mu}-\nu\frac{dS}{d\nu}\ldots\equiv 0,$$

$$\begin{vmatrix} 2S, & \dfrac{dS}{d\lambda}, & \dfrac{dS}{d\mu}, & \dfrac{dS}{d\nu}, & \ldots \\ \dfrac{dS}{d\lambda}, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\ \dfrac{dS}{d\mu}, & (2.1)^2, & 0, & (2.3)^2, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \equiv 0,$$

or

$$\begin{vmatrix} 0, & \dfrac{dS}{d\lambda}, & \dfrac{dS}{d\mu}, & \dfrac{dS}{d\nu}, & \ldots \\ \dfrac{dS}{d\lambda}, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\ \dfrac{dS}{d\mu}, & (2.1)^2, & 0, & (2.3)^2, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$\equiv (-2)^{n+1}\,(n!)^2\,V^2R^2S,$$

and therefore $S = 0$ is equivalent to

$$\begin{vmatrix} 0, & \frac{dS}{d\lambda}, & \frac{dS}{d\mu}, & \frac{dS}{d\nu}, & \dots \\ \frac{dS}{d\lambda}, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ \frac{dS}{d\mu}, & (2.1)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

If $a+b+c\,... = 0$,

$$a\frac{dS}{d\lambda}+b\frac{dS}{d\mu}+c\frac{dS}{d\nu}+... = 0,$$

or $$\left(\lambda\frac{d}{da}+\mu\frac{d}{db}+\nu\frac{d}{dc}+...\right)S\,(a, b, c\,...) = 0,$$

represents a linear locus through the circumcentre $(\lambda', \mu', \nu'\,...)$ at right angles (XVIII.) to

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \&c. \quad\text{......................(1).}$$

(This result is made use of in Art. XXVI.)

If $$\frac{\lambda-\lambda'}{a'} = \frac{\mu-\mu'}{b'} = \frac{\nu-\nu'}{c'} = \&c. \quad\text{......................(2)}$$

be another linear locus through $(\lambda', \mu', \nu'\,...)$, this will be at right angles to (1) if

$$\left(a'\frac{d}{da}+b'\frac{d}{db}+c'\frac{d}{dc}+...\right)S\,(a, b, c\,...) = 0,$$

and this is the condition that (1) and (2) should be at right angles.

Also, if p_1, p_2 be two distinct roots of the equation to determine the principal axes (XXX.), and a_1, b_1, c_1, &c., a_2, b_2, c_2, &c., the corresponding values of a, b, c, &c.,

$$\frac{d}{da_1}\left\{U\,(a_1, b_1, c_1\,...)-p_1\,S\,(a_1, b_1, c_1\,...)\right\}$$
$$= \frac{d}{db_1}\left\{U\,(a_1, b_1, c_1\,...)-p_1\,S\,(a_1, b_1, c_1\,...)\right\}$$
$$= \&c. \equiv h,$$

and
$$\frac{d}{da_2}\{U(a_2, b_2, c_2 \ldots) - p_2 S(a_2, b_2, c_2 \ldots)\}$$
$$= \frac{d}{db_2}\{U(a_2, b_2, c_2 \ldots) - p_2 S(a_2, b_2, c_2 \ldots)\}$$
$$= \&c. \equiv k;$$
therefore
$$\left\{a_2 \frac{d}{da_1} + b_2 \frac{d}{db_1} + c_2 \frac{d}{dc_1} + \ldots\right\}\{U(a_1, b_1, c_1 \ldots) - p_1 S(a_1, b_1, c_1 \ldots)$$
$$= h(a_2 + b_2 + c_2 \ldots) = 0,$$
and
$$\left\{a_1 \frac{d}{da_2} + b_1 \frac{d}{db_2} + c_1 \frac{d}{dc_2} + \ldots\right\}\{U(a_2, b_2, c_2 \ldots) - p_2 S(a_2, b_2, c_2 \ldots)$$
$$= k(a_1 + b_1 + c_1 + \ldots) = 0;$$
therefore $(p_1 - p_2)\left(a_2 \frac{d}{da_1} + b_2 \frac{d}{db_1} + c_2 \frac{d}{dc_1} \ldots\right) S(a_1, b_1, c_1 \ldots) = 0;$

and therefore, if p_1 and p_2 be distinct, the corresponding princi
axes are at right angles.

In two Appendices to the *Reprint* from the *Educational Times* have generalized a number of interesting theorems in geometry two and three dimensions, which had been set as questions in t periodical,—viz., in Appendix III., Vol. XLVII., Questions 5809, 70 7069 (Mr. Edwardes); 5828 (Mons. Darboux); 8204 (Miss Gordo and in Appendix III., Vol. XLVIII., Questions 7488 and 7533 (Profes Hudson and Mr. Walker); 6386 (Mr. McCay); 3228 and 9 (Professor Wolstenholme), see also Art. XXVII. above; 6392 (Elliott); 2119 (Professor Wolstenholme); 9093 and 9170 (Roberts), see also Art. XXVI. above.

On a Method in the Analysis of Curved Lines. Part III.

Mr. J. J. Walker.

[*Read December 8th,* 1887.]

§ VIII. *A Generalisation of some foregoing Results.*

In the first two papers which have appeared in the *Proceedings* (Vol. IX., pp. 226–242, and Vol. XVI., pp. 215–223), in certain results of operating on a ternary form u with

$$p\,.\,p-1\,...\,p-r+1\,.\,D^r = \begin{vmatrix} l & m & n \\ \alpha & \beta & \gamma \\ \partial_x & \partial_y & \partial_z \end{vmatrix}^r,$$

p being the order of u, and r any positive integer not greater than p, xyz were subject to the relations

$$\alpha x+\beta y+\gamma z = \text{a constant, say } \Delta;$$

$$lx+my+nz = 0,$$

viz., xyz were the trilinear coordinates of a point on the transversal L or $lx+my+nz$, and $\alpha\beta\gamma$ the sines of the angles of the triangle of reference, so that, to a factor,

$$m\gamma-n\beta,\quad n\alpha-l\gamma,\quad l\beta-m\alpha,$$

were the coordinates of the point in which the transversal L met the line at infinity, or the "direction coordinates" of that transversal, as they might appropriately be called.

Considering xyz as perfectly unrestricted parameters of the linear ternary forms

$$\Delta \equiv \alpha x+\beta y+\gamma z,$$

$$L \equiv lx+my+nz,$$

in connexion with a form u of any degree p; or, more generally, with two forms u, u' of orders p, p' respectively; I proceed to investigate the result

$$uD^2u'+u'D^2u-2Du\,Du',$$

which in the former of the two papers* referred to, when the vari-bles were restricted by the conditions

$$\Delta = \text{constant}, \quad L = 0,$$

was shown to coincide with Dr. Salmon's;

$$\overline{l12}^2 uu',\dagger \text{ multiplied by } \Delta^2, \text{ i.e., } (\alpha x+\beta y+\gamma z)^2.$$

For shortness, using

$$\mu, \nu \text{ for } m\gamma - n\beta,\ na - l\gamma,\ la - m\beta \text{ respectively,}$$

$$a \ldots f \ldots \text{ for } \frac{1}{p \cdot p-1} \frac{\partial^2 u}{\partial x^2} \ldots \frac{1}{p \cdot p-1} \frac{\partial^2 u}{\partial y \partial z} \ldots,$$

$$a' \ldots f' \ldots \text{ for } \frac{1}{p' \cdot p'-1} \frac{\partial u'}{\partial x^2} \ldots \frac{1}{p' \cdot p'-1} \frac{\partial^2 u'}{\partial y \partial z} \ldots,$$

the result in full is, the operator now being written more briefly,

$$p \cdot p-1 \ldots p-r+1\, D^r = \left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z}\right)^r,$$

$$\begin{aligned}
&(ax^2+by^2+cz^2+2fyz+2gzx+2hxy) \\
&(a'\lambda^2+b'\mu^2+c'\nu^2+2f'\mu\nu+2g'\nu\lambda+2h'\lambda\mu) \\
+&(a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy) \\
&(a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu) \\
-2&\left\{(ax+hy+gz)\,\lambda+(hx+by+fz)\,\mu+(gx+fy+cz)\,\nu\right\} \\
&\left\{(a'x+h'y+g'z)\,\lambda+(h'x+b'y+f'z)\,\mu+(g'x+f'y+c'z)\,\nu\right\},
\end{aligned}$$

which reduces identically to

$$\Sigma \left\{(bc'+b'c-2ff')(y\nu-z\mu)^2+2\,(gh'+g'h-af'-a'f\,)(z\lambda-x\nu)(x\mu-y\lambda)\right\}$$

But also, identically,

$$\left.\begin{aligned} y\nu - z\mu &= l\Delta - \alpha L \\ z\lambda - x\nu &= m\Delta - \beta L \\ x\mu - y\lambda &= n\Delta - \gamma L \end{aligned}\right\};$$

consequently the result above is, otherwise,

$$\begin{aligned}\Sigma \{&(bc'+b'c-2ff')(l^2\Delta^2-2l\alpha\Delta L+\alpha^2L^2) \\ &+2\,(gh'+g'h-af'-a'f)(mn\Delta^2-\overline{m\gamma+n\beta}\,\Delta L+\beta\gamma L^2)\},\end{aligned}$$

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* *Proceedings*, Vol. IX., p. 233.

† *Higher Algebra*, p. 145, 4th ed., l being substituted for a.

or, finally,

$$\begin{aligned} & uD^2u'+u'D^2u-2Du\,Du' \\ &\quad = \Delta^2\left\{(bc'+b'c-2ff')\,l^2+\ldots+2(gh'+g'h-af'-a'f)\,mn+\ldots\right\} \\ &\quad -2\Delta L\left\{\begin{aligned} &\{(bc'+b'c-2ff')\,l+(fg'+f'g-ch'-c'h)\,m \\ &\qquad\qquad +(hf'+h'f-bg'-b'g)\,n\}\,\alpha \\ &+\{(fg'+f'g-ch'-c'h)\,l+(ca'+c'a-2gg')\,m \\ &\qquad\qquad +(gh'+g'h-af'-a'f)\,n\}\,\beta \\ &+\{(hf'+h'f-bg'-b'g)l+(gh'+g'h-af'-a'f)m \\ &\qquad\qquad +(ab'+a'b-2hh')\,n\}\,\gamma \end{aligned}\right\} \\ &\quad +L^2\left\{(bc'+b'c-2ff')\,\alpha^2+\ldots+2\,(gh'+g'h-af'-a'f)\,\beta\gamma+\ldots\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots(37). \end{aligned}$$

In particular $\qquad uD^2u-(Du)^2$

$$\begin{aligned} &= \Delta^2\{(bc-f^2)\,l^2+\ldots+2\,(gh-af)\,mn\ldots\} \\ &\quad -2\Delta L\left[\{(bc-f^2)l+(fg-ch)\,m+(hf-bg)\,n\}\,\alpha+\{\ldots\}\,\beta+\{\ldots\}\,\gamma\right] \\ &\quad +L^2\{(bc-f^2)\,\alpha^2+\ldots+2\,(gh-af)\,\beta\gamma+\ldots\}\ \ldots\ldots\ldots\ldots\ldots\ldots(38). \end{aligned}$$

Now, if $\Phi=$ the particular value of $uD^2u'+\ldots$ for $lx+my+nz=0$, viz., $\Phi=\Delta^2\{(bc'+b'c-2ff')\,l^2+\ldots+2\,(gh'+g'h-af'-a'f)\,mn+\ldots\}$, then the general value (ϕ) may be written

$$\begin{aligned} \phi = \Delta^2\Phi-\Delta L\left(\alpha\frac{\partial\Phi}{\partial l}+\beta\frac{\partial\Phi}{\partial m}+\gamma\frac{\partial\Phi}{\partial n}\right) & \\ +\frac{L^2}{1.2}\left(\alpha^2\frac{\partial^2\Phi}{\partial l^2}+\ldots+2\beta\gamma\frac{\partial^2\Phi}{\partial m\,\partial n}+\ldots\right) & \ \ldots\ldots\ldots(39). \end{aligned}$$

This remark suggested the general theorem which follows:—

Let the result of the operation

$$\phi\,(D,D',D''\ldots)\,uu'u''\ldots=\Delta^i\Phi,$$

[u, u', u''... being any ternary forms, while D operates on u only, D' on u' only ...], for values of xyz satisfying $lx+my+nz=0$, then for all values of xyz,

$$\begin{aligned} &\phi\,(D\ldots)\,u\ldots \\ &= \Delta^i\Phi-\Delta^{i-1}L\left(\alpha\frac{\partial}{\partial l}+\beta\frac{\partial}{\partial m}+\gamma\frac{\partial}{\partial n}\right)\Phi+\frac{\Delta^{i-2}L^2}{1.2}\left(\alpha^2\frac{\partial^2}{\partial l^2}+\ldots\right)\Phi-\ldots+\ldots \\ &\quad +(-1)^r\frac{\Delta^{i-r}L^r}{1.2\ldots r}\left(\alpha\frac{\partial}{\partial l}+\beta\frac{\partial}{\partial m}+\gamma\frac{\partial}{\partial n}\right)^r\Phi+\ldots \\ &\quad +(-1)^i\frac{L^i}{1.2\ldots i}\left(\alpha\frac{\partial}{\partial l}+\ldots\right)^i\Phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40). \end{aligned}$$

To prove this, let

$$\phi = \Delta^i\Phi + A_1\Delta^{i-1}L + A_2\Delta^{i-2}L^2 + \ldots + A_iL^i,$$

—a form plainly justified by the condition $\phi = \Delta^i\Phi$, for all value xyz satisfying

$$lx + my + nz = 0,$$

and the consideration that the general value of ϕ must be sym trical as regards the terms of $\alpha x + \beta y + \gamma z$ and $lx + my + nz$,—then

$$\left.\begin{aligned} \frac{\partial\phi}{\partial l} &= \Delta^i\frac{\partial\Phi}{\partial l} + x\Delta^{i-1}A_1 + L\,(\ldots) \\ \frac{\partial\phi}{\partial m} &= \Delta^i\frac{\partial\Phi}{\partial m} + y\Delta^{i-1}A_1 + L\,(\ldots) \\ \frac{\partial\phi}{\partial n} &= \Delta^i\frac{\partial\Phi}{\partial n} + z\Delta^{i-1}A_1 + L\,(\ldots) \end{aligned}\right\}$$

for all values of xyz, and

$$\alpha\frac{\partial\phi}{\partial l} + \beta\frac{\partial\phi}{\partial m} + \gamma\frac{\partial\phi}{\partial n} = \Delta^i\left(\alpha\frac{\partial\Phi}{\partial l} + \beta\frac{\partial\Phi}{\partial m} + \gamma\frac{\partial\Phi}{\partial n}\right) + \Delta^iA_1 + L\,(\ldots).$$

Now every power of $\alpha\frac{\partial}{\partial l} + \beta\frac{\partial}{\partial m} + \gamma\frac{\partial}{\partial n}$ is an "annihilator" of ϕ is evident: hence, determining A_1 by supposing xyz to satisfy L:

$$-A_1 = \alpha\frac{\partial\Phi}{\partial l} + \beta\frac{\partial\Phi}{\partial m} + \gamma\frac{\partial\Phi}{\partial n}.$$

Next, let the general value of A_r, as far as r = any positive integer

$$\frac{(-1)^r}{1.2\ldots r}\left(\alpha\frac{\partial}{\partial l} + \beta\frac{\partial}{\partial m} + \gamma\frac{\partial}{\partial n}\right)^r\Phi,$$

and observe that the only term in

$$\left(\alpha\frac{\partial}{\partial l} + \beta\frac{\partial}{\partial m} + \gamma\frac{\partial}{\partial n}\right)^q(L^q A_q),$$

which does not contain L as a factor, is

$$[illegible]$$

$$[illegible] \left(\alpha\frac{\partial}{\partial l} + \beta\frac{\partial}{\partial m} + \gamma\frac{\partial}{\partial n}\right)^{[illegible]} A_q.$$

This being the case, and operating with

$$\left(a\frac{\partial}{\partial l}+\beta\frac{\partial}{\partial m}+\gamma\frac{\partial}{\partial n}\right)^{r+1},$$

$$\left(a\frac{\partial}{\partial l}+\dots\right)^{r+1}\phi = 0 = \Delta^i\Big[\Big\{1-(r+1)+\frac{(r+1)\,r}{1.2}-\dots$$

$$\dots+(-1)^r\frac{(r+1)\,r\dots 2}{1.2\dots r}\Big\}\left(a\frac{\partial}{\partial l}+\dots\right)^{r+1}\Phi$$

$$+r+1.r.r-1\dots 2.1\,A_{r+1}\Big]+L\,(\dots),$$

or
$$0=\Delta^i\Big[\{(1-1)^{r+1}-(-1)^{r+1}\}\left(a\frac{\partial}{\partial l}+\dots\right)^{r+1}\Phi$$

$$+(r+1)\,r\dots 2.1\,A_{r+1}\Big]+L\,(\dots).$$

As this holds for all values of xyz, let these variables satisfy

$$L\equiv lx+my+nz=0,$$

then
$$A_{r+1}=(-1)^{r+1}\left(a\frac{\partial}{\partial l}+\beta\frac{\partial}{\partial m}+\gamma\frac{\partial}{\partial n}\right)^{r+1}\Phi\div 1.2\dots r.\overline{r+1},$$

and this form, having been shown to hold for $r=0$, is thus proved to hold for all positive integer values of r.

Thus the theorem (40) is established generally.

§ IX. *Expressions for the Contravariants of the Cubic.*

Supposing, now, u to be, in particular, a cubic ternary form, viz., let

$$u\equiv a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3a_3x^2z+3b_1y^2x+3b_3y^2z$$
$$+3c_1z^2x+3c_2z^2y+6exyz,$$

according to the notation of the *Higher Plane Curves*, 1st ed.; then consider in connection with it the binary cubic

$$\rho^3D^3u+3\rho^2\rho'D^2u+3\rho\rho'^2Du+\rho'^3u=0\dots\dots\dots\dots\dots\dots(41),$$

the roots of which are, for values of xyz satisfying

$$lx+my+nz=0,$$

proportional* to the segments on the line L between the point (xyz),

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* *Proceedings*, Vol. IX., p. 227 (4).

and the points in which it meets the curve $u = 0$, on the furthe supposition that the coordinates xyz are subject to the relation

$$\alpha x + \beta y + \gamma z = \Delta, \text{ a constant.}$$

Hence the discriminant of the binary cubic (41) must, to a factor give the reciprocal of u, and this factor may be determined by findin the coefficient of the leading term $b_3^2 c_3^2 l^6$ in that discriminant. As fa then as the terms

$$b_3 y^3 + c_3 z^3$$

in u, the discriminant of (41), viz.,

$$(uD^3u - Du\,D^2u)^2 - 4\,\{uD^2u - (Du)^2\}\,\{Du\,D^3u - (D^2u)^2\}$$

(dropping suffixes)

$$= \{(by^3+cz^3)(b\mu^3+c\nu^3) - (by^2\mu+cz^2\nu)(by\mu^2+cz\nu^2)\}^2$$
$$-4\,\{(by^3+cz^3)(by\mu^2+cz\nu^2) - (b^2y^4\mu^2+c^2z^4\nu^2+2bcy^2z^2\mu\nu)^2\}$$
$$\times\,\{(by^2\mu+cz^2\nu)(b\mu^3+c\nu^3) - (b^2y^2\mu^4+c^2z^2\nu^4+2bcyz\mu^2\nu^2\}$$
$$= \{bc\,(y^3\nu^3+z^3\mu^3-y^2z\mu\nu^2-z^2y\mu^2\nu)+\ldots \text{ or } bc\,(y\nu+z\mu)(y\nu-z\mu)^2+\ldots\}^2$$
$$-4\,\{bc\,(y^3z\nu^2+z^3y\mu^2-2\mu\nu y^2z^2)+\ldots\}\,\{bc\,(z^2\mu^3\nu+y^2\nu^3\mu-2yz\mu^2\nu^2)+\ldots\}$$
$$\text{or } -4\,\{bcyz\,(y\nu-z\mu)^2+ \ldots\ldots\ldots\ldots\}\,\{bc\mu\nu\,(y\nu-z\mu)^2+ \ldots\ldots\ldots\ldots\}$$
$$= b_3^2 c_3^2\,(y\nu-z\mu)^6+\ldots$$
$$= \Delta^6 b_3^2 c_3^2 l^6+\ldots, \quad \text{(p. 484)},$$

for values of xyz which make $L = 0$.

Hence *the reciprocal* (v) *of* u *is equal to the discriminant of the binary cubic* (41), *divided by* Δ^6, *for values of the parameters* xyz *which satisfy*

$$lx+my+nz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42).$$

For all values of xyz, therefore, the discriminant of (41), according to the theorem (40) proved in § VIII., is equal to

$$\Delta^6 v - \Delta^5 L\left(\alpha\frac{\partial v}{\partial l}+\beta\frac{\partial v}{\partial m}+\gamma\frac{\partial v}{\partial n}\right)+\frac{\Delta^4 L^2}{1.2}\left(\alpha^2\frac{\partial^2 v}{\partial l^2}+\ldots\right)-\ldots$$
$$\ldots + L^6\,(b_3^2 c_3^2 \alpha^6+\ldots) \ldots\ldots\ldots\ldots\ldots\ldots (43).$$

To represent the "Pippian" or "Cayleyan" of u by a function of the form $\phi\,(D\ldots)\,u\ldots$, I refer back to my paper in *Proceedings*, Vol. IX.,

p. 233, where I have proved that the determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ Du_1 & Du_2 & Du_3 \\ D^2u_1 & D^2u_2 & D^2u_3 \end{vmatrix} \div \Delta^2$$

is equal to the cubic contravariant of the three quadric forms

$$u_1 = \tfrac{1}{3}\frac{\partial u}{\partial x}, \quad u_2 = \tfrac{1}{3}\frac{\partial u}{\partial y}, \quad u_3 = \tfrac{1}{3}\frac{\partial u}{\partial z},$$

provided xyz satisfy $\qquad lx+my+nz = 0.$

This determinant, therefore, with the proviso just mentioned, represents (*Higher Plane Curves*, p. 190, 3rd ed.)

$$-\Delta^3 P;^*$$

while, *for all values of xyz*,

$$|\, u_1, Du_2, D^2u_3 \,| = -\Delta^3 P + \Delta^2 L\left(\alpha\frac{\partial P}{\partial l} + \beta\frac{\partial P}{\partial m} + \gamma\frac{\partial P}{\partial n}\right)$$
$$-\frac{\Delta L^2}{1.2}\left(\alpha^2\frac{\partial^2 P}{\partial l^2}+\ldots\right) + L^3 P_a \quad \ldots\ldots\ldots\ldots(44),$$

P_a standing for P when l, m, n have been replaced by α, β, γ respectively.

The foregoing methods of representing the contravariants will be found very convenient for establishing relations which hold among them and other non-fundamental concomitants of the cubic form u. Thus the important relations (*Phil. Trans.*, 1888, A., pp. 158–160),

$$\text{Discriminant of } \overline{112}^2\, u = \tfrac{1}{4}P^2,$$
$$\text{Reciprocal of} \quad ,, \quad = -\tfrac{1}{4}\text{ Reciprocal of } u,$$

due to Prof. Cayley, may be established generally with less labour than by actual verification for the canonical form of u.

To complete the representation of the fundamental invariants of the cubic u, and the linear L, it remains to find the expression for the "Quippian" Q, defined by its discoverer Cayley as the first evectant of the sextic invariant of u.

Consider, in connexion with u, its Hessian $\mathfrak{H}$; then if, as above,

$$u_1 = \partial_x u/3, \quad u_2 = \partial_y u/3, \quad u_3 = \partial_z u/3,$$

* The numerical value of the coefficient and its sign may readily be verified by means of the canonical u.

and now also $\mathfrak{h}_1 = \partial_x \mathfrak{h}/3, \quad \mathfrak{h}_2 = \partial_y \mathfrak{h}/3, \quad \mathfrak{h}_3 = \partial_z \mathfrak{h}/3,$

$$6\Sigma \begin{vmatrix} \mathfrak{h}_1 & u_2 & u_3 \\ D\mathfrak{h}_1 & Du_2 & Du_3 \\ D^2\mathfrak{h}_1 & D^2u_2 & D^2u_3 \end{vmatrix} / \Delta^2,$$

for values of xyz which satisfy $L = 0$, will be a contravariant, dently of the fifth order in the coefficients of u, and cubic in l, which may, by the aid of the canonical form of u be ident with $-Q$. For (again dropping suffixes)

$$u = ax^3 + by^3 + cz^3 + 6exyz,$$

$$\mathfrak{h} = -e^2u + (abc + 8e^3)\, xyz,$$

$$3\mathfrak{h}_1 = -3e^2u_1 + (abc + 8e^3)\, yz.$$

Thus the determinant above becomes, for values of xyz which sat $L = 0$, (44),

$$18e^2P + (abc + 8e^3)\, \Sigma \begin{vmatrix} 2yz & by^2 + 2ezx & cz^2 + 2exy \\ z\mu + y\nu & ez\lambda + by\mu + cz\nu & ey\lambda + ex\mu + cz\nu \\ 2\mu\nu & b\mu^2 + 2e\nu\lambda & c\nu^2 + 2e\lambda\mu \end{vmatrix} / \Delta$$

the value of which is, identically,

$$18e^2P + (abc + 8e^3)\, \Sigma \left\{ -bc\,(y\nu - z\mu)^3 \right.$$
$$\left. + 4e^2\,(y\nu - z\mu)(z\lambda - x\nu)(x\mu - y\lambda) \right\} / \Delta^3,$$

or since, when $L = 0$ (p. 484),

$$y\mu - z\nu = l\Delta, \quad z\lambda - x\nu = m\Delta, \quad x\mu - y\lambda = n\Delta,$$

$$-\left\{ -18e^2P + (abc - 8e^3)(\Sigma bcl^3 - 12e^2lmn) \right\} = -Q.$$

Hence, for all values of xyz, (40),

$$6\Sigma \mid \mathfrak{h}_1\, Du_2\, D^2u_3 \mid = -\Delta^3 Q + \Delta^2 L \left(\alpha \frac{\partial Q}{\partial l} + \beta \frac{\partial Q}{\partial m} + \gamma \frac{\partial Q}{\partial n} \right)$$

$$- \frac{\Delta L^2}{1.2} \left(\alpha^2 \frac{\partial^2 Q}{\partial l^2} + \ldots \right) + L_3^3 Q_{\alpha\beta\gamma} \ \ldots\ldots\ldots\ldots (4$$

It is of interest to observe that $\mathfrak{h}_1$ is obtained by operating on u_1 w

$$\tfrac{1}{2} \left\{ (bc - f^2) \frac{\partial^2}{\partial x^2} + \ldots \right\};$$

and similarly $\mathfrak{h}_2$ from $u_2 \ldots$; hence the result of the same operator

$$u_1\,(Du_2D^2u_3 - Du_3D^2u_2) + u_2\,(Du_3D^2u_1 - Du_1D^2u_3)$$
$$+ u_3\,(Du_1D^2u_2 - Du_2D^2u_1)$$

will be found to be exactly equal to the sinister of (45). Also the dexter of (45) will be equal to the result of operating on that of (44); and thus may be obtained the relation

$$Q = \mathrm{a}\frac{\partial^2 P}{\partial l^2} + \dots + 2\mathrm{f}\frac{\partial^2 P}{\partial m\,\partial n} + \dots \qquad \dots\dots\dots\dots (46),$$

if $\mathrm{a}x^2 + \dots + 2\mathrm{f}yz + \dots \equiv (bc - f^2)\,l^2 + \dots + 2\,(gh - af)\,mn + \dots$

§ X. *Transformations of Formulæ.*

If, for shortness, as before,

$$\left.\begin{aligned} m\gamma - n\beta &= \lambda \\ n\alpha - l\gamma &= \mu \\ l\beta - m\alpha &= \nu \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\text{i.}),$$

so that, identically,

$$\left.\begin{aligned} y\nu - z\mu &= l\Delta - \alpha L \\ z\lambda - x\nu &= m\Delta - \beta L \\ x\mu - y\nu &= n\Delta - \gamma L \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (\text{ii.}),$$

where

$$\left.\begin{aligned} \Delta &\equiv \alpha x + \beta y + \gamma z \\ L &\equiv lx + my + nz \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (\text{iii.}),$$

and, u being of order p,

$u_1 \dots u_3 \dots, a \dots, f \dots$ stand for $\partial_x u/p \dots, \partial_x^2 u/p\,.\,p-1,\ \partial_{yz}^2/p\,.\,p-1$;

then, by definition,

$$\begin{aligned} pDu &= (\lambda\partial_x + \mu\partial_y + \nu\partial_z)(xu_1 + yu_2 + zu_3) \\ &= \lambda u_1 + \mu u_2 + \nu u_3 + (p-1)(xDu_1 + yDu_2 + zDu_3) \\ &= Du + (p-1)(xDu_1 + \dots), \end{aligned}$$

or

$$\begin{aligned} Du &= xDu_1 + yDu_2 + zDu_3 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\text{iv.}), \\ &= x^2Da + y^2Db + z^2Dc + 2yzDf + 2zxDg + 2xyDh \dots\dots\dots (\text{v.}); \end{aligned}$$

and so on.

Again,

$$\begin{aligned} u_2Du_3 - u_3Du_2 &= (xh + yb + zf)(\lambda g + \mu f + \nu c) \\ &\qquad - (xg + yf + zc)(\lambda h + \mu b + \nu f) \\ &= (bc - f^2)(y\nu - z\mu) + (fg - ch)(z\lambda - x\nu) + (hf - bg)(x\mu - y\lambda), \end{aligned}$$

by (ii.),

$$\begin{aligned} &= \Delta\left\{(bc - f^2)\,l + (fg - ch)\,m + (hf - bg)\,n\right\} \\ &\qquad - L\left\{(bc - f^2)\,\alpha + (fg - ch)\,\beta + (hf - bg)\,\gamma\right\} \dots (47). \end{aligned}$$

If $$s \equiv (bc-f^2)\,l^2+\ldots+2\,(gh-af)\,mn+\ldots \qquad \ldots\ldots\ldots($$

and

$$s_1=\frac{\partial s}{\partial l}/2,\quad s_2=\frac{\partial s}{\partial m}/2,\quad s_3=\frac{\partial s}{\partial n}/2,\quad s_{11}=\frac{\partial^2 s}{\partial l^2}/2\ldots s_{23}=\frac{\partial^2 s}{\partial m\,\partial n}/2$$

then the equation above may be written

$$\left.\begin{array}{l} W_1=u_3Du_2-u_2Du_3=\Delta s_1-L\,(s_{11}\alpha+s_{12}\beta+s_{13}\gamma) \\ \text{similarly} \\ W_2=u_1Du_3-u_3Du_1=\Delta s_2-L\,(s_{21}\alpha+s_{22}\beta+s_{23}\gamma) \\ W_3=u_2Du_1-u_1Du_2=\Delta s_3-L\,(s_{31}\alpha+s_{32}\beta+s_{33}\gamma) \end{array}\right\}\ldots\ldots(4$$

and $$lW_1+mW_2+nW_3=\Delta s-L\,(s_1\alpha+s_2\beta+s_3\gamma)\ldots\ldots\ldots\ldots\ldots(4$$

Observing that, generally, if ϕ, χ, ψ are homogeneous function xyz of orders p, q, r respectively, and if

$$\phi=\chi\psi,$$

$$\left.\begin{array}{ll} \text{then} & pD\phi=q\psi D\chi+r\chi D\psi, \\ \text{and hence} & \\ \multicolumn{2}{l}{p\,.\,\overline{p-1}D^2\phi=q\,.\,\overline{q-1}\psi D^2\chi+r\,.\,\overline{r-1}\chi D^2\psi+2qrD\chi D\psi} \end{array}\right\}\ldots\ldots($$

thus u being of order p, (50),

$$\left.\begin{array}{l} (2p-3)\,DW_1=(p-2)(u_3D^2u_2-u_2D^2u_3)=(p-2)(-V_1),\text{say,} \\ (2p-3)\,DW_2=(p-2)(u_1D^2u_3-u_3D^2u_1)=(p-2)(-V_2)\quad ,, \\ (2p-3)\,DW_3=(p-2)(u_2D^2u_1-u_1D^2u_2)=(p-2)(-V_3)\quad ,, \end{array}\right\}$$
$$\ldots\ldots\ldots(5$$

If, in particular, u is a ternary cubic form, similarly,

$$\left.\begin{array}{l} 3D^2W_1=-DV_1=(Du_3D^2u_2-Du_2D^2u_3)=U_1,\ \text{say} \\ 3D^2W_2=-DV_2=(Du_1D^2u_3-Du_3D^2u_1)=U_2\quad ,, \\ 3D^2W_3=-DV_3=(Du_2D^2u_1-Du_1D^2u_2)=U_3\quad ,, \end{array}\right\}\ldots\ldots(5$$

Hence, and from (49),

$$-(lV_1+mV_2+nV_3)=2\Delta Ds-2L\,(\alpha Ds_1+\beta Ds_2+\gamma Ds_3)\ldots\ldots(5$$

and (38), $$\Delta^2 s=uD^2u-(Du)^2+L\,\{\ldots\},$$

whence, (50)

$$2\Delta^2 Ds = uD^3u + 3Du\, D^2u - 4Du\, D^2u + 3LD\,\{\ldots\}$$
$$= uD^3u - Du\, D^2u + 3LD\,\{\ldots\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(54),$$

so that, *for values of xyz which satisfy $L = 0$,*

$$-\Delta\,(lV_1 + mV_2 + nV_3) = uD^3u - Du D^2u \ldots\ldots\ldots\ldots(55).$$

Similarly, for all values of xyz,

$$lU_1 + mU_2 + nU_3 = \Delta D^2 s - L\,(\alpha D^2 s_1 + \beta D^2 s_2 + \gamma D^2 s_3) \ldots\ldots(56),$$

while, from (54),

$$2\Delta^2 D^2 s = 3Du\, D^3u - 2\,(D^2u)^2 - Du D^3 u + 6LD^2\,\{\ldots\}.$$

or $$\Delta^2 D^2 s = Du D^3 u - (D^2u)^2 + 3LD^2\,\{\ldots\};$$

so that, for values of xyz which satisfy $L = 0$,

$$\Delta\,(lU_1 + mU_2 + nU_3) = Du\, D^3u - (D^2u)^2 \ldots\ldots\ldots\ldots (57).$$

From (42), (49), (55), (57), v being the reciprocal of u,

$$\Delta^4 v = (lV_1 + mV_2 + nV_3)^2 - 4\,(lW_1 + \ldots)(lU_1 + \ldots) \quad \ldots\ldots (58),$$

Again, since (44) $-\Delta^2 P$ is equal to any one of the three expressions

$$\left.\begin{array}{l} u_1 U_1 + Du_1 V_1 + D^2u_1 W_1 \\ u_2 U_2 + Du_2 V_2 + D^2u_2 W_2 \\ u_3 U_3 + Du_3 V_3 + D^2u_3 W_3 \end{array}\right\},$$

hence, by (48), for values of xyz satisfying $L = 0$,

$$-\Delta^2 P = s_1 D^2u_1 + u_1 D^2 s_1 - 2Ds_1 Du;$$

by (38),

$$-P = (\mathrm{b}_1 c_1 + \mathrm{c}_1 b_1 - 2\mathrm{f}_1 e)\, l^2 + \ldots + 2(\mathrm{g}_1 a_2 + \mathrm{h}_1 a_3 - \mathrm{a}_1 e - \mathrm{f}_1 a)\, mn + \ldots \; (59),$$

or $$-\Delta^2 P = s_2 D^2 u_2 + u_2 D^2 s_2 - 2Ds_2 Du_2$$

$$-P = (\mathrm{b}_2 c_2 + \mathrm{c}_2 b - 2\mathrm{f}_2 b_3)\, l^2 + \ldots + 2\,(\mathrm{g}_2 b_1 + \mathrm{h}_2 e - \mathrm{a}_2 b_3 - \mathrm{f}_2 a_2)\, mn + \ldots \; (60),$$

or $$-\Delta^2 P = s_3 D^2 u_3 + u_3 D^2 s_3 - 2Ds_3 Du_3,$$

$$-P = (\mathrm{b}_3 c + \mathrm{c}_3 b_3 - 2\mathrm{f}_3 c_2)\, l^2 + \ldots + 2\,(\mathrm{g}_3 e_1 + \mathrm{h}_3 c_1 - \mathrm{a}_3 c_2 - \mathrm{f}_3 a_3)\, mn + \ldots \; (61),$$

where $$s \equiv \mathrm{a}x^2 + \ldots + 2\mathrm{f}yz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (\text{vii.})$$

$$u_1 = ax^2 + b_1 y^2 + c_1 z^2 + 2eyz + 2a_3 zx + 2a_2 xy,$$

$$u_2 = a_2 x^2 + by^2 + c_3 z^2 + 2b_3 yz + 2ezx + 2b_1 xy,$$

$$u_3 = a_3x^2 + b_3y^2 + cz^2 + 2c_3yz + 2c_1zx + 2exy,$$

and $$\frac{\partial \mathrm{a}}{\partial l} = 2\mathrm{a}_1, \quad \frac{\partial \mathrm{a}}{\partial m} = 2\mathrm{a}_2, \quad \frac{\partial \mathrm{a}}{\partial n} = 2\mathrm{a}_3 \; \ldots \; \frac{\partial \mathrm{h}}{\partial n} = 2\mathrm{h}_3.$$

Since also, identically, (48), (51), (52),

$$u_1(lU_1+mU_2+nU_3)+Du_1(lV_1+mV_2+nV_3)+D^2u_1(lW_1+mW_2+nW$$
$$= l(u_1U_1+Du_1V_1+D^2u_1W_1);$$

which (44), for values of xyz satisfying L or $lx+my+nz=0$,

$$= -\Delta^2 lP,$$

by (49), (51), (52), for similar values of xyz,

$$-\Delta^2 lP = sD^2u_1 + u_1D^2s - 2DsDu, \text{ whence (38)},$$

$$-lP = (\mathrm{b}c_1+\mathrm{c}b_1-2\mathrm{f}e)\,l^2+\ldots+2\,(\mathrm{g}a_3+\mathrm{h}a_2-\mathrm{a}e-\mathrm{f}a)\,mn+\ldots \quad (62$$

and, similarly,

$$-mP = (\mathrm{b}c_3+\mathrm{c}b-2\mathrm{f}b_3)\,l^2+\ldots+2\,(\mathrm{g}b_1+\mathrm{h}e-\mathrm{a}b_3-\mathrm{f}a_2)\,mn+\ldots \quad (63$$

$$-nP = (\mathrm{b}c+\mathrm{c}b_3-2\mathrm{f}c_2)\,l^2+\ldots+2\,(\mathrm{g}e+\mathrm{h}c_1-\mathrm{a}c_2-\mathrm{f}a_3)\,mn+\ldots \quad (64$$

Observing that

$$u_3U_1+Du_3V_1+D^2u_3W_1 \equiv 0,$$

or any similar formula in which the suffixes are unlike,

$$(\mathrm{b}_1c_3+\mathrm{c}_1b-2\mathrm{f}_1b_1)\,l^2+\ldots+2\,(\mathrm{g}_1b_1+\mathrm{h}_1e-\mathrm{a}_1b_1-\mathrm{f}a)\,mn+\ldots \equiv 0 \; \ldots \quad (65$$

or the contravariant quadrics of any two of the sets

$$s_1,\ s_2,\ s_3 \text{ and } u_1,\ u_2,\ u_3,$$

selected with unlike suffixes; from which, and (63), (62), it follo[ws] that

$$\left.\begin{aligned} -m\frac{\partial P}{\partial l} &= 2\,\{(\mathrm{b}c_3+\mathrm{c}b-2\mathrm{f}b_3)\,l+(\mathrm{f}e+\mathrm{g}b_3-\mathrm{c}b_1-\mathrm{h}e_2)\,m \\ &\qquad +(\mathrm{h}b_3+\mathrm{f}b_1-\mathrm{b}e-\mathrm{g}b)\,n\} \\ -l\frac{\partial P}{\partial m} &= 2\,\{(\mathrm{f}a_3+\mathrm{g}e-\mathrm{c}a_2-\mathrm{h}c_1)\,l+(\mathrm{c}a_1+\mathrm{a}c_1-2\mathrm{g}a_3)\,m \\ &\qquad +(\mathrm{g}a_2+\mathrm{h}a_3-\mathrm{a}e-\mathrm{f}a)\,n\} \end{aligned}\right\} \ldots \quad (66$$

and so on.

But, by (62), (59),

$$-l\partial P/\partial l = -P+2\,\{(\mathrm{b}c_1+\mathrm{c}b_1-2\mathrm{f}e)\,l+(\mathrm{f}a_3+\mathrm{g}e-\mathrm{c}a_2-\mathrm{h}c_1)\,m$$
$$+(\mathrm{h}e+\mathrm{f}a_2-\mathrm{b}a_3-\mathrm{g}b_1)\,n\},$$

whence

$$-l\,\partial^2P/\partial l^2 = 2\,(\mathrm{b}c_1+\mathrm{c}b_1-2\mathrm{f}e)+4\,\{(\mathrm{b}_1c_1+\mathrm{c}_1b_1-2\mathrm{f}_1e)\,l$$
$$+(\mathrm{f}_1a_3+\mathrm{g}_1e-\mathrm{c}_1a_2-\mathrm{h}c_1)\,m+(\mathrm{h}_1e+\mathrm{f}_1a_2-\mathrm{b}_1a_3-\mathrm{g}_1b_1)\,n\}\ldots (67);$$

while, from (66),

$$-m\partial^2P/\partial l\partial m-\partial P/\partial l=2\,(\mathrm{f}e+\mathrm{g}b_3-\mathrm{c}b_1-\mathrm{h}c_2)+4\,\{(\mathrm{b}_2c_2+\mathrm{c}_2b-2\mathrm{f}_2b_3)\,l$$
$$+(\mathrm{f}_2e+\mathrm{g}_2b_3-\mathrm{c}_2b_1-\mathrm{h}_2c_2)\,m+(\mathrm{h}_2b_3+\mathrm{f}_2b_1-\mathrm{b}_2e-\mathrm{g}_2b)\,n_2\},$$

$$-n\partial^2P/\partial l\partial n-\partial P/\partial l = 2\,(\mathrm{h}c_2+\mathrm{f}e-\mathrm{b}c_1-\mathrm{g}b_3)+4\,\{(\mathrm{b}_3c+\mathrm{c}_3b_3-2\mathrm{f}_3c_2)\,l$$
$$+(\mathrm{f}_3c_1+\mathrm{g}c_2-\mathrm{c}_3e-\mathrm{h}_3c)\,m+(\mathrm{h}_3c_2+\mathrm{f}_3e-\mathrm{b}_3c_1-\mathrm{g}_3b_3)\,n\}.$$

From the last three equations,

$$\left.\begin{aligned}
-\partial P/\partial l &= \Sigma\,\{(\mathrm{b}_1c_1+\mathrm{c}_1b_1-2\mathrm{f}_1e)\,l+(\mathrm{f}_1a_3+\mathrm{g}_1e-\mathrm{c}_1a_2-\mathrm{h}_2c_1)\,m\\
&\qquad\qquad+(\mathrm{h}_1e+\mathrm{f}_1a_2-\mathrm{b}_1a_3-\mathrm{g}_1b_1)\,n\}\\
&\text{and, similarly,}\\
-\partial P/\partial m &= \Sigma\,\{\mathrm{f}_1a_3+\mathrm{g}_1e-\mathrm{c}_1a_2-\mathrm{h}_1c_1)\,l+(\mathrm{c}_1a+\mathrm{a}_1c_1-2\mathrm{g}_1a_3)\,m\\
&\qquad\qquad+(\mathrm{g}_1a_2+\mathrm{h}_1a_3-\mathrm{a}_1e-\mathrm{f}_1a)\,n\}\\
-\partial P/\partial n &= \Sigma\,\{(\mathrm{h}_1e+\mathrm{f}_1a_2-\mathrm{b}_1a_3-\mathrm{g}_1b_1)\,l+(\mathrm{g}_1a_3+\mathrm{h}_1a_3-\mathrm{a}_1e-\mathrm{f}_1a)\,m\\
&\qquad\qquad+(\mathrm{a}_1b_1+\mathrm{b}_1a-2\mathrm{h}_1a_2)\,n\}
\end{aligned}\right\}\ \ldots\ldots\ldots (68).$$

To proceed to the second differential coefficients of P: from (67),

$$-l\,\partial^3P/\partial l^3-\partial^2P/\partial l^2 = 8\,(\mathrm{b}_1c_1'+\ldots)+4\,\{(\mathrm{b}_{11}c_1+\ldots)\,l$$
$$+(\mathrm{f}_{11}a_3+\ldots)\,m+(\mathrm{h}_{11}e+\ldots)\,n\};$$

but, (59), $-\partial^2P/\partial l^2 = 2\,(\mathrm{b}_1c_1+\ldots)+4\,\{(\mathrm{b}_{11}c_1+\ldots)\,l$

$$+(\mathrm{f}_{11}a_3+\ldots)\,m+(\mathrm{h}_{11}e+\ldots)n\},$$

so that $$-l\partial^3P/\partial l^3 = 6\,(\mathrm{b}_1c_1+\mathrm{c}_1b_1-2\mathrm{f}_1e)\ \ldots\ldots\ldots\ldots (69).$$

Again, (66),

$$-m\,\partial^3P/\partial l^2\partial m-\partial^2P/\partial l^2=4\,(\mathrm{b}_2c_2+\mathrm{c}_2b-2\mathrm{f}_2b_3)+4\,(\mathrm{f}_1e+\mathrm{g}_1b_3-\mathrm{c}_1b_1-\mathrm{h}_1c_2)$$
$$+4\,\{(\mathrm{b}_{12}c_2+\mathrm{c}_{12}b-2\mathrm{f}_{12}b_3)\,l+(\mathrm{f}_{12}e+\mathrm{g}_{12}b_3-\mathrm{c}_{12}b_1-\mathrm{h}_{12}c_2)\,m+(\mathrm{h}_{12}b_3+\ldots)\,n\},$$

while, (60), $-\partial^2P/\partial l^2=2\,(\mathrm{b}_2c_2+\mathrm{c}_2b-2\mathrm{f}_2b_3)+4\,\{(\mathrm{b}_{12}c_2+\mathrm{c}_{12}b-2\mathrm{f}_{12}b_3)\,l$

$$+(\mathrm{f}_{12}e+\ldots)\,m+(\mathrm{h}_{12}b_3+\ldots)\,n\},$$

so that

$$-m\,\partial^3P/\partial l^2\partial m= 2(\mathrm{b}_2c_2+\mathrm{c}_2\,b-2\mathrm{f}_2b_3)+4\,(\mathrm{f}_1e+\mathrm{g}_1b_3-\mathrm{c}_1b_1-\mathrm{h}_1c_2)\ldots (70).$$

Similarly, (64), (61),

$$-n\partial^3 P/\partial l^2\partial n = 2\,(\mathrm{b}_3 c+\mathrm{c}_3 b_3-2\mathrm{f}_3 c_2)+4\,(\mathrm{h}_1 c_2+\mathrm{f}_1 e-\mathrm{b}_1 c_1-\mathrm{g}_1 b_2) \ldots\text{(7}$$

Adding to this (69), (70), there results

$$-\frac{\partial^2 P}{\partial l^2}=2(\mathrm{b}_1 c_1+\mathrm{c}_1 b_1-2\mathrm{f}_1 e)+2(\mathrm{b}_2 c_2+\mathrm{c}_2 b-2\mathrm{f}_2 b_3)+2(\mathrm{b}_3 c+\mathrm{c}_3 b_3-2\mathrm{f}_3 c_2)$$

Analogously

$$-\frac{\partial^2 P}{\partial m^2}=2(\mathrm{c}_1 a+\mathrm{a}_1 c-2\mathrm{g}_1 a_3)+2(\mathrm{c}_2 a_2+\mathrm{a}_2 c_2-2\mathrm{g}_2 e)+2(\mathrm{c}_3 a_3+\mathrm{a}_3 c-2\mathrm{g}_3 c_1)$$

$$-\frac{\partial^2 P}{\partial n^2}=2(\mathrm{a}_1 b_1+\mathrm{b}_1 a-2\mathrm{h}_1 a_2)+2(\mathrm{a}_2 b+\mathrm{b}_2 a_2-2\mathrm{h}_2 b_1)+2(\mathrm{a}_3 b_3+\mathrm{b}_3 a_3-2\mathrm{h}_3 e)$$

$$\ldots\ldots\ldots\text{(71}$$

For the other second differential coefficients of P, a like meth might be employed; but, as rather shorter, and to illustrate the use the equations of which (65) is a type and which may be written brie

$$[s_3,\,u_3]=[s_3,\,u_2]=[s_1,\,u_3]=[s_3,\,u_1]=[s_2,\,u_1]=[s_1,\,u_2]\equiv 0,$$

I proceed as follows. From (59)

$$-\partial^2 P/\partial m\partial n=2\,(\mathrm{g}_1 a_2+\mathrm{h}_1 a_3-\mathrm{a}_1 e-\mathrm{f}_1 a)+2\,\{(\mathrm{f}_{31} a_3+\ldots)+(\mathrm{h}_{12} e+\ldots)$$

$$+2\,\{(\mathrm{c}_{31} a+\ldots)+(\mathrm{g}_{12} a_2+\ldots)\}\,m+2\,\{(\mathrm{g}_{31} a_3+\ldots)+(\mathrm{a}_{12} b_1+\ldots)\}$$

$$\ldots\ldots\ldots\ldots\text{(72}$$

But, identically,

$$2\,(\mathrm{f}_{31} a_3+\mathrm{g}_{31} e-\mathrm{c}_{31} a_3-\mathrm{h}_{31} c_1)=-2\,(\mathrm{c}_{31} a_2+\mathrm{a}_{31} c_3-2\mathrm{g}_{31} e_3)$$

$$-2\,(\mathrm{g}_{31} e+\mathrm{h}_{31} c_1-\mathrm{a}_{31} c_3-\mathrm{f}_{31} a_3$$

and the coefficient of m^2n in $[s_1,\,u_2]\equiv 0$ gives

$$-2\,(c_{31} a_3+\mathrm{a}_{31} c_3-2\mathrm{g}_{31} e)=4\,(\mathrm{g}_{12} b_1+\mathrm{h}_{12} e-\mathrm{a}_{12} b_3-\mathrm{f}_{12} a_2)\,;$$

also, identically,

$$2\,(\mathrm{h}_{12} e+\mathrm{f}_{12} a_2-\mathrm{b}_{12} a_3-\mathrm{g}_{12} b_1)=-2\,(\mathrm{a}_{12} b_3+\mathrm{b}_{12} a_3-2\mathrm{h}_{12} e)$$

$$-2\,(\mathrm{g}_{12}\mathrm{b}_1+\mathrm{h}_{12} e-\mathrm{a}_{12}\mathrm{b}_3-\mathrm{f}_{12} a_2$$

and the coefficient of mn^2 in $[s_1,\,u_3]\equiv 0$, gives

$$-2\,(\mathrm{a}_{12}\mathrm{b}_3+\mathrm{b}_{12} a_3-2\mathrm{h}_{12} e)=4\,(\mathrm{g}_{31} e+\mathrm{h}_{31} c_1-\mathrm{a}_{31} c_3-\mathrm{f}_{31} a_3).$$

From the addition of the preceding four equalities,

$$2\left\{(\mathrm{f}_{31}a_3+\ldots)+(\mathrm{h}_{12}e+\ldots)\right\} = 2\left\{(\mathrm{g}_{12}b_1+\mathrm{h}_{12}e-\mathrm{a}_{12}b_3-\mathrm{f}_{12}a_2)\right.$$
$$\left.+(\mathrm{g}_{31}e+\mathrm{h}_{31}c_1-\mathrm{a}_{31}c_2-\mathrm{f}_{31}a_3)\right\}\ldots\ldots\ldots\ldots(73).$$

Again, from the vanishing of the coefficient of lm^2 in $[s_3, u_1]\equiv 0$,

$$\begin{aligned}\mathrm{c}_{31}a+\mathrm{a}_{31}c_1-2\mathrm{g}_{31}a_3 &= -2\,(\mathrm{f}_{23}a_3+\mathrm{g}_{23}e-\mathrm{c}_{23}a_2-\mathrm{h}_{23}c_1)\\ &= \;\;2\,(\mathrm{g}_{23}e+\mathrm{h}_{23}c_1-\mathrm{a}_{23}c_2-\mathrm{f}_{23}a_3)\\ &\quad +2\,(\mathrm{c}_{23}a_2+\mathrm{a}_{23}c_2-2\mathrm{g}_{23}e),\end{aligned}$$

which latter term, being the coefficient of m^3 in $[s_3, u_2]\equiv 0$, vanishes; while equating coefficients of m^2n in $[s_1, u_1]=[s_2, u_2]=P$,

$$\mathrm{c}_{31}a+\mathrm{a}_{31}c-2\mathrm{g}_{31}a_3+2\,(\mathrm{g}_{12}a_2+\mathrm{h}_{12}a_3-\mathrm{a}_{12}e-\mathrm{f}_{12}a)$$
$$=\mathrm{c}_{23}a_2+\mathrm{a}_{23}c_2-2\mathrm{g}_{23}e+2\,(\mathrm{g}_{22}b_1+\mathrm{h}_{22}e-\mathrm{a}_{22}\mathrm{b}_3-\mathrm{h}_{22}a_2).$$

of which $\mathrm{c}_{23}a_2+\ldots$ vanishes identically, as pointed out just above.

Hence

$$2\,(\mathrm{c}_{31}a+\mathrm{a}_{31}c-2\mathrm{g}_{31}a_3)+2\,(\mathrm{g}_{12}a_2+\mathrm{h}_{12}a_3-\mathrm{a}_{12}e-\mathrm{f}_{12}a)$$
$$=2\,(\mathrm{g}_{22}b_1+\mathrm{h}_{22}e-\mathrm{a}_{22}b_3-\mathrm{h}_{22}a_2)+2\,(\mathrm{g}_{23}e+\mathrm{h}_{23}c_1-\mathrm{a}_{23}c_2-\mathrm{f}_{23}\mathrm{a}_3)\ldots(74).$$

Similarly, by interchange of letters and suffixes, the coefficient of n (explicit) in $-\partial^2P/\partial m\partial n$ above is shown to be equal to

$$2\,(\mathrm{g}_{23}b_1+\mathrm{h}_{23}e-\mathrm{a}_{23}b_3-\mathrm{h}_{23}a_2)+2\,(\mathrm{g}_{33}e+\mathrm{h}_{33}c_1-\mathrm{a}_{33}c_2-\mathrm{f}_{33}a_3)\ldots(75).$$

Adding equations (73), (74), (75), multiplied by l, m, n respectively, to (72), there results

$$\left.\begin{aligned}-\partial^2P/\partial m\partial n &= 2\,(\mathrm{g}_1a_2+\mathrm{h}_1a_3-\mathrm{a}_1e-\mathrm{f}_1a)+2\,(\mathrm{g}_2b_1+\mathrm{h}_2e-\mathrm{a}_2b_3-\mathrm{f}_2a_2)\\ &\quad+2\,(\mathrm{g}_3e+\mathrm{h}_3c_1-\mathrm{a}_3c_2-\mathrm{f}_3a_3)\\ &\text{Analogously}\\ -\partial^2P/\partial n\partial l &= 2\,(\mathrm{h}_1e+\mathrm{f}_1a_2-\mathrm{b}_1a_3-\mathrm{g}_1b_1)+2\,(\mathrm{h}_2b_3+\mathrm{f}_2b_1-\mathrm{b}_2e-\mathrm{g}_2b)\\ &\quad+2\,(\mathrm{h}_3c_2+\mathrm{f}_3e-\mathrm{b}_3c_1-\mathrm{g}_3b_3),\\ -\partial^2P/\partial l\partial m &= 2\,(\mathrm{f}_1a_3+\mathrm{g}_1e-\mathrm{c}_1a_2-\mathrm{h}_1c_1)+2\,(\mathrm{f}_2e+\mathrm{g}_2b_3-\mathrm{c}_2b_1-\mathrm{h}_2c_2)\\ &\quad+2\,(\mathrm{f}_3c_1+\mathrm{g}_3c_2-\mathrm{c}_3e-\mathrm{h}_3c)\end{aligned}\right\}\ldots\ldots(76).$$

Since

$$-(l^2\partial^2P/\partial l^2+\ldots+2mn\partial^2P/\partial m\partial n+\ldots)=-6P,$$

which, by (59), (60), (61),

$$= 2l^2\Sigma(b_1c_1+\ldots)+\ldots+4mn\Sigma(g_1a_2+\ldots)+\ldots,$$

the values for the second differential coefficients of P, above in gated, might have been anticipated from, but are not necessaril plied in (59), (60), (61); viz., they result only from the special v of a ... f ... as functions of the coefficients of the cubic. It ma remarked that some of the coefficients of $[s_1, u_1]$ ... have inti relations of interest, but not necessary to the above investigat thus, considering the vanishing coefficients of l^2m in $[s_2, u_1]$ an in $[s_3, u_1]$,

$$2(f_{31}a_3+g_{31}e-c_{31}a_2-h_{31}c_1) = 2(h_{12}e+f_{12}a_2-b_{12}a_3-g_{12}b_1)\ldots\ldots($$

viz., each is equal to

$$-b_{23}c_1-c_{23}b_1+2f_{23}e;$$

but in what has preceded I have confined myself to results wh have found essential.

It has been shown that, for values of xyz satisfying L (45), (51), (52),

$$-\Delta^3 Q = 6\Sigma(U_1\mathfrak{h}_1+V_1D\mathfrak{h}_1+W_1D^2\mathfrak{h}_1),$$

$3\mathfrak{h}_1$, $3\mathfrak{h}_2$, $3\mathfrak{h}_3$ being the first differential coefficients of $\mathfrak{h}$, the He of u, say,

$$\mathfrak{h} = \mathfrak{a}x^3+\mathfrak{b}y^3+\mathfrak{c}z^3+\ldots+6\mathfrak{e}xyz;$$

hence, precisely as for P,

$$-Q = 6\Sigma\{(\mathfrak{b}_1c_1+\mathfrak{c}_1b_1-2\mathfrak{e}f_1)l^2+\ldots\}\ \ldots\ldots\ldots\ldots($$

In the next place I investigate some transformations I have fo useful, involving the coefficients of the reciprocal

$$A\xi^2+\ldots+2F\eta\zeta+\ldots \equiv \sigma,$$

of
$$s \equiv (bc-f^2)l^2+\ldots+2(gh-af)mn+\ldots$$
$$\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

a ..., f ... being $\dfrac{\partial^2u}{\partial x^2}\Big/6\ldots$, $\dfrac{\partial^2u}{\partial y\partial z}\Big/6$; u a ternary cubic form.

The values of a ... f ... in terms of the coefficients of u are g in full, *Phil. Trans.*, 1888, A., p. 157, where they are represented $u_{11}\ldots u_{23}/2\ldots$; but it is sufficient here to state that

$$a = \text{reciprocal of } \frac{\partial u}{\partial x}\Big/3 \text{ or } u_1\ldots,$$

$2\mathrm{f} =$ contravariant quadric of $\dfrac{\partial u}{\partial y}\Big/3$ or u_2, and $\dfrac{\partial u}{\partial z}\Big/3$ or u_3 ....

Hence, by (38), *for values of xyz satisfying* $lx+my+nz=0$,

$$\Delta^2\mathrm{a} = u_1D^2u_1-(Du_1)^2 \quad \text{.......................(79)},$$

and, by (37), $\quad 2\Delta^2\mathrm{f} = u_2D^2u_3+u_3D^2u_2-2Du_2Du_3$ ....(80);

and $\quad \Delta^4A \equiv \{u_2D^2u_2-(Du_2)^2\}\{u_3D^2u_3-(Du_3)^2\}$

$$-(u_2D^2u_3+u_3D^2u_2-2Du_2Du_3)^2/4,$$

or (48, 51, 52) $\quad 4\Delta^4A \equiv 4W_1U_1-V_1^2$ .... (81).

But (48) $\quad W_1 = \Delta s_1 - L\,(as_{11}+\ldots)$,

and (50) $\quad V_1 = -\,3DW_1 = -\,2\Delta Ds_1 - L\,(\ldots)$,

$$U_1 = \;\; 3D^2W_1 = \Delta D^2s_1 - L\,(\ldots),$$

so that (81) for values of xyz, satisfying $L=0$,

$$\Delta^2A = s_1D^2s_1-(Ds_1)^2,$$

or (38) $\quad A = (\mathrm{b}_1\mathrm{c}_1-\mathrm{f}_1^2)\,l^2+\ldots+2\,(\mathrm{g}_1\mathrm{h}_1-\mathrm{a}_1\mathrm{f}_1)\,mn+\ldots$ ...(82).

Similarly, $\quad 4\Delta^4F = 2W_2U_3+2W_3U_2-V_2V_3$,

$$2\Delta^2F = s_2D^2s_3+s_3D^2s_2-2Ds_2Ds_3,$$

or, by (37),

$$2F = (\mathrm{b}_2\mathrm{c}_3+\mathrm{b}_3\mathrm{c}_2-2\mathrm{f}_2\mathrm{f}_3)\,l^2+\ldots+2\,(\mathrm{g}_2\mathrm{h}_3+\mathrm{g}_3\mathrm{h}_2-\mathrm{a}_2\mathrm{f}_3-\mathrm{a}_3\mathrm{f}_2)\,mn+\ldots$$

..............(83).

In these formulæ (82, 83), as above,

$$s_1 = \frac{\partial s}{\partial l}\Big/2 \ldots = \mathrm{a}_1x^2+\ldots,\quad s_2 = \frac{\partial s}{\partial m}\Big/2 = \mathrm{a}_2x^2+\ldots,\quad s_3 = \ldots.$$

From (82), (83), and the analogous formulæ,

$$lA+mH+nG = \{(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)\,l^2+\ldots$$

$$\ldots+2\,(\mathrm{gh}_1+\mathrm{hg}_1-\mathrm{af}_1-\mathrm{fa}_1)\,mn+\ldots\}/2 \ldots(84),$$

with two other analogous formulæ.

Now, by definition,

$$A = \mathrm{bc}-\mathrm{f}^2 \ldots\; F = \mathrm{gh}-\mathrm{af} \ldots\; \mathrm{b}_1 = \frac{\partial \mathrm{b}}{\partial l}\Big/2 \ldots,$$

so that (84) may otherwise be written

$$4\,(lA+mH+nG) = l^2\frac{\partial A}{\partial l}+\ldots+2mn\frac{\partial F}{\partial l}+\ldots \quad \text{......(85)},$$

with similar values for

$$4(lH+mB+nF),\quad 4(lG+mF+nC).$$

Differentiating (84) with respect to l, and attending to (82), to the fact, resulting from definition,

$$\frac{l}{2}\frac{\partial A}{\partial l}=(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)\,l,\quad \frac{m}{2}\frac{\partial H}{\partial l}=(\mathrm{fg}_1+\mathrm{gf}_1-\mathrm{ch}_1-\mathrm{hc}_1)\,m,$$

$$n\frac{\partial G}{\partial l}=(hf_1+\ldots)\,n,$$

$$l\frac{\partial A}{\partial l}+m\frac{\partial H}{\partial l}+n\frac{\partial G}{\partial l}=2A+\{(\mathrm{bc}_{11}+\mathrm{cb}_{11}-2\mathrm{ff}_{11})\,l^2+\ldots\}\ldots\ldots(8$$

Analogously,

$$l\frac{\partial H}{\partial m}+m\frac{\partial B}{\partial m}+n\frac{\partial F}{\partial m}=2B+\{(\mathrm{bc}_{22}+\mathrm{cb}_{22}-2\mathrm{ff}_{22})\,l^2+\ldots\}\ldots\ldots(8$$

$$l\frac{\partial G}{\partial n}+m\frac{\partial F}{\partial n}+n\frac{\partial C}{\partial n}=2C+\{(\mathrm{bc}_{33}+\mathrm{cb}_{33}-2\mathrm{ff}_{33})\,l^2+\ldots\}\ldots\ldots(8$$

Again, differentiating (84) with respect to m, or the analog formula for $lH+mB+nF$ with respect to l, either

$$l\frac{\partial A}{\partial m}+m\frac{\partial H}{\partial m}+n\frac{\partial G}{\partial m},$$

or $$l\frac{\partial H}{\partial l}+m\frac{\partial B}{\partial l}+n\frac{\partial F}{\partial l}=2H+\{(\mathrm{bc}_{12}+\mathrm{cb}_{12}-2\mathrm{ff}_{12})\,l^2+\ldots\}\ldots\ldots(8$$

and analogously

$$l\frac{\partial A}{\partial n}+m\frac{\partial H}{\partial n}+n\frac{\partial G}{\partial n},$$

or $$l\frac{\partial G}{\partial l}+m\frac{\partial F}{\partial l}+n\frac{\partial C}{\partial l}=2G+\{(\mathrm{bc}_{31}+\mathrm{cb}_{31}-2\mathrm{ff}_{31})\,l^2+\ldots\}\ldots\ldots(9$$

$$l\frac{\partial G}{\partial m}+m\frac{\partial F}{\partial m}+n\frac{\partial C}{\partial m},$$

or $$l\frac{\partial H}{\partial n}+m\frac{\partial B}{\partial n}+n\frac{\partial F}{\partial n}=2F+\{(\mathrm{bc}_{23}+\mathrm{cb}_{23}-2\mathrm{ff}_{23})\,l^2+\ldots\}\ldots\ldots(9$$

It is to be observed that the dexters of the six equations last prov

are equal to, for values of xyz satisfying $lx+my+nz=0$,

$$2\left\{s_1 D^2 s_1-(Ds_1)^2\right\}/\Delta^2+(sD^2 s_{11}+s_{11}D^2 s-2DsDs_{11})/\Delta^2 \quad \ldots(92),$$

$$\ldots(s_2 D^2 s_3+s_3 D^2 s_2-2Ds_2 Ds_3)/\Delta^2+(sD^2 s_{23}+s_{23}D^2-2DsDs_{23})/\Delta^2 \ldots(93),$$

wherein $\quad s_1=\dfrac{\partial s}{\partial l}\Big/2\,\ldots,\quad s_{11}=\left\{\dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial^2 u}{\partial z^2}-\left(\dfrac{\partial^2 u}{\partial y\partial z}\right)^2\right\}\Big/6\,\ldots,$

$$s_{23}=\left\{\frac{\partial^2 u}{\partial z\partial x}\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y\partial z}\right\}\Big/6\,\ldots,$$

and
$$s=l^2 s_{11}+\ldots 2mns_{23}+\ldots,$$

$$\Delta=\alpha x+\beta y+\gamma z,$$

$$2Ds=(m\gamma-n\beta)\frac{\partial s}{\partial x}+\ldots,$$

$$2.1.D^2 s=(m\gamma-n\beta)^2\frac{\partial^2 s}{\partial x^2}+\ldots.$$

The equality of the pairs of formulæ

$$l\frac{\partial A}{\partial m}+m\frac{\partial H}{\partial m}+n\frac{\partial G}{\partial m},\quad l\frac{\partial H}{\partial l}+m\frac{\partial B}{\partial l}+n\frac{\partial F}{\partial l}\,\ldots,$$

which is apparent from the double formula (89) above, may be verified for any special forms of u.

For the canonical form, *e.g.*,

$$ax^3+by^3+cz^3+6exyz,$$

the actual value of either of the first pair is found to be

$$\left\{c\,(abc+20e^3)\,l^3m^2+e^2\,(7bcl^3+7cam^3+abn^3)\,n+6e\,(abc+2e^3)\,lmn^2\right\}/2.$$

The differential coefficients of v, the reciprocal of u, may now be readily obtained in terms of a … f …, the coefficients of s (p. 498).

For, since (p. 489)

$$v=-4\,(Al^2+\ldots+2Fmn+\ldots),$$

$$\frac{\partial v}{\partial l}=-8\,(Al+Hm+Gn)-4\left(l^2\frac{\partial A}{\partial l}+\ldots+2mn\frac{\partial F}{\partial l}+\ldots\right),$$

$$\left.\begin{array}{ll}\text{or (85)} & \dfrac{1}{6}\dfrac{\partial v}{\partial l}=-4\,(Al+mH+nG)\\ \textit{i.e.}\text{ (84)} & \qquad=-2\left\{(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)\,l^2+\ldots\right\}\\ \multicolumn{2}{l}{\text{or (37), for values of } xyz \text{ which satisfy } lx+my+nz=0}\\ & \qquad=-2\,(sD^2 s_1+s_1D^2 s-2DsDs_1)/\Delta^2\end{array}\right\}\ldots\ldots\ldots(94).$$

$$\left.\begin{aligned}\text{Again,}\quad A' &= \frac{1}{6}\frac{\partial^3 v}{\partial l^3} = -4\left(A + l\frac{\partial A}{\partial l} + m\frac{\partial H}{\partial l} + n\frac{\partial G}{\partial l}\right)\\ \text{or (86)}\quad A' + 12A &= -4\{(\mathrm{bc}_{11} + \mathrm{cb}_{11} - 2\mathrm{ff}_{11})\, l^2 + \dots\}\\ \textit{i.e.}\ (37)\quad &= -4\,(sD^2s_{11} + s_{11}D^2s - 2Ds\,Ds_{11}) / \Delta^{2*}\end{aligned}\right\}\dots($$

Similarly,

$$\left.\begin{aligned}\frac{1}{6}\frac{\partial^3 v}{\partial m^3} = B' &= -4\left(B + l\frac{\partial H}{\partial m} + m\frac{\partial B}{\partial m} + n\frac{\partial F}{\partial m}\right)\\ \text{or (87)}\quad B' + 12B &= -4\{(\mathrm{bc}_{22} + \mathrm{cb}_{22} - 2\mathrm{ff}_{22})\, l^2 + \dots\}\\ \textit{i.e.}\ (37)\quad &= -4\,(sD^2s_{22} + s_{22}D^2s - 2Ds\,Ds_{22}) / \Delta^{2*}\end{aligned}\right\}\dots\dots($$

$$\left.\begin{aligned}\frac{1}{6}\frac{\partial^3 v}{\partial l\,\partial m} = H' &= -4\left(H + l\frac{\partial A}{\partial m} + m\frac{\partial H}{\partial m} + n\frac{\partial G}{\partial m}\right)\\ &= -4\left(H + l\frac{\partial H}{\partial l} + m\frac{\partial B}{\partial l} + n\frac{\partial F}{\partial l}\right)\\ \text{or}\quad H' + 12H &= -4\{(\mathrm{bc}_{12} + \mathrm{cb}_{12} - 2\mathrm{ff}_{12})\, l^2 + \dots\}\\ \textit{i.e.}\ (38)\quad &= -4\,(sD^2s_{12} + s_{12}D^2s - 2Ds\,Ds_{12}) / \Delta^{2*}\end{aligned}\right\}\dots\dots(9$$

From the preceding group of formulæ, observing that

$$s_{11} = \frac{\partial \mathfrak{h}}{\partial a} \dots 2s_{12} = \frac{\partial \mathfrak{h}}{\partial h},$$

$$\Sigma\,(A' + 12A)\,a + 2\Sigma\,(F' + 12F)\,f = -12\,(sD^2\mathfrak{h} + \mathfrak{h}D^2s - 2Ds\,D\mathfrak{h}) / \Delta$$

$$\dots\dots\dots\dots\dots(9$$

$$= -12\,\{(\mathrm{bc}' + \mathrm{cb}' + 2\mathrm{f}f')\, l^2 + \dots\}$$

if $\qquad 6a' = \partial^2\mathfrak{h}/\partial x^2 \dots,\quad 6f' = \partial^2\mathfrak{h}/\partial y\partial z \dots.$

Such transformations as those given (47) ... (98), might be multipli almost indefinitely, but I here confine myself to those which I ha hitherto found to have useful applications in the employment of t Method.

* For values of xyz satisfying $lx + my + nz = 0$ only.

June 14*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Dr. C. Taylor read a paper "On the determination of the Circular Points at Infinity."

Prof. Hill followed with a paper "On the c- and p- Discriminants of Integrable Differential Equations of the first order."

The Secretary communicated papers by Lord Rayleigh, "On Point-, Line-, and Plane-Sources of Sound"; by Mr. Fortey, "Note on Rationalisation"; by Prof. G. B. Mathews, "Applications of Elliptic Functions to the Theory of Twisted Quartics."

Prof. Greenhill communicated remarks on "Coefficients of Induction and Capacity, and allied Problems" (in continuation of a former paper).

The following papers were taken as read:—

"Electrical Oscillations," Prof. J. J. Thomson;

and a demonstration of the theorem "that the equation

$$x^3+y^3+z^3 = 0$$

cannot be solved in integers," Mr. J. R. Holt.

The following presents were received:—

Cabinet Likeness of Mr. E. B. Elliott.

"Proceedings of the Royal Society," Vol. XLIII., No. 265; Vol. XLIV., Nos. 266 and 267.

"Educational Times" for June.

"On Integration by means of selected Values of the Function," by W. S. B. Woolhouse (extracted from Vol. XXVIII. of the "Journal of the Institute of Actuaries,") 8vo pamphlet.

"Proceedings of the Canadian Institute," Toronto, No. 149, April, 1888.

"Annual Report of the Canadian Institute," Session 1886—1887; Toronto, 1888.

"Annals of Mathematics," Vol. 4, No. 1.

"Bulletin des Sciences Mathématiques," Tome XII., May, 1888.

"Atti della Reale Accademia dei Lincei — Rendiconti," Vol. IV., Fasc. 2, 3, and 4; 1888.

"Reale Istituto Lombardo di Scienze e Lettere — Rendiconti," Serie II., Vols. XVIII. and XIX.; Memorie, Vol. XV., Fasc. IV.; Vol. XVI., Fasc. I.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," No. 57.

"Annali di Matematica," Serie II., Tomo XVI., Fasc. I.

"Acta Mathematica," XI., 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band, XII., Stück, 5, 1888.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," I.—XX.

"Annales de la Faculté des Sciences de Toulouse," Tome II., Année 1888.

"Memorias de la Sociedad Cientifica—Antonio Alzate," Tomo I., Nos. 9 and 10.

On Point-, Line-, and Plane-Sources of Sound.

By Lord RAYLEIGH, F.R.S.

[*Read June 14th*, 1888.]

The velocity-potential at a distance ρ from a simple source of sound is*

$$\phi = \frac{\Phi_1 e^{ik(at-\rho)}}{4\pi a^2 \rho} \quad \text{.............................} \quad (1),$$

where $-a^{-2}\Phi_1 e^{ikat}$ represents the rate at which fluid is being introduced at the source at time t. In order to apply this to a linear source of unit intensity, coincident with the axis of y, we have to imagine that the introduction of fluid along the element dy is equal to $dy\,e^{ikat}$; so that, if for the sake of brevity we omit the time factor e^{ikat}, we may take as the velocity potential

$$\phi = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{e^{-ik\rho}\,dy}{\rho} \quad \text{........................} \quad (2).$$

If r be the distance of the point at which ϕ is to be estimated from the axis of y,

$$\rho^2 = r^2 + y^2,$$

and

$$\phi = -\frac{1}{2\pi}\int_r^{\infty} \frac{e^{-ik\rho}\,d\rho}{\sqrt{(\rho^2 - r^2)}} = -\frac{1}{2\pi}\int_1^{\infty} \frac{e^{-ikrv}\,dv}{\sqrt{(v^2-1)}} \quad \text{.........} \quad (3),$$

if $\rho = rv$.

The relation of (3) to Bessel's functions is best studied by the method of Lipschitz.† Consider the integral $\int \frac{e^{-rw}\,dw}{\sqrt{(1+w^2)}}$, where w is a complex variable of the form $u+iv$. If we represent, as usual, simultaneous pairs of values of u and v by the coordinates of a point, the integral will vanish when taken round any closed circuit not including the points $w = \pm i$. The first circuit we have to consider is that enclosed by the axes of u and v, and the quadrant of a circle whose centre is the origin and whose radius is infinite. It is easy to see that along this quadrant the integral

* *Theory of Sound*, § 277.

† Crelle, Bd. LVI., 1859.

ultimately vanishes, so that the result is the same whether we integrate from 0 to ∞ along the axis of u or from 0 to $i\infty$ along the axis of v. Thus

$$\int_0^\infty \frac{e^{-ru}\,du}{\sqrt{(1+u^2)}} = \int_0^{i\infty} \frac{e^{-r(iv)}\,d(iv)}{\sqrt{(1+i^2v^2)}} = i\int_0^1 \frac{e^{-irv}\,dv}{\sqrt{(1-v^2)}} + \int_1^\infty \frac{e^{-irv}\,dv}{\sqrt{(v^2-1)}} \quad \ldots\ (4).$$

In like manner, the integral along the axis of u from 0 to ∞ is equal to that along the course from 0 to i along the axis of v, and then to infinity along a line through i parallel to u. Thus

$$\int_0^\infty \frac{e^{-ru}\,du}{\sqrt{(1+u^2)}} = \int_0^i \frac{e^{-irv}\,d(iv)}{\sqrt{(1-v^2)}} + \int_0^\infty \frac{e^{-r(u+i)}\,du}{\sqrt{\{1+(u+i)^2\}}}$$

$$= i\int_0^1 \frac{e^{-irv}\,dv}{\sqrt{(1-v^2)}} + \int_0^\infty \frac{e^{-ir}\,e^{-ru}\,du}{\sqrt{(2iu+u^2)}} \quad \ldots\ldots\ldots\ldots\ (5).$$

By comparison of (4), (5), or at once by equating the results of integrating from the point i to $i\infty$, and to $\infty+i$, we get

$$\int_1^\infty \frac{e^{-irv}\,dv}{\sqrt{(v^2-1)}} = \int_0^\infty \frac{e^{-ir}\,e^{-ru}\,du}{\sqrt{(2iu+u^2)}} = \frac{e^{-ir}}{\sqrt{(2ir)}}\int_0^\infty \frac{e^{-\beta}\,\beta^{-\frac{1}{2}}\,d\beta}{\sqrt{\left(1+\frac{\beta}{2ir}\right)}}$$

$$= \left(\frac{\pi}{2ir}\right)^{\frac{1}{2}} e^{-ir}\left\{1-\frac{1^2}{1.8ir}+\frac{1^2.3^2}{1.2.(8ir)^2}-\frac{1^2.3^2.5^2}{1.2.3(8ir)^3}+\ldots\right\} \ldots\ldots\ (6).$$

This is the series in descending powers of r by which is expressed the effect of a linear source at a great distance.

Equation (4) may be written in the form

$$\int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+r^2)}} = i\int_0^{\frac{1}{2}\pi} e^{-ir\cos\theta}\,d\theta + \int_0^1 \frac{e^{-irv}\,dv}{\sqrt{(v^2-1)}};$$

or, if we put, as usual,

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(r\cos\theta)\,d\theta = J_0(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (7),$$

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(r\cos\theta)\,d\theta = K_0(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (8),$$

and separate the real and imaginary parts,

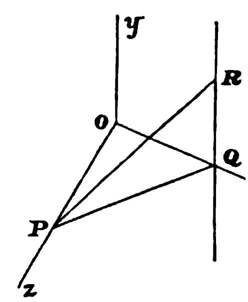
$$\int_0^1 \frac{\cos(rv)\,dv}{\sqrt{(v^2-1)}} = \int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+r^2)}} - \frac{\pi}{2}.K_0(r) \quad \ldots\ldots\ldots\ (9),$$

$$\int_0^1 \frac{\sin(rv)\,dv}{\sqrt{(v^2-1)}} = \frac{\pi}{2}\,.\,J_0(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1$$

the latter giving Mehler's integral expressive of the Bessel's funct of order zero.*

By integrating the effect of a linear source, parallel to y, w respect to a perpendicular coordinate x, we may obtain the effect o source uniformly distributed over a plane. If the rate of introducti of fluid over the area $dx\,dy$ be $dx\,dy\,e^{ikat}$, the value of ϕ at a poi distant z from the plane, will be found by integrating (3) with respect to x, connected with r and z by the relation

$$r^2 = z^2 + x^2\,;$$



see Fig., in which

$$RQ = y, \quad PR = \rho, \quad OQ = x,$$

$$PQ = r, \quad OP = z.$$

Thus

$$\phi = -\frac{1}{\pi}\int_0^\infty dx \int_1^\infty \frac{e^{-ikrv}\,dv}{\sqrt{(v^2-1)}}$$

$$= -\frac{1}{\pi}\int_z^\infty \frac{r\,dr}{\sqrt{(r^2-z^2)}}\int_1^\infty \frac{e^{-ikrv}\,dv}{\sqrt{(v^2-1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11$$

The result of a uniform plane source is of course a train of pla waves issuing from it symmetrically in both directions. On t positive side $\phi = Ae^{-ikz}$, where A is a constant readily determine For $\frac{d\phi}{dz}\,(z=0) = -ikA$; and this, representing the half of t rate of introduction of fluid per unit area, is by supposition equ to $\frac{1}{2}$. Thus

$$\phi = \frac{i}{2k}e^{-ikz} = \frac{i}{2k}\cos kz + \frac{1}{2k}\sin kz \quad \ldots\ldots\ldots\ldots \quad (12$$

Comparing the two expressions for ϕ, and having regard to (9) a (10), we see that

$$\int_z^\infty \frac{J_0(kr)\,r\,dr}{\sqrt{(r^2-z^2)}} = \frac{\cos kz}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13$$

$$\int_z^\infty \frac{r\,dr}{\sqrt{(r^2-z^2)}}\left\{\int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+k^2r^2)}} - \frac{\pi}{2}\,.\,K_0(kr)\right\} = \frac{\sin kz}{k} \quad \ldots \quad (14$$

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* Math. Ann. v., p. 141.

If we use the series (6), the identity may be written

$$-\frac{1}{\pi}\int_z^\infty \frac{r\,dr}{\sqrt{(r^2-z^2)}}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1-\frac{1^2}{1\,.\,8\,.\,ikr}+\dots\right\} = \frac{i}{2k}\,e^{-ikz} \;\dots\;(15).$$

This equation is easily verified when kz (and therefore kr) is great. Under these circumstances the series may be replaced by its first term; also with sufficient approximation

$$\frac{\sqrt{r}}{\sqrt{(r^2-z^2)}} = \frac{1}{\sqrt{2}\,.\,\sqrt{(r-z)}},$$

since only those elements for which r differs little from z contribute sensibly to the integral.

Some Applications of Elliptic Functions to the Theory of Twisted Quartics. *By* Professor G. B. Mathews.

[*Read June 14th,* 1888.]

The general theory of curves whose coordinates may be expressed by elliptic functions of a parameter is given by Clebsch (*Crelle*, t. 64, pp. 210—270), and Klein ("Ueber die Elliptischen Normalcurven der n^{ten} Ordnung," etc.; Leipzig, 1885). The special application to twisted quartics has been discussed by Laguerre (*Liouville*, 2nd series, t. 15, p. 193), Harnack (*Math. Annalen*, tt. 12, 15), Westphal (*ibid.*, t. 13), Cayley (*ibid.*, t. 25), and Lange (*Schlömilchs Zeitschrift*, t. 28, pp. 1, 65).

The notation adopted in this paper is that of Halphen's "Traité des Fonctions elliptiques et de leurs Applications": viz., $\wp u$ is Weierstrass's elliptic function, defined by the differential equation

$$(\wp' u)^2 = 4\wp^3 u - g_2\,\wp u - g_3,$$

and the periods of $\wp u$ are denoted by 2ω, $2\omega'$.

1. Suppose the homogeneous coordinates of a point in space are given by

$$x : y : z : t = \wp' u : \wp^2 u : \wp u : 1,$$

where u is a variable parameter.

(This is Klein's "canonical representation" of the quartic: see memoir above quoted, § 7).

The locus of (x, y, z, t) is obtained as the complete intersectio the surfaces

$$S \equiv x^2 - 4yz + g_2 zt + g_3 t^2 = 0,$$

and
$$V \equiv z^2 - yt = 0,$$

$V = 0$ is one of the four cones which contain the quartic: $S =$ may be called its associated quadric, and is defined by the circu stance that the Φ-invariant of S and V vanishes.

2. The discriminant of $S - \lambda V$ is

$$4(\lambda^3 - 4g_2\lambda - 16g_3) = 4(\lambda - 4e_1)(\lambda - 4e_2)(\lambda - 4e_3):$$

so that the equations of the other three quadric cones which cont the quartic are

$$S - 4e_1 V = 0, \quad S - 4e_2 V = 0, \quad S - 4e_3 V = 0.$$

The coordinates of the vertices of the four cones are given by

$$\begin{aligned} x : y : z : t &= 1 : 0 : 0 : 0, \\ &\ 0 : e_2 e_1 + e_1 e_3 - e_2 e_3 : e_1 : 1, \\ &\ 0 : e_1 e_2 + e_2 e_3 - e_3 e_1 : e_2 : 1, \\ &\ 0 : e_2 e_3 + e_3 e_1 - e_1 e_2 : e_3 : 1. \end{aligned}$$

Hence the equations of the planes of which each contains th vertices are

$$X = 0, \quad Y = 0, \quad Z = 0, \quad T = 0,$$

where
$$\begin{aligned} \mu X &= x, \\ \mu Y &= y - 2e_1 z - (e_1^2 + e_2 e_3)\, t, \\ \mu Z &= y - 2e_2 z - (e_2^2 + e_3 e_1)\, t, \\ \mu T &= y - 2e_3 z - (e_3^2 + e_1 e_2)\, t, \end{aligned}$$

μ being an arbitrary multiplier, which may, when convenient, put $= 1$.

Taking the tetrahedron formed by these planes as a new tetrahedı of reference, we obtain Klein's "singular" representation of t

quartic, which may be expressed in the following equivalent forms:

$$X : Y : Z : T$$

$$= \wp' u : \wp^2 u - 2e_1 \wp u - (e_1^2 + e_2 e_3)$$
$$: \wp^2 u - 2e_2 \wp u - (e_2^2 + e_3 e_1)$$
$$: \wp^2 u - 2e_3 \wp u - (e_3^2 + e_1 e_2)$$

$$= \wp' u : \left(\wp u - \wp . \frac{\omega}{2}\right)\left(\wp u - \wp . \frac{3\omega}{2}\right)$$
$$: \left(\wp u - \wp . \frac{\omega+\omega'}{2}\right)\left(\wp u - \wp . \frac{3(\omega+\omega')}{2}\right)$$
$$: \left(\wp u - \wp . \frac{\omega'}{2}\right)\left(\wp u - \wp . \frac{3\omega'}{2}\right)$$

$$= A \sigma u \sigma (u-\omega)\, \sigma (u-\omega')\, \sigma (u-\omega-\omega')$$
$$: B\sigma \left(u - \frac{\omega}{2}\right) \sigma \left(u - \frac{3\omega}{2}\right) \sigma \left(u - \frac{\omega+2\omega'}{2}\right) \sigma \left(u - \frac{3\omega+2\omega'}{2}\right)$$
$$: C\sigma \left(u - \frac{\omega+\omega'}{2}\right) \sigma \left(u - \frac{3\omega+3\omega'}{2}\right) \sigma \left(u - \frac{\omega+3\omega'}{2}\right) \sigma \left(u - \frac{3\omega+\omega'}{2}\right)$$
$$: D\sigma \left(u - \frac{\omega'}{2}\right) \sigma \left(u - \frac{3\omega'}{2}\right) \sigma \left(u - \frac{2\omega+3\omega'}{2}\right) \sigma \left(u - \frac{2\omega+\omega'}{2}\right),$$

where A, B, C, D are numerical constants (*cf.* Klein, *l.c.*, § 9).

3. The singular tetrahedron is, of course, the common self-conjugate tetrahedron of all the quadric surfaces which contain the quartic: and it may be verified that, if

$$G = g_2^3 - 27 g_3^2 = 16\,(e_2-e_3)^2\,(e_3-e_1)^2\,(e_1-e_2)^2,$$
$$\sqrt{G} = 4\,(e_1-e_2)(e_1-e_3)(e_2-e_3),$$

then

$$\sqrt{G} \,.\, S \equiv \sqrt{G} \,.\, X^2 + 4e_1\,(e_2-e_3)\, Y^2 + 4e_2 (e_3-e_1)\, Z^2 + 4e_3\,(e_1-e_2)\, T^2,$$
$$\sqrt{G} \,.\, V \equiv (e_2-e_3)\, Y^2 + (e_3-e_1)\, Z^2 + (e_1-e_2)\, T^2.$$

Hence the equations of the other three cones are derived in the form

$$V' \equiv (e_2-e_3)\, X^2 + Z^2 - T^2 = 0,$$
$$V'' \equiv (e_3-e_1)\, X^2 + T^2 - Y^2 = 0,$$
$$V''' \equiv (e_1-e_2)\, X^2 + Y^2 - Z^2 = 0.$$

The quadric surface associated with V' is

$$S' \equiv 4e_1(e_2-e_3)X^2+4(e_2-e_3)Y^2+4e_3Z^2-4e_2T^2=0,$$

and similarly the equations $S''=0$, $S'''=0$ of the quadrics associ with V'' and V''' may be written down.

4. There are 32 collineations which transform the quartic i itself (*cf.* Harnack, *Math. Ann.*, XII.): these may be grouped follows.

There are 8 collineations which simultaneously transform S, V i themselves; these are obtained by variations of signs from

$$X'=\pm X, \quad Y'=\pm Y, \quad Z'=\pm Z, \quad T'=\pm T.$$

There are 8 collineations which interchange (S, V) with $(S',$ and (S'', V'') with (S''', V'''); viz., these are derived by variat of sign from

$$X'=\pm Y,$$

$$Y'=\pm\sqrt{(e_1-e_2)(e_1-e_3)}\,X,$$

$$Z'=\pm\sqrt{(e_1-e_3)}\,T,$$

$$T'=\pm\sqrt{(e_1-e_3)}\,Z.$$

Similarly, there are two other sets of 8 collineations; one int changing (S, V) with (S'', V'') and (S', V') with (S''', V'''), and t other interchanging (S, V) with (S''', V''') and (S', V') with $(S'',$ (*cf.* also Lange, *l.c.*, p. 79).

5. The osculating plane at any point $(X'Y'Z'T')$ on the int section of

$$aX^2+bY^2+cZ^2+dT^2=0,$$

and

$$a'X^2+b'Y^2+c'Z^2+d'T^2=0,$$

is (Salmon, *Solid Geometry*, Art. 393)

$$(ab'-a'b)(ac'-a'c)(ad'-a'd)\,X'^3X+\ldots+\ldots+\ldots=0.$$

Applying this to

$$(e_1-e_2)X^2+Y^2-Z^2 \qquad =0,$$

$$(e_1-e_3)X^2 \qquad +Z^2-T^2=0,$$

we get $(e_2-e_3)(e_3-e_1)(e_1-e_2)\,X'^3X$

$$+(e_2-e_3)\,Y'^3Y+(e_3-e_1)\,Z'^3Z+(e_1-e_2)\,T'^3T=0.$$

In particular, the perosculating planes which go through the vertex of V are

$$(e_2-e_3)\,Y+(e_3-e_1)\,Z+(e_1-e_2)\,T=0,$$

$$-(e_2-e_3)\,Y+(e_3-e_1)\,Z+(e_1-e_2)\,T=0,$$

$$(e_2-e_3)\,Y-(e_3-e_1)\,Z+(e_1-e_2)\,T=0,$$

$$(e_2-e_3)\,Y+(e_3-e_1)\,Z-(e_1-e_2)\,T=0,$$

or, in the other notation, $t=0,$

$$y-2e_1z+e_1^2t=0,$$

$$y-2e_2z+e_2^2t=0,$$

$$y-2e_3z+e_3^2t=0.$$

Applying the transformation which interchanges X, Y, the equations of four more perosculating planes are obtained in the form

$$(e_2-e_3)\,X\pm(e_1-e_2)^{\frac{1}{2}}\,Z\pm(e_1-e_3)^{\frac{1}{2}}\,T=0,$$

and hence, by cyclical permutation, the other eight, viz.,

$$(e_3-e_1)\,X\pm(e_2-e_3)^{\frac{1}{2}}\,T\pm(e_2-e_1)^{\frac{1}{2}}\,Y=0,$$

$$(e_1-e_2)\,X\pm(e_3-e_1)^{\frac{1}{2}}\,Y\pm(e_3-e_2)^{\frac{1}{2}}\,Z=0.$$

The correctness of these last results may be verified independently. Suppose, for instance, we take the point for which $u=\frac{\omega_1}{2}$.

Then for this point

$$Y'=\wp^2-2e_1\wp-(e_1^2+e_2e_3)=0,$$

$$Z'=\wp^2-2e_2\wp-(e_2^2+e_3e_1)=Z'-Y'$$

$$=2\,(e_1-e_2)\,\wp+(e_1^2-e_2^2)-e_3\,(e_1-e_2)$$

$$=2\,(e_1-e_2)(\wp-e_3)$$

$$=2\,(e_1-e_2)(e_1-e_3)^{\frac{1}{2}}\left\{(e_1-e_3)^{\frac{1}{2}}+(e_1-e_2)^{\frac{1}{2}}\right\}.$$

Similarly $T'=2\,(e_1-e_3)(e_1-e_2)^{\frac{1}{2}}\left\{(e_1-e_3)^{\frac{1}{2}}+(e_1-e_2)^{\frac{1}{2}}\right\},$

$$X'=\wp'=-2\,(e_1-e_2)^{\frac{1}{2}}\,(e_1-e_3)^{\frac{1}{2}}\left\{(e_1-e_3)^{\frac{1}{2}}+(e_1-e_2)^{\frac{1}{2}}\right\},$$

(*cf.* Halphen, p. 54).

Substituting in the equation

$$(e_2-e_3)(e_3-e_1)(e_1-e_2)\,X'^3X+\ldots=0,$$

we get, on reduction,

$$(e_2-e_3)^{\frac{1}{2}} X-(e_1-e_2)^{\frac{1}{2}} Z+(e_1-e_3)^{\frac{1}{2}} T=0.$$

The other cases may be similarly treated.

The osculating plane at the point whose parameter is u meets quartic again at a point whose parameter is $-3u$; so that the e tion of the osculating plane gives rise to the identity

$$(e_2-e_3)(e_3-e_1)(e_1-e_2)\,\wp'^3(u)\,\wp'(3u)$$

$$=(e_2-e_3)\left\{\wp^2 u-2e_1\wp u-(e_1^2+e_2e_3)\right\}^3\left\{\wp^2(3u)-2e_1\wp(3u)-(e_1^2+e_2$$

$$+\text{two similar terms},$$

which, of course, may be also expressed as a formula in σ-functio

6. Let u, v be the parameters of any two points on the qua Then, if we take a normal tetrahedron of reference, the coordinat any point on the line uv are given by

$$\begin{aligned} x &= \lambda\wp' u+\mu\wp' v, \\ y &= \lambda\wp^2 u+\mu\wp^2 v, \\ z &= \lambda\wp u+\mu\wp v, \\ t &= \lambda+\mu. \end{aligned}$$

Hence
$$\begin{aligned} S &\equiv x^2-4yz+g_2zt+g_3t^2 \\ &= \lambda\mu\left\{2\wp' u\wp' v-(4\wp u\wp v-g_2)(\wp u+\wp v)+2g_3\right\}, \\ V &\equiv z^2-yt=-\lambda\mu(\wp v-\wp u)^2. \end{aligned}$$

Therefore $\dfrac{S}{V}=4\wp(u+v)$; so that, if $u+v=a$, a constant, the uv describes the quadric surface

$$S-4\wp(a)\,V=0;$$

and conversely, the generating lines of any quadric surface wh contains the quartic consist of lines uv as above for which $u+v=$: the double sign corresponding to the double system of generators.

7. The consideration of lines uv for which $u-v=a$ presents m difficulty. It has been pointed out by Laguerre and Harnack tha this case the line uv describes a ruled surface of the eighth or called a "quadricuspidal" by de la Gournerie ("Recherches les surfaces tétraédrales symétriques"), and Harnack has furt

remarked that the line uv meets the self-conjugate tetrahedron in four points of which the cross-ratio is constant.

It is convenient to begin with the determination of this cross-ratio.

If $(XYZT)$, $(X'Y'Z'T')$ are the two points on the quartic, one of the six cross-ratios of the four points where uv meets the tetrahedron is

$$\frac{XT'-X'T}{XZ'-X'Z} \cdot \frac{YZ'-Y'Z}{YT'-Y'T}.$$

Since we know beforehand that its value only depends upon the difference $u-v$, we may put $u=a$, $v=0$; then it is easily seen that

$$\frac{XT'-X'T}{XZ'-X'Z} \text{ reduces to } 1.$$

Moreover

$$\begin{aligned} YZ'-Y'Z = {} & \left\{\wp^2 u - 2e_1\wp u - (e_1^2+e_2e_3)\right\}\left\{\wp^2 v - 2e_2\wp v - (e_2^2+e_3e_1)\right\} \\ & - \left\{\wp^2 u - 2e_2\wp u - (e_2^2+e_3e_1)\right\}\left\{\wp^2 v - 2e_1\wp v - (e_1^2+e_2e_3)\right\} \\ = {} & 2(e_1-e_2)(\wp u - \wp v)\left\{\wp u\,\wp v - e_3(\wp u+\wp v) - (e_3^2+e_1e_2)\right\}, \end{aligned}$$

$$YT'-Y'T = 2(e_1-e_3)(\wp u-\wp v)\left\{\wp u\,\wp v - e_2(\wp u+\wp v) - (e_2^2+e_3e_1)\right\}.$$

Hence, when $v=0$ and $u=a$,

$$\frac{YZ'-Y'Z}{YT'-Y'T} = \frac{e_1-e_2}{e_1-e_3} \cdot \frac{\wp(a)-e_3}{\wp(a)-e_2}.$$

Writing $\wp$ instead of $\wp(u-v)$, the six cross-ratios in question are therefore

$$\sigma = \frac{e_1-e_2}{e_1-e_3} \cdot \frac{\wp-e_3}{\wp-e_2},$$

$$\frac{1}{\sigma} = \frac{e_1-e_3}{e_1-e_2} \cdot \frac{\wp-e_2}{\wp-e_3},$$

$$1-\sigma = \frac{e_3-e_2}{e_3-e_1} \cdot \frac{\wp-e_1}{\wp-e_2},$$

$$\frac{1}{1-\sigma} = \frac{e_3-e_1}{e_3-e_2} \cdot \frac{\wp-e_2}{\wp-e_1},$$

$$\frac{\sigma}{\sigma-1} = \frac{e_2-e_1}{e_2-e_3} \cdot \frac{\wp-e_3}{\wp-e_1},$$

$$\frac{\sigma-1}{\sigma} = \frac{e_2-e_3}{e_2-e_1} \cdot \frac{\wp-e_1}{\wp-e_3}.$$

It will be found that

$$\sigma^2-\sigma+1=\frac{1}{16}\cdot\frac{12g_2\wp^2+36g_3\wp+g_2^2}{(e_1-e_3)^2(\wp-e_3)^2},$$

$$(\sigma+1)(2-\sigma)(1-2\sigma)=-\frac{1}{32}\cdot\frac{216g_3\wp^3+36g_2^2\wp^2+54g_2g_3\wp+(54g_3^2-}{(e_1-e_3)^3(\wp-e_3)^3}$$

So that the six cross-ratios are the roots of the equation

$$\frac{4(\sigma^2-\sigma+1)^3}{(\sigma+1)^2(2-\sigma)^2(1-2\sigma)^2}=\frac{(12g_2\wp^2+36g_3\wp+g_2^2)^3}{\{216g_3\wp^3+36g_2^2\wp^2+54g_2g_3\wp+(54g_3^2-g_2^3}$$

Supposing that G does not vanish, the only values of $\wp$ for wl $\sigma=0,\ 1,\ \infty$ are e_1, e_2, e_3; and, in confirmation of this, we have identity

$$(12g_2\wp^2+36g_3\wp+g_2^2)^3-\{216g_3\wp^3+36g_2^2\wp^2+54g_2g_3\wp+(54g_3^2-g_2^3)$$
$$=108G(4\wp^3-g_2\wp-g_3)^2.$$

The cross-ratios of the four points where uv meets the self-conju tetrahedron are equal to the cross-ratios of the four planes which be drawn through the line uv so as to touch the quartic elsewhere

Putting $u-v\equiv 0$, the line uv becomes a tangent to the quartic; values of the cross-ratios are $\frac{e_1-e_2}{e_1-e_3}$, etc.; and the equation of wl they are the roots is

$$\frac{4(\sigma^2-\sigma+1)^3}{(\sigma+1)^2(2-\sigma)^2(1-2\sigma)^2}=\frac{g_2^3}{27g_3^2}.$$

Hence, if $g_2=0$, the four points form an equianharmonic, an $g_3=0$, a harmonic system.

The equation last written may be otherwise obtained. Namely, four planes through the tangent at $(X'Y'Z'T')$, which touch quartic elsewhere, are the four-tangent planes at that point to cones V, V', V'', V'''. Their equations are

$$u_1\equiv(e_2-e_3)\,Y'Y+(e_3-e_1)\,Z'Z+(e_1-e_2)\,T'T=0,$$
$$u_2\equiv(e_2-e_3)\,X'X+Z'Z-T'T=0,$$
$$u_3\equiv(e_3-e_1)\,X'X+T'T-Y'Y=0,$$
$$u_4\equiv(e_1-e_2)\,X'X+Y'Y-Z'Z=0.$$

Obviously $$u_2=-u_3-u_4,$$

also $$u_1=-e_1u_2-e_2u_3-e_3u_4$$
$$=(e_1-e_2)\,u_3+(e_1-e_3)\,u_4.$$

This gives $\frac{e_1-e_3}{e_1-e_2}$ as one value of the cross-ratio of the planes; and the rest follows as before.

8. Proceeding now to the question of the quadricuspidals. The simplest case to begin with is the one in which $v = u+\omega_1$. In the notation of last article,

$$ZT'-Z'T = 2(e_2-e_3)(\wp u-\wp v)\left\{\wp u\wp v-e_1(\wp u+\wp v)-(e_1^2+e_2e_3)\right\} = 0.$$

Thus the line uv meets one edge of the self-conjugate tetrahedron. It is easily proved that it meets the opposite edge; in fact,

$$\begin{aligned} X' &= \wp'(u+\omega_1) = -\frac{(e_1-e_2)(e_1-e_3)\,\wp' u}{(\wp u-e_1)^2} \\ &= -\frac{(e_1-e_2)(e_1-e_3)}{(\wp u-e_1)^2}X, \\ Y' &= \wp^2(u+\omega)-2e_1\wp(u+\omega)-(e_1^2+e_2e_3) \\ &= \{\wp(u+\omega)-e_1\}^2-(e_1-e_2)(e_1-e_3) \\ &= \frac{(e_1-e_2)^2(e_1-e_3)^2}{(\wp u-e_1)^2}-(e_1-e_2)(e_1-e_3) \\ &= -\frac{(e_1-e_2)(e_1-e_3)}{(\wp u-e_1)^2}\{\wp^2 u-2e_1\wp u-(e_1^2+e_2e_3)\} \\ &= -\frac{(e_1-e_2)(e_1-e_3)}{(\wp u-e_1)^2}Y. \end{aligned}$$

Therefore $$XY'-X'Y = 0;$$

that is, the line uv meets the line $(X=0,\ Y=0)$.

Hence, in order to get the equation of the surface described by uv, we have only to eliminate $X'Y'Z'T'$ from the equations

$$XY'-X'Y = 0 \quad\ldots\ldots\ldots\ldots(\alpha),$$

$$ZT'-Z'T = 0 \quad\ldots\ldots\ldots\ldots(\beta),$$

$$(e_2-e_3)X'^2+Z'^2-T'^2 = 0 \quad\ldots\ldots\ldots\ldots(\gamma),$$

$$(e_3-e_1)X'^2+T'^2-Y'^2 = 0 \quad\ldots\ldots\ldots\ldots(\delta).$$

Equations (α), (β) give

$$X' = \lambda X,\quad Z' = \mu Z,$$
$$Y' = \lambda Y,\quad T' = \mu T.$$

Substituting in (γ), (δ),

$$\lambda^2 (e_2-e_3)\, X^2+\mu^2 (Z^2-T^2) = 0,$$

$$\lambda^2 \left\{(e_3-e_1)\, X^2-Y^2\right\} +\mu^2 T^2 = 0;$$

hence
$$(e_2-e_3)\, X^2 T^2-(e_3-e_1)\, X^2 Z^2+ (e_3-e_1)\, X^2 T^2$$
$$+Y^2 Z^2-Y^2 T^2 = 0,$$

or
$$(e_1-e_3)\, X^2 Z^2-(e_1-e_2)\, X^2 T^2+Y^2 (Z^2-T^2) = 0.$$

Similarly
$$(e_2-e_1)\, X^2 T^2-(e_2-e_3)\, X^2 Y^2+Z^2 (T^2-Y^2) = 0,$$
$$(e_3-e_2) X^2 Y^2-(e_3-e_1) X^2 Z^2+T^2 (Y^2-Z^2) = 0,$$

are the other surfaces corresponding to $v \equiv u+\omega_2$ and $v \equiv u$ respectively.

These quartic surfaces are de la Gournerie's "limiting qua cuspidals": the quadricuspidal, in fact, reduces to a quartic sur counted twice.

9. The next special case to be considered is $v \equiv u$; the surfa clearly the developable generated by the tangents of the quartic.

Now Salmon (S. G., Art. 218) gives the equation of the develop formed by the tangent lines of the intersection of the two qua U, V in the form

$$4\,(\Theta UV-Q'U-\Delta V^2)(\Theta' UV-QV-\Delta' U^2)$$
$$=(\Phi UV-QU-Q'V)^2$$

(where Q, Q' have been written for Salmon's T, T', to avoid fusion).

In the present case we may put

$$U = (e_3-e_1)\, X^2-Y^2+T^2,$$

$$V = (e_3-e_2)\, X^2-Z^2+T^2;$$

whence

$$\Delta = 0, \quad \Theta = e_3-e_1, \quad \Phi = 3e_3, \quad \Theta' = e_3-e_2, \quad \Delta' = 0,$$

$$Q = (e_3-e_1)(e_3-e_2)\, X^2+(e_3-e_2)\, T^2,$$

$$Q' = (e_3-e_1)(e_3-e_2)\, X^2+(e_3-e_1)\, T^2.$$

Substituting and expanding, the result is

$$
\begin{aligned}
0 = {} & (e_2-e_3)^4 X^4Y^4+(e_3-e_1)^4 X^4Z^4+(e_1-e_2)^4 X^4T^4 \\
& +(e_2-e_3)^2 Z^4T^4+(e_3-e_1)^2 T^4Y^4+(e_1-e_2)^2 Y^4Z^4 \\
& -2(e_1-e_2)^2(e_1-e_3)^2 X^4Z^2T^2-2(e_2-e_3)^2(e_2-e_1)^2 X^4T^2Y^2 \\
& -2(e_3-e_1)^2(e_3-e_2)^2 X^4Y^2Z^2 \\
& -2(e_1-e_2)^2(e_3-e_2) X^2Z^2T^4-2(e_1-e_3)^2(e_2-e_3) X^2T^2Z^4 \\
& -2(e_2-e_3)^2(e_1-e_3) X^2T^2Y^4-2(e_2-e_1)^2(e_3-e_1) X^2Y^2T^4 \\
& -2(e_3-e_1)^2(e_2-e_1) X^2Y^2Z^4-2(e_3-e_2)^2(e_1-e_2) X^2Z^2Y^4 \\
& -2(e_1-e_2)(e_1-e_3) Y^4Z^2T^2-2(e_2-e_3)(e_2-e_1) Z^4T^2Y^2 \\
& -2(e_3-e_1)(e_3-e_2) T^4Y^2Z^2 \\
& -54e_1e_2e_3 X^2Y^2Z^2T^2.
\end{aligned}
$$

This being a covariant surface, its equation must be unaltered by any of the 32 collineations so often referred to; the verification of this affords a test of the correctness of the work.

10. Comparing the results thus obtained, it seems probable that the following is the correct equation of the quadricuspidal corresponding to $v \equiv u+a$: $\wp$ denoting the function $\wp(a)$:—

$$
\begin{aligned}
0 = {} & (e_2-e_3)^4(\wp-e_1)^2 X^4Y^4+(e_3-e_1)^4(\wp-e_2)^2 X^4Z^4 \\
& \qquad +(e_1-e_2)^4(\wp-e_3)^2 X^4T^4 \\
& +(e_2-e_3)^2(\wp-e_1)^2 Z^4T^4+(e_3-e_1)^2(\wp-e_2)^2 T^4Y^4 \\
& \qquad +(e_1-e_2)^2(\wp-e_3)^2 Y^4Z^4 \\
& -2(e_1-e_2)^2(e_1-e_3)^2(\wp-e_2)(\wp-e_3) X^4Z^2T^2 \\
& -2(e_2-e_3)^2(e_2-e_1)^2(\wp-e_3)(\wp-e_1) X^4T^2Y^2 \\
& -2(e_3-e_1)^2(e_3-e_2)^2(\wp-e_1)(\wp-e_2) X^4Y^2Z^2 \\
& -2(e_1-e_2)^2(e_3-e_2)(\wp-e_3)(\wp-e_1) X^2Z^2T^4 \\
& -2(e_1-e_3)^2(e_2-e_3)(\wp-e_1)(\wp-e_2) X^2T^2Z^4 \\
& -2(e_2-e_3)^2(e_1-e_3)(\wp-e_1)(\wp-e_2) X^2T^2Y^4 \\
& -2(e_2-e_1)^2(e_3-e_1)(\wp-e_2)(\wp-e_3) X^2Y^2T^4 \\
& -2(e_3-e_1)^2(e_2-e_1)(\wp-e_2)(\wp-e_3) X^2Y^2Z^4 \\
& -2(e_3-e_2)^2(e_1-e_2)(\wp-e_3)(\wp-e_1) X^2Z^2Y^4 \\
& -2(e_1-e_2)(e_1-e_3)(\wp-e_2)(\wp-e_3) Y^4Z^2T^2 \\
& -2(e_2-e_3)(e_2-e_1)(\wp-e_3)(\wp-e_1) Z^4T^2Y^2 \\
& -2(e_3-e_1)(e_3-e_2)(\wp-e_1)(\wp-e_2) T^4Y^2Z^2 \\
& -\tfrac{3}{8}(36g_3\wp'+4g_2^2\wp+3g_2g_3) X^2Y^2Z^2T^2.
\end{aligned}
$$

At any rate, this is a covariant surface: when $v \equiv u$, it redu it should, to the equation of the developable generated by ta to the quartic; and when $v-u \equiv \omega_1$, ω_2, or ω_3, it reduces to the sponding quartic surface, counted twice.

11. The aggregate of all lines which meet the quartic once up a complex, of which the equation may be found as follows:—

Let $(XYZT)$, $(X'Y'Z'T')$ be two points in space, and put

$$XY'-X'Y = p_{12}, \quad XZ'-X'Z = p_{13}, \quad XT'-X'T = p_{14},$$
$$ZT'-Z'T = p_{34}, \quad TY'-T'Y = p_{42}, \quad YZ'-Y'Z = p_{23},$$

so that
$$p_{23}p_{14}+p_{34}p_{12}+p_{42}p_{13} = 0$$
identically.

Any point on the line joining $(XYZT)$ and $(X'Y'Z'T')$ is give
$$\xi : \eta : \zeta : \tau = \lambda X+\mu X' : \lambda Y+\mu Y' : \lambda Z+\mu Z' : \lambda T+\mu T'.$$

If the line meets the quartic, it will be possible to find a poin which lies simultaneously on the cones
$$(e_2-e_3)\, X^2+Z^2-T^2 = 0,$$
$$(e_3-e_1)\, X^2+T^2-Y^2 = 0.$$

Hence we have to eliminate $\lambda : \mu$ from the equations
$$\lambda^2 \left\{(e_2-e_3)\, X^2+Z^2-T^2\right\}+2\lambda\mu \left\{(e_2-e_3)\, XX'+ZZ'-TT'\right\}$$
$$+\mu^2 \left\{(e_2-e_3)\, X'^2+Z'^2-T'^2\right\} = 0,$$
and
$$\lambda^2 \left\{(e_3-e_1)\, X^2-Y^2+T^2\right\}+2\lambda\mu \left\{(e_3-e_1)\, XX'-YY'+TT'\right\}$$
$$+\mu^2 \left\{(e_3-e_1)\, X'^2-Y'^2+T'^2\right\} = 0.$$

The result is
$$B^2-4AC = 0,$$
where
$$A = \left\{(e_2-e_3)\, XX'+ZZ'-TT'\right\}\left\{(e_3-e_1)\, X'^2-Y'^2+T'^2\right\}$$
$$-\left\{(e_3-e_1)\, XX'-YY'+TT'\right\}\left\{(e_2-e_3)\, X'^2+Z'^2-T'^2\right\}$$
$$= -(e_2-e_3)\, X'Y'p_{12}-(e_3-e_1)\, X'Z'p_{13}-(e_1-e_2)\, X'T'p_{14}$$
$$+Z'T'p_{34}+T'Y'p_{42}+Y'Z'p_{23},$$
$$C = -(e_2-e_3)\, XYp_{12}-(e_3-e_1)\, XZp_{13}-(e_1-e_2)\, XTp_{14}$$
$$+ZTp_{34}+TYp_{42}+YZp_{23},$$
$$B = (e_2-e_3)(XY'+X'Y)\, p_{12}+(e_3-e_1)(XZ'+X'Z)\, p_{13}$$
$$+(e_1-e_2)(XT'+X'T)\, p_{14}$$
$$-(ZT'+Z'T)\, p_{34}-(TY'+T'Y)\, p_{42}-(YZ'+Y'Z)\, p_{23}.$$

Substituting and expanding, the result is

$$\begin{aligned}0 = {} & (e_2-e_3)^2 p_{12}^4+(e_3-e_1)^2 p_{13}^4+(e_1-e_2)^2 p_{14}^4+p_{34}^4+p_{42}^4+p_{23}^4 \\ & -2(e_1-e_2)(e_1-e_3)\, p_{13}^2 p_{14}^2-2(e_2-e_3)(e_2-e_1)\, p_{14}^2 p_{12}^2 \\ & -2(e_3-e_1)(e_3-e_2)\, p_{12}^2 p_{13}^2 \\ & -2(e_2-e_3)(p_{13}p_{42}-p_{14}p_{23})\, p_{12}p_{34} \\ & -2(e_3-e_1)(p_{14}p_{23}-p_{12}p_{34})\, p_{13}p_{42} \\ & -2(e_1-e_2)(p_{12}p_{34}-p_{13}p_{42})\, p_{14}p_{23} \\ & -2(e_2-e_3)\, p_{12}^2\,(p_{23}^2+p_{42}^2) \\ & -2(e_3-e_1)\, p_{13}^2\,(p_{23}^3+p_{34}^2) \\ & -2(e_1-e_2)\, p_{14}^2\,(p_{34}^2+p_{42}^2) \\ & -2p_{23}^2 p_{42}^2-2p_{34}^2 p_{42}^2-2p_{42}^2 p_{23}^2.\end{aligned}$$

Regarding $(X'Y'Z'T')$ as a fixed point, this gives the equation of the quartic cone projecting the curve from that point.

12. If the line meets the quartic *twice*,

$$A = 0, \quad B = 0, \quad C = 0$$

simultaneously. Hence also

$$X^2 . A + X'^2 . C - XX' . B = 0,$$

which reduces to

$$p_{13}p_{14}p_{34}+p_{14}p_{12}p_{42}+p_{12}p_{13}p_{23} = 0.$$

Similarly $$Y^2 . A + Y'^2 . C - YY' . B = 0$$

gives $$(e_1-e_2)\, p_{12}p_{43}p_{14}+(e_1-e_3)\, p_{12}p_{23}p_{13}+p_{23}p_{42}p_{34} = 0,$$

and we also have

$$(e_2-e_3)\, p_{12}p_{13}p_{23}+(e_2-e_1)\, p_{13}p_{34}p_{42}+p_{23}p_{34}p_{42} = 0,$$

$$(e_3-e_1)\, p_{13}p_{14}p_{34}+(e_3-e_2)\, p_{12}p_{14}p_{42}+p_{23}p_{34}p_{42} = 0.$$

Of these complexes only two, of course, are independent; viz., multiplying the first by (e_2-e_1), and adding to the second, we get

$$(e_2-e_3)\, p_{12}p_{23}p_{13}+(e_2-e_1)\, p_{13}p_{14}p_{34}+p_{23}p_{34}p_{42} = 0,$$

i.e., the third complex; and so for the other one.

Taking the complexes

$$p_{12}p_{14}p_{34} + p_{14}p_{13}p_{42} + p_{12}p_{13}p_{23} = 0,$$
$$(e_1-e_2)\,p_{12}p_{42}p_{14} + (e_1-e_3)\,p_{12}p_{13}p_{23} + p_{23}p_{34}p_{42} = 0,$$

the congruence determined by them contains as part of itsel three linear congruences

$$\left.\begin{matrix} p_{12} = 0 \\ p_{34} = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} p_{13} = 0 \\ p_{42} = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} p_{14} = 0 \\ p_{23} = 0 \end{matrix}\right\},$$

i.e., the three special linear congruences formed by lines which two opposite edges of the self-conjugate tetrahedron. Reje these, we have left a congruence formed by lines which mee quartic twice. This is of the sixth order. The six lines pa through a given point consist of four tangents to the quartic, two lines drawn from the point "to the two apparent d points."

In a given plane there are six lines, viz., the lines joining tw two the four points where the plane meets the quartic.

It does not seem possible to obtain the congruence of lines me the quartic twice as the *complete* intersection of two complexes; as a curve is not necessarily the complete intersection of two surf

Electrical Oscillations on Cylindrical Conductors. By J. Thomson, M.A., F.R.S., Cavendish Professor of Experim Physics, Cambridge.

[*Read June 14th, 1888.*]

Of all the questions at issue between the various theories of [illegible] the most fundamental and important [illegible] closed circ [illegible] the same equation [illegible] components of an in [illegible]

At [illegible] direct evidence on the p [illegible] as pos [illegible] to see whe [illegible] them.

this paper I consider the problem of the conduction of electricity along wires, taking first Maxwell's theory and then v. Helmholtz's dielectric theory.

I had previously considered several cases of this problem, assuming Maxwell's theory, in a paper read before the *Mathematical Society*, June 10th, 1886.

I also consider some cases in which, assuming the electro-magnetic theory of Light, the "compressibility" of the current might be expected to produce measurable effects. The various theories of electro-dynamics may conveniently be divided into two classes:—(*a*) those which neglect the action of the dielectric altogether, or, in other words, which only take into account the ordinary conduction currents flowing through conductors; and (*b*) those in which changes in the electro-motive force acting on the dielectric are supposed to be accompanied by changes in the state of the dielectric, which produce electro-dynamic effects similar to those produced by the passage of electricity through conductors.

It hardly, I think, requires further experiments to prove that those theories which neglect the action of the dielectric altogether must be fallacious. For when an electro-motive force, produced we may suppose by a moving magnet, acts on a dielectric medium, it polarises it, producing changes which are accompanied by changes in its physical state. From this it follows from the principle of action and reaction, as I pointed out in my report on "Electrical Theories" (*British Association Report*, 1886), that changes in polarisation must produce magnetic forces; they may similarly be shown to produce all the effects of electric currents. It is true that it is only in material dielectrics that we have evidence of the change in state produced by an electro-motive force; but there is considerable reason to believe that the polarisation produced by an electro-motive force is primarily an affair of the ether, and not of the material dielectric medium; in other words, that though the nature of the dielectric does effect electric phenomena, the effect is rather of the nature of a modification of a larger effect than the production of a new one.

For example, though Maxwell's theory that the square of the refractive index is proportional to the specific inductive capacity is not in all cases, and especially for the fatty oils, confirmed by experiment, the agreement is amply sufficient to prove that the dielectric influences electric phenomena to an extent strictly comparable with that to which light phenomena are influenced by the refracting medium, and that presumably the molecules of the dielectric merely modify the effect produced by the ether.

If this is so, then we may extend the reasoning previously applied

to material dielectrics to the ether, and conclude that change of p sation, whether in an ordinary dielectric or the ether, produces el dynamic effects.

But, even if we grant that changes in the polarisation of dielectric produce electro-dynamic effects, there are still se questions left unsettled. In the first place, what is the rel between the rate of change of the electro-motive force acting dielectric, and the current which would produce the same ele dynamic effect? Maxwell assumes that, if X is the electro-m force, K the specific inductive capacity, the equivalent current is

$$\frac{K}{4\pi}\frac{dX}{dt}.$$

This value of the multiplier of dX/dt was taken by Maxwell to with his theory that all circuits are closed, or that electricity f like an incompressible fluid.

v. Helmholtz, adopting another and more general theory, reg the dielectric as the exact analogue of a magnetisable substance, supposes that when X, an electro-motive force parallel to the axis acts on an element of a dielectric, the polarisation produced ma represented by a distribution of electricity of surface density over one and $-X/\epsilon$ over the other of the two faces of the elem When the state of the polarisation alters, he supposes that the e valent current is one which, if it flowed through the element, w produce the required changes in the distribution of electricity the faces. If the circuits are not closed, then, as far as our pre knowledge extends, the expressions for the vector potential due distribution of electric currents are indeterminate, but v. Helmh has shown that this indeterminateness can be adequately represe by the use of a new constant k which cannot be negative.

Thus the points in the theories that are unsettled are,—first, whe all circuits are closed. If they are, we have a complete theory; if are not, we require to know the relation between the rate of cha of the electro-motive force, and the current which would produce same electro-dynamic effect. Secondly, if the circuits are not clo we require to know the value of v. Helmholtz's constant k.

The following pages contain investigations of some problem electro-dynamics according to both Maxwell's and v. Helmho theory, made with especial reference to the points on which theories differ.

The notation is the sa sed in the previ same subject, and is as f

F, G, H are the components of the vector potential.

a, b, c those of magnetic induction.

α, β, γ those of magnetic force.

X, Y, Z those of electro-motive force.

$\dot{f}$, $\dot{g}$, $\dot{h}$ those of the currents whose electro-dynamic action is the same as that of the change in the polarisation of the dielectric.

u, v, w the components of the ordinary currents through conductors.

ϕ is the electro-static potential.

K the specific inductive capacity.

μ the magnetic permeability.

σ the specific resistance of a conductor.

k a quantity such that the components of the vector potential due to the currents u, v, w through an element $dx\,dy\,dz$ are expressed by the equation

$$F = \left\{ \frac{u}{r} + \tfrac{1}{2}(1-k)\left(u\frac{d^2r}{d\xi\,dx} + v\frac{d^2r}{d\xi\,dy} + w\frac{d^2r}{d\xi\,dz} \right) \right\} dx\,dy\,dz,$$

where r is the distance of the point $(\xi\eta\zeta)$ at which the potential is reckoned from the element $dx\,dy\,dz$. v. Helmholtz has shown that this is the most general expression for the vector potential consistent with the expression for the magnetic force due to a closed circuit, agreeing with those given by Ampère, which have been amply confirmed by experiment.

For convenience of reference, I will now write down the equations we shall require, dividing them into three classes.

(*a*) Equations which are true on either theory.

(*b*) Equations which are true on Maxwell's theory, but not on Helmholtz's.

(*c*) Equations which are true on Helmholtz's theory, but not on Maxwell's.

Equations true on either theory :—

$$a = \frac{dH}{dy} - \frac{dG}{dz} \quad \text{.............................(1)},$$

$$X = -\frac{dF}{dt} - \frac{d\phi}{dx} \quad \text{.............................(2)},$$

$$\sigma u = X \quad \text{....................................(3)},$$

$$\mu\alpha = a \quad \text{....................................(4)}.$$

At the boundary of two different media, the components of the ve potential are continuous, as are the magnetic induction at right an to the surface, and the magnetic force parallel to the surface.

Equations true on Maxwell's theory, but not on Helmho theory :—

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d}{dx}(u+\dot{f})+\frac{d}{dy}(v+\dot{g})+\frac{d}{dz}(w+\dot{h})=0 \quad \ldots\ldots\ldots\ldots$$

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi\,(u+\dot{f}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f=\frac{4\pi}{K}X \text{ in a dielectric} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots($$

From these follow—

$$\nabla^2\varphi=0 \text{ in both dielectrics and conductors,}$$

$$\frac{1}{\mu K}\nabla^2 F=\frac{d^2F}{dt^2}+\frac{d^2\varphi}{dx\,dt} \text{ in a dielectric} \quad \ldots\ldots\ldots\ldots\ldots($$

$$\frac{\sigma}{4\pi\mu}\nabla^2 F=\frac{dF}{dt}+\frac{d\phi}{dx} \text{ in a conductor} \ldots\ldots\ldots\ldots\ldots\ldots(1$$

Equations which are true on Helmholtz's theory, but not on Ma well's :—

$$f=\frac{X}{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=-A^2k\frac{d\phi}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1$$

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=A\left(\frac{d^2\phi}{dx\,dt}-4\pi\,(u+\dot{f})\right)\ldots\ldots\ldots\ldots\ldots\ldots(1$$

where A is the ratio of the measures of a quantity of electricity electro-static and electro-magnetic systems.

From these follow—

$$4\pi\left(\frac{d}{dx}(u+\dot{f})+\frac{d}{dy}(v+\dot{g})+\frac{d}{dz}(w+\dot{h})\right)=\frac{d}{dt}\nabla^2\varphi \quad \ldots\ldots(14$$

and in a dielectric, since

$$\frac{\dot{f}}{\epsilon}=-\frac{d^2F}{dt^2}-\frac{d^2\varphi}{dt\,dx},$$

and therefore

$$\frac{1}{\epsilon}\left\{\frac{d\dot{f}}{dx}+\frac{d\dot{g}}{dy}+\frac{d\dot{h}}{dz}\right\} = -\frac{d^2}{dt^2}\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right)-\frac{d}{dt}\nabla^2\phi,$$

we get
$$\nabla^2\phi = \frac{A^2k4\pi\epsilon}{1+4\pi\epsilon}\frac{d^2\phi}{dt^2} \quad\text{...........................(15).}$$

In a conductor,
$$\sigma u = -\frac{dF}{dt}-\frac{d\phi}{dx},$$

$$\sigma\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) = -\frac{d}{dt}\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right)-\nabla^2\phi,$$

and therefore
$$\left(\frac{\sigma}{4\pi}\frac{d}{dt}+1\right)\nabla^2\phi = Ak\frac{d^2\phi}{dt^2} \quad\text{....................(16).}$$

From equations (1) and (13), we see that, in the dielectric,

$$\frac{1}{4\pi\mu t}\nabla^2 F = \frac{d^2F}{dt^2}+\frac{d^2\phi}{dx\,dt}\left(\frac{1+4\pi\epsilon}{4\pi\epsilon}-\frac{k}{4\pi\epsilon\mu}\right) \quad\text{............(17);}$$

in a conductor,

$$\frac{\sigma}{4\pi\mu}\nabla^2 F = \frac{dF}{dt}-\left(\frac{\sigma k}{4\pi\mu}-\frac{\sigma}{4\pi}\right)\frac{d^2\phi}{dx\,dt}+\frac{d\phi}{dx} \quad\text{............(18).}$$

On comparing these equations, we see at once that the equations for ϕ are fundamentally different in the two systems. According to Maxwell's theory, $\nabla^2\phi$ vanishes both in conductors and dielectrics, so that according to this theory ϕ is propagated with an infinite velocity, while according to v. Helmholtz's theory, since

$$\nabla^2\phi = A^2k\frac{4\pi\epsilon}{1+4\pi\epsilon}\frac{d^2\phi}{dt^2},$$

ϕ is propagated with the velocity

$$\{1+4\pi\epsilon/A^2k4\pi\epsilon\}^{\frac{1}{2}}.$$

It is interesting to consider the different results to which these theories lead. Take, first, the case of two concentric spheres whose potentials vary as ϵ^{ipt}.

The results of the theory, when $\nabla^2\phi = 0$, are too well known to require quotation.

When, however,
$$\nabla^2\phi = \frac{4A^2\pi k\epsilon}{1+4\pi\epsilon}\frac{d^2\phi}{dt^2},$$

or, say,
$$\nabla^2\phi = v_0^2\frac{d^2\phi}{dt^2},$$

where v_0 is the velocity of propagation of the electro-static pote
we have for the case of two spheres, if $\phi \propto \epsilon^{ipt}$,

$$\frac{d^2}{dr^2}(r\phi)+\frac{p^2}{v_0^2}(r\phi)=0, \text{ or } r\phi = A\epsilon^{ipr/v_0}+B\epsilon^{-ipr/v_0}.$$

Let a and b be the radii of the spheres, $\phi_1\epsilon^{ipt}$ and $\phi_2\epsilon^{ipt}$
potentials; then, inside the inner sphere, since ϕ does not becom
finite when $r=0$, we must have

$$\phi = \frac{a\phi_1 \sin\left(\frac{p}{v_0}r\right)\epsilon^{ipt}}{\sin\frac{p}{v_0}a}\,\frac{1}{r};$$

outside the outer sphere, we have

$$\phi = \frac{b\phi_2\epsilon^{i\{pt-(r-b)p/v_0\}}}{r};$$

between the two spheres,

$$\phi = \frac{(A\epsilon^{ipr/v_0}+B\epsilon^{-ipr/v_0})\,\epsilon^{ipt}}{r};$$

to determine A and B, we have

$$A\epsilon^{ipa/v_0}+B\epsilon^{-ipa/v_0} = a\phi_1,$$
$$A\epsilon^{ipb/v_0}+B\epsilon^{-ipb/v_0} = b\phi_2.$$

From these we deduce

$$\phi = \frac{\left\{a\phi_1\sin\frac{p}{v_0}(r-b)-b\phi_2\sin\frac{p}{v_0}(r-a)\right\}}{r\sin\frac{p}{v_0}(a-b)}\,\epsilon^{ipt}.$$

If σ is the density of electricity on the inner sphere, since
equals the difference between the forces just inside and outside
have

$$4\pi\sigma = \frac{p}{v_0}\phi_1\epsilon^{ipt}\left(\cot\frac{p}{v_0}a-\cot\frac{p}{v_0}(a-b)+\frac{p}{v_0}\phi_2\epsilon^{ipt}\frac{b}{a\sin\frac{p}{v_0}(a-\right.$$

If σ' is the surface density on the outer sphere, we find similarly

$$4\pi\sigma' = \frac{p}{v_0}\phi_1\epsilon^{ipt}\frac{a}{b\sin\frac{p}{v_0}(a-b)}+\frac{p}{v_0}\phi_2\epsilon^{ipt}\left(i+\cot\frac{p}{v_0}(b-a)\right)$$

We see from the equations that the coefficient of capacity of the

spheres (defined as in Maxwell, Vol. I., p. 102) is

$$\frac{p}{v_0}\,\frac{ab}{\sin\frac{p}{v_0}(a-b)},$$

and also that the spheres may be charged, even though their potentials are equal. We also see that the density on the outer sphere is not in the same phase as the potential; the force inside the inner sphere is also finite.

By making the radius of the spheres infinite and their difference finite and equal to h, we get the case of two parallel planes. We easily find that, if the planes are at right angles to the axis of x, and if the equation to the plane whose potentials are ϕ_1 and ϕ_2 are $x = 0$ and $x=h$ respectively, the potential outside the planes when x is negative is

$$\phi_1 \exp. \frac{ip}{v_0}(v_0 t+x),$$

or, taking the real part only,

$$\phi_1 \cos\frac{p}{v_0}(pt+x);$$

between the planes,

$$\frac{\phi_1 \sin\frac{p}{v_0}(h-x)+\phi_2 \sin\frac{p}{v_0}x}{\sin\frac{p}{v_0}h};$$

outside the planes in the region where x is positive,

$$\phi_1 = \phi_2 \cos p\left(t-\frac{(x-l)}{v_0}\right).$$

The capacity of unit area of the planes is

$$\frac{1}{4\pi}\,\frac{p}{v_0 \sin\frac{ph}{v_0}}.$$

Now $p/v_0 = 2\pi/\lambda$, where λ is the wave-length of the electrical vibrations, so that the capacity of unit area of the plate

$$= \frac{1}{2\lambda \sin\frac{2\pi h}{\lambda}};$$

thus, when the length of the electrostatic waves is comparable with the distance between the plates, the ordinary expressions cease to represent the capacity.

It will be noticed that the force outside the planes no vanishes, but, when x is positive, equals

$$\frac{p}{\omega}\phi_2 \sin p\left(t-\frac{x-l}{v_0}\right),$$

or

$$\frac{2\pi}{\lambda}\phi_2 \sin p\left(t-\frac{x-l}{v_0}\right).$$

In order for λ to be of moderate size, the time of vibrations be very rapid, but Hertz has succeeded in getting electrical sy whose vibrations are so rapid that, for a disturbance propagated the velocity of light, λ would not be much more than a metre. I case the force outside the plates might be expected to be finit would, however, on account of the extremely rapid rate of revers difficult to detect, unless we could get some substance which gl when acted upon by a varying electromotive force. The rarifie inside an electric lamp seems to do this to some extent, and mig of service in proving or disproving the existence of an electro force outside two infinite parallel planes.

Another way in which it might perhaps be done would be to r the vibration from some plane at constant potential, and thus stationary vibration which we may represent by

$$\phi = \phi_2 \cos\frac{2\pi}{\lambda}x \cos pt.$$

and there will be a force acting on a small insulated condu equal per unit volume to

$$\phi_2 \frac{2\pi^2}{\lambda^3}\cos\frac{2\pi}{\lambda}x \sin\frac{2\pi}{\lambda}x \cos^2 pt,$$

the mean value of which is

$$\frac{\phi_2^2}{2}\frac{\pi^2}{\lambda^3}\sin\frac{4\pi x}{\lambda}.$$

The maximum value of this is the same as the force which w be exerted on the same conductor if placed outside a sphere of r $\lambda/2\pi$ maintained at the potential ϕ_2.

We might thus either prove the existence of this force, in w case electricity cannot move like an incompressible fluid, or else an inferior limit to the velocity of propagation of the potential.

We shall now go on [illegible] the propagation [illegible] lo metallic wire.

It is convenient to distinguish two cases, one where the magnetic force is along the wire, and the currents in the wire flow in circles whose planes are at right angles to the axis of the wire, and whose centres are along it.

In this case we may put, if the axis of the wire is taken as the axis of z,

$$F = \frac{d\chi}{dy} + A\frac{d\phi}{dx},$$

$$G = -\frac{d\chi}{dx} + A\frac{d\phi}{dy},$$

where A is a constant, and χ and ϕ functions of r and z; and, if we suppose that we are considering a wave whose period is $2\pi/p$ and whose wave-length is $2\pi/m$, all the quantities will vary as $\epsilon^{i(mz+pt)}$.

Thus
$$a = \frac{dH}{dy} - \frac{dG}{dz} = im\frac{d\chi}{dx},$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} = im\frac{d\chi}{dy},$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} = -\frac{d^2\chi}{dx^2} - \frac{d^2\chi}{dy^2}.$$

From equations of type (9), we have in a dielectric, on Maxwell's theory,

$$\frac{1}{\mu\kappa}\nabla^2 a = \frac{d^2 a}{dt^2};$$

on v. Helmholtz's, we have similarly

$$\frac{1}{4\pi\epsilon\mu}\nabla^2 a = \frac{d^2 a}{dt^2};$$

or, if v be the velocity with which the electro-dynamic influence is propagated, we have on either theory

$$v^2\nabla^2 a = \frac{d^2 a}{dt^2},$$

with similar equations for b and c.

In a conductor,
$$\frac{\sigma}{4\pi\mu}\nabla^2 a = -\frac{da}{dt}.$$

Since a varies as $\epsilon^{i(mz+pt)}$, these equations may be written, substituting for a its value $im\, d\chi/dx$,

$$\frac{d^2\chi}{dr^2} + \frac{1}{r}\frac{d\chi}{dr} - \left(m^2 - \frac{p^2}{v^2}\right)\chi = 0 \text{ in the dielectric,}$$

and $$\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}-\left(m^2+\frac{4\pi\mu ip}{\sigma}\right)\chi=0 \text{ in the conductor.}$$

Let $$m^2-\frac{p^2}{v^2}=\kappa^2,$$

$$m^2+\frac{4\pi\mu ip}{\sigma}=n^2.$$

Then $$\chi = AI_0(i\kappa r) \text{ in the dielectric,}$$
$$= BJ_0(inr) \text{ in the wire,}$$

where $I_0(x)$ and $J_0(x)$ are the Bessel's functions of zero order which vanish when $x=\infty$ and when $x=0$, respectively. Thus

$$J_0(inr)=1+\frac{n^2r^2}{4}+\frac{1}{2!\,2!}\frac{n^4r^4}{4^2}+\frac{1}{3!\,3!}\frac{n^6r^6}{4^3}+\ldots,$$

and $$I_0(i\kappa r)=2\left\{\frac{\kappa^2r^2}{4}+\frac{1}{2!\,2!}\frac{\kappa^4r^4}{4^2}\left(1+\frac{1}{2}\right)\right.$$
$$\left.+\frac{1}{3!\,3!}\frac{\kappa^6r^6}{4^3}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\ldots\right\}$$
$$-\left(1.16+\log\frac{\kappa^2r^2}{4}\right)J_0(i\kappa r).$$

The magnetic induction along the radius is $Bin\,J_0'(inr)$ in the wire, and $Ai\kappa\,I_0'(i\kappa r)$ in the dielectric. At the surface of separation of the wire and the dielectric these must be equal, so that, if a is the radius of the wire,

$$Bin\,J_0'(ina)=Ai\kappa\,I_0'(i\kappa a) \quad\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Again, since $$c=-\frac{d^2\chi}{dx^2}-\frac{d^2\chi}{dy^2},$$

we have $$c=-\kappa^2AI_0(i\kappa r) \text{ in the dielectric,}$$

and $$=-n^2BJ_0(inr) \text{ in the wire} \quad\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Since the magnetic force parallel to the surface is the same in the two media close to the surface of separation, we must have, if μ is the coefficient of magnetic permeability of the wire,

$$\frac{n^2B}{\mu}J_0(ina)=k^2A\,I_0(i\kappa a).$$

Eliminating A and B from equations (19) and (20), we get

$$\frac{ina}{\mu}\frac{J_0(ina)}{J_0'(ina)}=i\kappa a\frac{I_0(i\kappa a)}{I_0'(i\kappa a)}\quad\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

a relation connecting the wave-length $2\pi/m$ with the time of vibration $2\pi/p$.

Let us first consider the case of iron wires where μ is large; then since, in this case, the left-hand side of equation (21) will be small, hence κa must be small; and, when this is the case, we have approximately

$$\frac{i\kappa a\, I_0(i\kappa a)}{I_0'(i\kappa a)} = -\,\kappa^2 a^2 \log\frac{\kappa^2 a^2}{4}.$$

Let us first take the case when na is small, then

$$\frac{ina\, J_0(ina)}{J_0'(ina)} = -\,2\left\{1+\frac{n^2a^2}{8}\right\} \text{ approximately.}$$

Equation (21) therefore reduces to

$$\kappa^2 a^2 \log\frac{\kappa^2 a^2}{4} = \frac{2}{\mu}\left\{1+\left(m^2+\frac{4\pi\mu ip}{\sigma}\right)\frac{a^2}{8}\right\}.$$

To solve this equation let $\quad \kappa^2 a^2 = k\epsilon^{i\psi}$,

then, equating real and imaginary parts, we get

$$k\log\frac{k}{4}\cos\psi - k\psi\sin\psi = \frac{2}{\mu}\left(1+\frac{m^2a^2}{8}\right),$$

$$k\log\frac{k}{4}\sin\psi = \frac{\pi p a^2}{\sigma}.$$

Since k is very small, the solution of these equations is

$$k = -\frac{2}{\mu}\Big/\log 2\mu,$$

$$\psi = \frac{1}{2}\,\frac{p\mu a^2}{\sigma},$$

so that, since $\qquad \kappa^2 = m^2 - \dfrac{p^2}{v^2}$,

we get $\qquad m^2 - \dfrac{p^2}{v^2} = -\dfrac{2}{a^2\mu\log 2\mu}\left(1+\dfrac{ip\mu a^2}{2\sigma}\right);$

or, since p^2/v^2 is small compared with m^2, unless the vibrations are of great rapidity,

$$m = i\left\{\frac{2}{\mu a^2\log 2\mu}\right\}^{\frac{1}{2}}\left\{1+\frac{ip\mu a^2}{4\sigma}\right\} \quad \text{..................(22).}$$

This represents a disturbance whose wave-length is

$$\frac{8\pi\sigma}{\mathrm{a}p}\left\{\frac{\log 2\mu}{2\mu}\right\}^{\frac{1}{2}},$$

the velocity of propagation being therefore

$$\frac{4\sigma}{\mathrm{a}}\left\{\frac{\log 2\mu}{2\mu}\right\}^{\frac{1}{2}}.$$

The distance to which the disturbance travels before the ampli its vibration falls to $1/e$ of its original value is

$$\frac{\mathrm{a}}{2}\left(2\mu\log 2\mu\right)^{\frac{1}{2}};$$

this is small compared with the wave-length. So that the distu practically dies away before completing a single undulation.

If any arbitrary distribution of magnetic force be produced the wire, and if this by Fourier's theorem be expressed as a se harmonic terms, equation (22) shows that the harmonic term will, when the external force producing the arbitrary distribu removed, die away to $1/e$ after a time

$$\frac{\mu\mathrm{a}^2}{2\sigma\left(\frac{m^2\mathrm{a}^2}{2}\mu\log 2\mu+1\right)}.$$

Another case of interest is when $n\mathrm{a}$, instead of being small, i compared with unity, but small compared with μ. In this c will still be small, and since, when $n\mathrm{a}$ is large,

$$J_0'(in\mathrm{a}) = -iJ_0(in\mathrm{a}),$$

equation (21) becomes $\quad -\kappa^2\mathrm{a}^2\log\frac{\kappa^2\mathrm{a}^2}{4} = -\frac{1}{\mu}n\mathrm{a},$

or, approximately,

$$\kappa^2\mathrm{a}^2\log\frac{\kappa^2\mathrm{a}^2}{4} = \frac{1}{\mu}\left\{\frac{4\pi\mu ip}{\sigma}\right\}^{\frac{1}{2}}\mathrm{a},$$

or

$$\kappa^2\mathrm{a}^2\log\frac{\kappa^2\mathrm{a}^2}{4} = \left\{\frac{2\pi p\mathrm{a}^2}{\mu\sigma}\right\}^{\frac{1}{2}}(1+i)$$

$$= M(1+i), \text{ say.}$$

To solve this equation put

$$\kappa^2\mathrm{a}^2 = k\epsilon^{i\psi};$$

then, equating real and imaginary parts, we get

$$k\log\frac{k}{4}\cos\psi - k\psi\sin\psi = M,$$

$$k \log \frac{k}{4} \sin \psi = M.$$

An approximate solution of these equations is

$$\psi = \frac{\pi}{4},$$

$$k = -m\sqrt{2} / \log \frac{M}{2\sqrt{2}};$$

hence $$m^2 - \frac{p^2}{v^2} = \frac{1}{a} \left\{ \frac{4\pi p}{\mu\sigma} \right\}^{\frac{1}{2}} \epsilon^{\frac{1}{4}i\pi} / \log \left\{ \frac{4\mu\sigma}{\pi p a^2} \right\}^{\frac{1}{2}}.$$

Neglecting p^2/v^2 as before, the solution of this is

$$m = \frac{\frac{1}{a^{\frac{1}{2}}} \left\{ \frac{4\pi p}{\mu\sigma} \right\}^{\frac{1}{4}} \epsilon^{\frac{1}{8}i\pi}}{\left\{ \frac{1}{2} \log \frac{4\mu\sigma}{\pi p a^2} \right\}^{\frac{1}{2}}}.$$

This represents a wave-motion whose wave-length is

$$\frac{2\pi a^{\frac{1}{2}} \left(\frac{1}{2} \log \frac{4\mu\sigma}{\pi p a^2} \right)^{\frac{1}{2}}}{\left\{ \frac{4\pi p}{\mu\sigma} \right\}^{\frac{1}{4}} \cos \frac{\pi}{8}},$$

propagated with velocity

$$a^{\frac{1}{2}} p^{\frac{3}{4}} \mu^{\frac{1}{4}} \sigma^{\frac{1}{4}} \left\{ \frac{1}{4\pi^{\frac{1}{2}}} \log \frac{4\mu\sigma}{\pi p a^2} \right\}^{\frac{1}{2}} \sec \frac{\pi}{8},$$

and dying away to $1/e$ of its original amplitude after traversing a

distance $$\cot \frac{\pi}{8} \text{ (wave-length).}$$

To reduce this to numbers, let us take the case of an iron rod 1 centimetre in radius, and for which $\sigma = 10^4$, $\mu = 500$; then, for vibrations in the magnetic force at the rate of 100 per second, the velocity of propagation is in round numbers between 5000 and 6000 centimetres per second. The wave-length, therefore, is about 5 or 6 centimetres, and the amplitude of the vibration would die away to $1/e$ of its original value after traversing about 13 or 14 centimetres.

The preceding cases only apply to iron wires, as for wires of all other metals we may put $\mu = 1$; in this case equation (21) becomes

$$in \frac{J_0(ina)}{J_0'(ina)} = i\kappa \frac{I_0(i\kappa a)}{I_0'(i\kappa a)} \quad \text{.....................(23).}$$

We can see that there cannot be disturbance propagated whose w length is a large multiple of the diameter of the wire, for in this κa is small, and the right-hand side of (23) very small; but the hand side of (23) cannot be small, unless ina is nearly equal where ξ is a root of the equation $J_0(x) = 0$. The smallest value and therefore of ina, is about 2·5, which is not, as is required by supposition, small.

When κa is large, an approximate solution of (23) is

$$i\kappa a = \eta,$$

where η is a root of the equation

$$I_0'(x) = 0,$$

so that, in this case, we have a disturbance fading away to $1/e$ of value, after traversing a distance comparable with the diamete the wire; so that in this case the disturbance is practically not pagated at all.

The two theories lead to identical results in this case. We sl therefore go on to investigate the case when the current is lo tudinal, and the lines of magnetic force transverse.

In my previous paper, I investigated several cases of this probl assuming Maxwell's theory. In that paper, however, as Mr. Heavi has pointed out, μ has been left out of one of the boundary conditi which was taken as dH/dr continuous instead of $\frac{1}{\mu}dH/dr$, whe is the magnetic permeability of the wire. The effect of this cha in the boundary condition is to introduce a μ under the differer coefficient of a J_0 wherever it occurs in a final equation. T change will not affect the results arrived at for copper wires, nor the velocity of propagation of a disturbance along an iron wire; the latter, however, it does affect the rate at which the vibrations away: in fact they will only travel $\frac{1}{\mu}$ times the distance statec that paper, and hence a disturbance will travel further along a co wire than along an iron one instead of the reverse.

In the case of a current flowing longitudinally along a met wire surrounded by a dielectric, we have, on Maxwell's theory *Proc. of Math. Soc.*, XVII., p. 312),

in the wire. $\phi = AJ_0(imr)\,\epsilon^{imz}\epsilon^{ipt}$.

$$H = \left(BJ_0(inr) - \frac{m}{p}AJ_0(inr)\right)\epsilon^{imz}\epsilon^{ipt}.$$

[redacted]

$$F = \frac{d\chi}{dx}, \qquad G = \frac{d\chi}{dy},$$

$$\chi = \left(-\frac{im}{n^2} BJ_0(inr) - \frac{1}{ip} AJ_0(imr)\right) \epsilon^{imz} \epsilon^{ipt};$$

in the dielectric,

$$\phi = A'I_0(imr)\, \epsilon^{imz} \epsilon^{ipt},$$

$$H = \left(DI_0(i\kappa r) - \frac{im}{ip} A'I_0(imr)\right) \epsilon^{imz} \epsilon^{ipt},$$

$$F = \frac{d\chi'}{dx}, \qquad G = \frac{d\chi'}{dy},$$

$$\chi' = \left(-\frac{im}{k^2} DI_0(i\kappa r) - \frac{A'}{ip} I_0(imr)\right) \epsilon^{imz} \epsilon^{ipt},$$

where
$$k^2 = m^2 + \frac{p^2}{v^2},$$

$$n^2 = m^2 + \frac{4\pi\mu ip}{\sigma},$$

v being the velocity with which electro-dynamic influence is propagated.

Since ϕ and H are continuous,

$$BJ_0(ina) = DI_0(i\kappa a).$$

Since the magnetic force parallel to the surface $\frac{1}{\mu}\left\{\frac{dH}{dr} - \frac{d^2\chi}{dr\,dz}\right\}$ is continuous, we have

$$\frac{in}{\mu} B \left\{1 - \frac{m^2}{n^2}\right\} J_0'(ina) = i\kappa D\left(1 - \frac{m^2}{k^2}\right) I_0'(i\kappa a),$$

or, eliminating B and D,

$$\frac{m^2-n^2}{\mu n} \frac{J_0'(ina)}{J_0(ina)} = \frac{m - k^2}{k} \frac{I_0'(i\kappa a)}{I_0(i\kappa a)} \quad \text{.................(24)},$$

an equation which is shown, in the paper already referred to, to denote, in general, a disturbance propagated along the wire with velocity v.

Let us now investigate the same problem on v. Helmholtz's theory.

In the dielectric, the equation for ϕ is

$$\nabla^2\phi = \frac{A^2 k 4\pi\epsilon}{1 + 4\pi\epsilon} \frac{d^2\phi}{dt^2},$$

of which the solution is

$$\phi = L' I_0 (iqr)\, \epsilon^{imz} \epsilon^{ipt},$$

where
$$q^2 = m^2 - p^2 \frac{A^2 k 4\pi\epsilon}{1+4\pi\epsilon},$$

in the wire the equation for ϕ is

$$\left(\frac{\sigma}{4\pi}\frac{d}{dt}+1\right)\nabla^2\phi = A^2 k^2 \frac{d^2\phi}{dt^2},^*$$

of which the solution is

$$\phi = L\, J_0 (iq'r)\, \epsilon^{imz} \epsilon^{ipt},$$

where
$$q'^2 = m^2 - \frac{A^2 k p^2}{\frac{\sigma}{4\pi} ip + 1}.$$

In the dielectric the equation for H is, see equation (17),

$$\frac{1}{4\pi\mu\epsilon}\nabla^2 H = \frac{d^2 H}{dt^2} + \frac{d^2\phi}{dz\,dt}\left(\frac{1+4\pi\epsilon}{4\pi\epsilon} - \frac{k}{4\pi\epsilon\mu}\right) \ldots\ldots\ldots$$

of which a particular integral is

$$H = \left(1+\frac{1}{4\pi\epsilon}\right)\frac{1}{p^2}\frac{d^2\phi}{dz\,dt}.$$

The complete solution of (25) is

$$H = M' I_0 (i\kappa' r) + \left(1+\frac{1}{4\pi\epsilon}\right)\frac{1}{p^2}\frac{d^2\phi}{dz\,dt},$$

where
$$\kappa'^2 = m - \frac{p^2}{v'^2},$$

and v' the rate of propagation of the electrodynamic action.

Since
$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = -A^2 k \frac{d\phi}{dt},$$

the values of F and G are given by the equations

$$F = \frac{d\chi'}{dx}, \quad G = \frac{d\chi'}{dy},$$

where
$$\chi' = -\frac{im}{k'^2} M' I_0 (ik'r) + \left(1+\frac{1}{4\pi\epsilon}\right)\frac{1}{p^2}\frac{d\phi}{dt}.$$

---

* In this equation σ is measured in electrostatic units.

In the wire the equation for H is, see equation (18),

$$\frac{\sigma}{4\pi\mu}\nabla^2 H = \frac{dH}{dt} - \left(\frac{\sigma K}{4\pi\mu} - \frac{\sigma}{4\pi}\right)\frac{d^2\phi}{dx\,dt} + \frac{d\phi}{dx},$$

of which a particular solution is

$$H = \frac{\left(\frac{\sigma}{4\pi} ip + 1\right)}{p^2}\,\frac{d^2\phi}{dz\,dt},$$

where
$$n^2 = m^2 + \frac{4\pi\mu ip}{\sigma}.$$

The values of F and G are given by the equations

$$F = \frac{d\chi'}{dx},\quad G = \frac{d\chi'}{dy},$$

where
$$X' = -\frac{im}{n^2} MJ_0(inr) + \frac{\left(\frac{\sigma ip}{4\pi} + 1\right)}{p^2}\,\frac{d\phi}{dt}.$$

If a is the radius of the wire, we have, since ϕ is continuous, as we pass from the wire to the dielectric,

$$L'I_0(iq\mathrm{a}) = LJ_0(iq'\mathrm{a}).$$

Since H is also continuous, we have

$$M'I_0(i\kappa\mathrm{a}) - \left(1 + \frac{1}{4\pi\epsilon}\right)\frac{m}{p} L'I_0(iq\mathrm{a})$$
$$= MJ_0(in\mathrm{a}) - \left(\frac{\sigma}{4\pi} ip + 1\right)\frac{m}{p} LJ_0(iq'\mathrm{a}).$$

Since F and G are continuous, we have

$$M'\frac{m}{\kappa'} I_0'(i\kappa\mathrm{a}) - \frac{q\left(1 + \frac{1}{4\pi\epsilon}\right)}{p} L'I_0'(iq\mathrm{a})$$
$$= M\frac{m}{n} J_0'(in\mathrm{a}) - \frac{q'}{p}\left(\frac{\sigma}{4\pi ip} + 1\right) LJ_0'(iq'\mathrm{a}).$$

The induction parallel to the surface of the wire is

$$MinJ_0'(in\,\mathrm{a}) - \frac{m^2}{n^2}\, in\, MJ_0'(in\,\mathrm{a})$$

in the wire, and
$$M'i\kappa' I_0'(i\kappa'\mathrm{a}) - \frac{m^2}{\kappa_1^2} M'I_0'(i\kappa'\mathrm{a})$$

in the dielectric. Since the magnetic force parallel to the surface is

continuous, we have

$$\frac{1}{\mu} M in\left(1-\frac{m^2}{n^2}\right) J_0'(in\mathrm{a}) = M' i\kappa' I_0'(i\kappa'\mathrm{a})\left(1-\frac{m^2}{\kappa'^2}\right).$$

From these equations, we get

$$M\left\{J_0(ina) - \frac{n}{\kappa'\mu}\frac{\left(1-\frac{m^2}{n^2}\right)}{1-\frac{m^2}{\kappa'^2}}\frac{J_0'(ina)\,I_0(i\kappa'a)}{I_0'(i\kappa'a)}\right\}$$

$$= \left(\frac{ip\sigma}{4\pi} - \frac{1}{4\pi\epsilon}\right)\frac{mL}{p} J_0(iq'a),$$

$$\frac{mM}{n} J_0'(in\mathrm{a})\left\{1 - \frac{n^2-m^2}{\mu(\kappa_1^2-m^2)}\right\}$$

$$= \frac{L}{p}\left\{\left(1+\frac{1}{4\pi\epsilon}\right) q\frac{I_0'(iq\mathrm{a})}{I_0(iq\mathrm{a})} J_0(iq'\mathrm{a}) - \left(1+\frac{ip\sigma}{4\pi}\right) q' J_0'(i\text{…}\right.$$

Eliminating M and L from these equations, we get

$$\frac{J_0(in\mathrm{a})}{J_0'(in\mathrm{a})} - \frac{n}{\kappa_1\mu}\frac{\left(1-\frac{m^2}{n^2}\right)}{\left(1-\frac{m^2}{\kappa_1^2}\right)}\frac{I_0(i\kappa'\mathrm{a})}{I_0'(i\kappa'\mathrm{a})}$$

$$= \frac{\frac{m^2}{n}\left\{1-\frac{n^2-m^2}{\mu(\kappa_1^2-m^2)}\right\}\left(\frac{ip\sigma}{4\pi}-\frac{1}{4\pi\epsilon}\right) ip\, J_0(iq'\mathrm{a})\, I_0(iq\text{…}}{\left(1+\frac{1}{4\pi\epsilon}\right) q J_0(iq\mathrm{a})\, I_0'(iq\mathrm{a}) - \left(\frac{\sigma}{4\pi} ip+1\right) q' I_0(iq\mathrm{a})\, J_0'\text{…}}$$

....

In Maxwell's theory, in the corresponding equation the right-side was the same, but the left-hand side vanished. To invest the magnitude of the right-hand side, we must remember that measured in electrostatic units, and that its value in these even for so good a conductor as copper, is of the order $1{\cdot}6\times$ so that we may without appreciable error, substitute unit $\frac{\sigma ip}{4\pi}+1$, and put $q'=q$.

Now, $$q^2 = m^2 - \frac{p^2}{v_0^2},$$

so that, unless the vibrations are so rapid that the wave-leng comparable with the radius of the wire, qa will be extremely

and we may therefore put

$$J_0(iqa) = 1, \quad I_0(iqa) = \log(\gamma iqa),$$

where $$\log\gamma = \cdot 577 - \log 2.$$

Again, $$n^2 - m^2 = \frac{4\pi ip}{\sigma}, \quad \text{and} \quad k_1^2 - m^2 = \frac{p^2}{v^2};$$

so that, unless the vibrations are comparable in rapidity with those of light, $n^2-m^2/\mu\,(\kappa_1^2-m^2)$ will be large compared with unity; so that we may write the right-hand side of the equation (25) as

$$\frac{1}{1+4\pi\epsilon}\,\frac{na}{\mu}\log(\gamma iqa),$$

and, unless na is large, this will be small compared with either term on the left-hand side of equation (25); so that, unless the vibrations are so rapid as to make na/μ very large, Maxwell's and v. Helmholtz's theory give the same results for the velocity of propagation of a disturbance along a conductor.

In the previous paper we proved that, for a wire at an infinite distance from another conductor, the velocity of propagation was the same as that of electro-dynamic actions.

Many experiments have been made to determine this velocity, with exceedingly discordant results, as the following table taken from Sir William Thomson's paper on the "Velocity of Electricity" (*Reprint*, Vol. II., p. 132), will show.

	Miles per Second.
Wheatstone, in 1834, with Copper Wire	288,000
Walker, in America, with Telegraph Iron Wire...	18,780
Mitchell ditto ditto	28,524
Fizeau and Gounelli, Copper Wire	112,680
Ditto Iron Wire	62,600
A. B. G. (copper), London and Brussels Telegraph	2,700
Ditto (copper), London and Edinburgh Telegraph	7,600
Induction Coils through 2,500 miles Atlantic Cable tested by heavy Needle Galvanometer, Queenstown, 1857 ...	1,430
Daniell's Battery through 3000 miles Atlantic Cable tested by Mirror Galvanometer, Devonport, 1888	3,000

To these we may add W. Siemens' determination (*Pogg. Ann.*, CLVII., p. 309, 1876) of the velocity through iron wire as $1{\cdot}8\times10^{10}$ centimetres per second, and Hertz's (Wiedemann's *Annalen*, XXXIV., p. 559, 1888) as 2×10^{10} centimetres per second. The discordance between

these results shows that the preceding theory is not applical the most obvious cause is the presence of conductors in the bourhood.

In my preceding paper I investigated the effect of capacity velocity, and showed that the action is or is not propagated w velocity of propagation of electro-dynamic action according as

$$\frac{\left(\begin{matrix}\text{capacity of a centimetre of wire} \\ \text{in electrostatic measure}\end{matrix}\right)\left(\begin{matrix}\text{resistance of a centimetre of} \\ \text{in electro-magnetic meas}\end{matrix}\right)}{p} \quad \ldots\ldots\ldots\ldots$$

is small or large compared with unity, $4\pi\mu ipa/\sigma$ being a quantity. If this quantity is large, the disturbance is propagated the wire with the velocity of propagation of electro-dynamic ac

Now, if we consider the case of an overhead telegraph wire radius is at the distance d from the ground, the capacity in el static measure of a centimetre is

$$\tfrac{1}{2}\log\frac{2d}{a},$$

so that, if the wire is ·25 of a centimetre in radius, and at a heig four metres from the ground, the capacity of a centimetre is 1/18. The resistance of such an iron wire is about 5×10^4 in el magnetic measure, so that, unless the direction of the current ch nearly 400 times a second, the quantity (26) would be larger unity, and the velocity of propagation in this case would not b same as that of electro-dynamic action.

In this case it is shown that the velocity of propagation is

$$v\left\{\frac{2p}{rc}\right\}^{\frac{1}{2}},$$

where r is the resistance of a centimetre of the wire, and capacity of the same length. As the velocity in this case depen the rate of reversal of the current, vibrations of different pitch travel at different rates; thus, if an arbitrary disturbance is municated to the wire, we cannot without further investigation at what rate it will travel. The axis of the wire being as b taken as the axis of z, let F be a function of z and t, represen some quantity which is propagated according to the same laws disturbance along the wire.

Then, since $\qquad F=\epsilon^{i(pt+mz)}$

will satisfy this condition, if

$$m^2=\frac{p}{v^2a^2}-\frac{ircp}{v^2},$$

where r is the resistance of unit length and c its capacity; so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}(a)\,\epsilon^{-m^2v^2t/rc}\cos m\,(z-a)\,dm\,da \quad \ldots\ldots\ldots \quad (27)$$

will satisfy the condition, and by Fourier's theorem will equal $F(x)$, when $F=0$. Thus, if $F(x)$ represents the value initially, the expression (27) will represent it after a time t, since

$$\int_{-\infty}^{+\infty}\epsilon^{-ax^2}\cos b\,x\,dx = \frac{\sqrt{\pi}}{\sqrt{a}}\,\epsilon^{-b^2/4a}.$$

We may write the expressions (27) as

$$\frac{\sqrt{rc}}{2\sqrt{\pi t v}}\int_{-\infty}^{+\infty}F(a)\,\epsilon^{-(z-a)^2 rc/4v^2t}\,da,$$

so that, if the disturbance were initially confined to a short space near the origin, writing F_1 for $\int F(a)\,dx$, we see that, after a time t and at a distance z, it will be represented by

$$\frac{\sqrt{rc}}{2\sqrt{\pi t}}\,v F_1\,\epsilon^{\frac{-z^2rc}{4v^2t}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

The disturbance at z will attain a maximum after a time T, where

$$T=\frac{z^2rc}{v^2}.$$

Thus $z/T = v^2/zrc$, and since this is the quantity which is measured when the velocity of propagation is measured by means of signals along telegraph wires, we see that the "velocity" will depend upon the length of the line, being greater the shorter the line. This will explain the discrepancy between the results of the various experiments made on the rate of propagation of signals along telegraph wires.

Since, however, the above theory only applies when $4\pi\mu p/\sigma$ is small, it will not explain why in Hertz's experiments, when periodic currents which alternated millions of times in a second were used, the velocity of propagation along the wire was not the same as that of electro-dynamic action, as in this case $4\pi\mu p/\sigma$ was large, and, whenever this is the case, the velocity along the wire is the same as the velocity of electro-dynamic action through the dielectric.

The velocity was determined by measuring the electrical wave-length and dividing by the period of vibration of the electrical system.

This system consisted of two plates of a condenser connected by wire, and the time of vibration of this system was deduced from t formula $2\pi\sqrt{Lc}$, where L is the coefficient of self-induction of t wire and c the capacity of the condenser. The coefficient of se induction L was calculated on the supposition that the current uniform throughout the length of the wire. This formula, howev as the following investigation shows, is only true for alternati currents when the electrical wave-length is very much greater th the length of the wire, and ceases to be accurate when, as in Hert experiments, the wave-length is only a small multiple of the length wire; the actual numbers in Hertz's experiments being about 3 to 1

Let us consider the case of two plates of a condenser connected l a wire, and let us suppose that the plates of the condenser we originally charged with equal quantities of positive and negati electricity. The quantity of positive electricity which flows out one plate of the condenser must be the same as the quantity negative which flows into the other.

Let us take the wire as the axis of z, the origin being at one end the wire. Then, if l is the length of the wire, we may, using the sam notation as before, put

$$H = B\left(\epsilon^{im(z-\frac{1}{2}l)} + \epsilon^{-im(z-\frac{1}{2}l)}\right) \epsilon^{ipt} J_0(inr) - \frac{1}{ip}\frac{d\phi}{dz},$$

$$\phi = \left\{A\,\epsilon^{im(z-\frac{1}{2}l)} + A'\epsilon^{-im(z-\frac{1}{2}l)}\right\} \epsilon^{ipt} J_0(imr).$$

Since
$$\sigma w = -\frac{dH}{dt} - \frac{d\phi}{dz}$$
$$= -Bip\left(\epsilon^{im(z-\frac{1}{2}l)} + \epsilon^{-im(z-\frac{1}{2}l)}\right) \epsilon^{ipt} J_0(inr),$$

the quantity of electricity which flows through a section at a distanc z from the origin

$$= \int_0^a w\, 2\pi r\, dr$$

$$= -\frac{Bip}{\sigma}\left\{\epsilon^{im(z-\frac{1}{2}l)} + \epsilon^{-im(z-\frac{1}{2}l)}\right\} \epsilon^{ipt} \int_0^a J_0(inr)\, 2\pi r\, dr;$$

but, since
$$\frac{d^2 J_0(inr)}{dr^2} + \frac{1}{r}\frac{dJ_0(inr)}{dr} - n^2 J_0(inr) = 0,$$

$$rJ_0(inr) = \frac{1}{n^2}\frac{d}{dr}\left(r\frac{dJ_0(inr)}{dr}\right).$$

Making this substitution, we see that the quantity of electricit

flowing through the section at a distance z from the origin

$$= -\frac{2Bip}{n^2\sigma}\cos m\left(z-\frac{l}{2}\right)\epsilon^{ipt}\,2\pi a\,\frac{d}{da}J_0(ina),$$

so that the quantity flowing out of the first condenser into the second

$$= -\frac{4\pi Bipa}{n^2\sigma}\cos\frac{ml}{2}\epsilon^{ipt}\frac{d}{da}J_0(ina) \;\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

If V_1 and V_2 are the potentials of the plates of the condensers, C the capacity, this must equal

$$-C\frac{d}{dt}(V_1-V_2).$$

We have, since F and G are continuous,

$$B\left(\frac{m}{n}J_0'(ina)-\frac{m}{k}\frac{I_0'(i\kappa a)}{I_0(i\kappa a)}J_0(ina)\right)=\frac{Am}{p}\left(J_0'(ima)-\frac{I_0'(ima)}{I_0(ima)}J_0(ima)\right)$$

$$= -\frac{Am}{p}\frac{1}{ima}\frac{1}{I_0(ima)},$$

and therefore

$$\frac{Bm}{n\mu}J_0'(ina)\left(\mu-\frac{m^2-n^2}{m^2-k^2}\right) = -\frac{A}{ipa}\frac{1}{I_0}(ima),$$

or, approximately, since k^2 is small compared with m^2,

$$B\frac{m}{n}J_0'(ina)\left(1+\frac{4\pi ip}{\sigma m^2}\right) = -\frac{A}{ipa}\frac{1}{I_0(ima)} \;\ldots\ldots\ldots\ldots (30).$$

The relation between B and A' can be found by writing $-m$ for m in this equation; hence $A' = -A$, so that

$$V_1 - V_2 = -4iA\sin\frac{ml}{2}\epsilon^{ipt},$$

and therefore

$$-C\frac{d}{dt}(V_1-V_2) = -4pCA\sin\frac{ml}{2}\epsilon^{ipt},$$

and this equals

$$-\frac{4\pi Bipa}{n^2\sigma}\cos\frac{ml}{2}\epsilon^{ipt}\frac{d}{da}J_0(ina).$$

Substituting the value of A/B from (30), we get, since $4\pi p/\sigma m^2$ is large compared with unity,

$$4p^2C\tan\frac{ml}{2}I_0(ima) = -m,$$

or $$4p^2 C \tan \gamma ima \log \gamma ima = -m \text{ .... (}$$

The ordinary formula when the wave-length is large compared the length of the wave is

$$2p^2 Cl \left(\log \frac{4l}{d}\right) = 1.$$

Thus the true value of p^2 is to the one reckoned on the hypotl that the currents are uniform, as

$$\frac{m\frac{l}{2}}{\tan\frac{ml}{2}} = \frac{\frac{\pi l}{\lambda}}{\tan\frac{\pi l}{\lambda}},$$

if λ is the electrical wave-length.

In Hertz's experiments, l was 60, λ 560, so that the correction wo be of the wrong sign to explain the discrepancy between theory experiment. If, however, there had been waves whose fronts were normal to the wire reflected from the sides of the wall, they wo diminish the wave-length on the wire, and so diminish the calcula velocity.

We may write equation (31) in the form

$$4\, ml \tan\frac{ml}{2} = \frac{l}{C' \log \gamma ima} \text{ ....(3}$$

where C' is the electrostatic measure of the capacity of C/v^2 whe is the velocity of propagation of the electro-dynamic influence. T equation serves to determine the electrical wave-length.

In the theories of light the normal disturbance is essentially volved in the question of reflection and refraction of light; it n therefore be of interest to examine this question from the point view of the two theories.

By referring to the beginning of this paper, it will be seen tl according to both theories,

$$F = X + A\frac{d\phi}{dx},$$

when X represents some function propagated with the rate of elect dynamic influence, and ϕ the electrostatic potential; the multipl A being different on the two theories.

Let us first consider the question of the reflection and refraction light polarised so that the electric disturbance is at right angles

the plane containing the incident and reflected ray; let us take this plane as the plane of xy, the equation to the reflecting surface being $y = 0$.

Then, since in this case the vector potential is parallel to the axis of z and $d\phi/dz$ vanishes, the two theories will agree; and we may put, if v is the velocity of propagation in the first medium, v' that in the second, λ and λ' the wave-lengths of the light in the first and second media respectively,

$$\left.\begin{aligned} H &= B \exp. i\frac{2\pi}{\lambda}(x\sin i + y\cos i - vt), \text{ for the incident light} \\ H &= B_1 \exp. i\frac{2\pi}{\lambda}(x\sin i - y\cos i - vt), \text{ for the reflected light} \\ H &= B_2 \exp. i\frac{2\pi}{\lambda_1}(x\sin r + y\cos r - vt), \text{ for the refracted light} \end{aligned}\right\} \ldots (33).$$

Since H is continuous at the surface $y = 0$,

$$B + B_1 = B_2 \ldots\ldots\ldots\ldots\ldots\ldots (34);$$

the magnetic induction parallel to the surface is $\frac{dH}{dy}$; and, since the magnetic force is continuous, we have, if μ_0 be the magnetic permeability in the upper, μ_1 in the refracting media,

$$\frac{1}{\mu_0}\left(B\frac{2\pi}{\lambda}\cos i - B_1'\frac{2\pi}{\lambda}\cos i\right) = \frac{1}{\mu_1} B_2 \frac{2\pi}{\lambda_1}\cos r,$$

$$B - B_1 = \frac{\mu_0}{\mu_1}\frac{\lambda}{\lambda_1}\frac{\cos r}{\cos i} B_2;$$

and if μ is the refractive index, since

$$\frac{\lambda}{\lambda_1} = \mu = \frac{\sin i}{\sin r},$$

we have
$$B - B_1 = \frac{\mu_0}{\mu_1}\frac{\sin i}{\sin r}\frac{\cos r}{\cos i}(B_2) \ldots\ldots\ldots\ldots (35).$$

Adding and subtracting equations (34) and (35), we get

$$2B = B_2\left(1 + \frac{\mu_0}{\mu_1}\frac{\sin i \cos r}{\sin r \cos i}\right) \ldots\ldots\ldots\ldots (36),$$

and
$$B_1 = B\frac{\left(\frac{\mu_1}{\mu_0}\frac{\sin r}{\cos r}\frac{\cos i}{\sin i} - 1\right)}{\frac{\mu_1}{\mu_0}\frac{\sin r}{\cos r}\frac{\cos i}{\sin i} + 1}.$$

If the media are non-magnetic, $\mu_1 = \mu_0$, and therefore

$$B_2 = \frac{2 \sin r \cos i}{\sin (r+i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad ($$

$$B_1 = \frac{\sin (i+r)}{\sin (i+r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad ($$

so that, if the intensity of the incident light is 1, that of the refle will be $\frac{\sin^2 (1-r)}{\sin^2 (1+r)}$ and of the refracted $\frac{4 \sin^2 i \cos^2 r}{\sin^2 (r+i)}$.

Let us now take the case when the electric force is in the plan incidence, and let

$$\phi = A_1 \exp.\, i \frac{2\pi}{\lambda} (x \sin i - y \cos i' - v_0 t), \text{ in the upper medium,}$$

$$\phi = A_2 \exp.\, i \frac{2\pi}{\lambda} (x \sin r' + y \cos r' - v_0' t), \text{ in the lower medium.}$$

Since ϕ is continuous, we have

$$A_1 = A_2.$$

Let the part of the vector potential propagated with the velo of electro-dynamic influence be

$$F' = B \exp.\, i \frac{2\pi}{\lambda} (x \sin i + y \cos i - vt) \cos i,$$

$$F_1' = B_1 \exp.\, i \frac{2\pi}{\lambda} (x \sin i - y \cos i - vt) \cos i,$$

$$G' = B \exp.\, i \frac{2\pi}{\lambda} (x \sin i + y \cos i - vt) \sin i \text{ for the incident r}$$

$$G_1' = B_1 \exp.\, i \frac{2\pi}{\lambda} (x \sin i - y \cos i - vt) \sin i \text{ for the reflected r}$$

$$F_2' = B_2 \exp.\, i \frac{2\pi}{\lambda} (x \sin r + y \cos r - vt) \cos r,$$

$$G_2' = B_2 \exp.\, i \frac{2\pi}{\lambda'} (x \sin r + y \cos i - vt) \sin r,$$

in the refracting substance.

[redacted]

To these we must add $\gamma\, d\phi/dx$ for F and $\gamma\, d\phi/dy$ for G in the upper, and $\gamma' \frac{d\phi}{dx} \cdot \gamma' \frac{d\phi}{dy}$ in the lower medium, where

$$\gamma = \left(1+\frac{1}{4\pi e}\right)\frac{1}{ip}, \quad \gamma' = \left(1+\frac{1}{4\pi e'}\right)\frac{1}{ip}.$$

Since F and G are continuous, we have

$$\cos i\,(B-B_1) - \cos r\, B_2 + (\gamma+\gamma')\frac{2\pi}{\lambda}\sin i\, A_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots(39),$$

$$\sin i\,(B+B_1) - \sin r\, B_2 - A_1\left(\gamma\frac{2\pi}{\lambda}\cos i + \gamma'\frac{2\pi}{\lambda'}\cos r'\right) = 0 \quad\ldots(40);$$

and, since the magnetic force parallel to the surface of separation is continuous,

$$\frac{2\pi}{\lambda\mu}(B+B_1) = \frac{2\pi}{\lambda_1}B_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

or for non-magnetic media

$$(B+B_1)\sin r - B_2 \sin i = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(42).$$

Solving the equations, we find

$$B_1 = \frac{B\tan(i-r)}{\tan(i+r)}\,\frac{\left\{1-\frac{\eta}{\xi}\tan(i+r)\right\}}{\left\{1+\frac{\eta}{\xi}\tan(i-r)\right\}},$$

$$B_2 = \frac{2B\cos i\sin r}{\sin(i+r)\cos(i-r)\left\{1+\frac{\eta}{\xi}\tan(i-r)\right\}},$$

where
$$\frac{\eta}{\xi} = \frac{\gamma-\gamma'}{\gamma\cot i' + \gamma'\cot r'}.$$

Now γ/γ' is real, so that if, as in this case, there is a change in phase produced by reflection, $\cot i$ or $\cot r'$ must be imaginary.

If v is the velocity of propagation of electro-dynamic influence, v_0 that of electrostatic potential,

$$\cos i = \sqrt{1-\frac{\sin^2 i\, v_0^2}{v^2}},$$

$$\cos r' = \sqrt{1-\frac{\sin^2 i\, v_0^2}{v^2}}.$$

2 N 2

These will not in general be imaginary unless v_0 is consid greater than v; but if v_0 is a large multiple of v, cos i and cos be purely imaginary, and

$$\frac{\eta}{\xi} = \frac{i(\gamma-\gamma')v}{\gamma+\gamma' v_0} = iM, \text{ say.}$$

Then

$$B_1 = \frac{B\tan(i-r)}{\tan(i+r)} \frac{\{1-iM\tan(1+r)\}}{\{1+iM\tan(1-r)\}} \quad \ldots\ldots\ldots$$

$$= B\left\{\frac{\cot^2(i+r)+M^2}{\cot^2(1-r)+M^2}\right\}^{\frac{1}{2}} \xi^{-i(\psi+\psi')};$$

thus the ratio of the intensities of the reflected light and inc light is

$$\frac{\cot^2(1+r)+M^2}{\cot^2(1-r)+M^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and, if ϵ is the change in phase,

$$\epsilon = \tan^{-1} M\tan(i+r)+\tan^{-1} M\tan(i-r) \quad \ldots\ldots\ldots$$

These are identical in form with Green's expressions, and Ja experiments seem to show that they represent the facts if M i garded as an independent constant very much smaller than the given by Green.

On Maxwell's theory $\gamma = \gamma'$, so that there would be no chan phase if the assumption is correct that the potential is continuo passing from the one medium to the other.

If, however, there were a layer on the surface charged with o site electricities on its two faces, so as to produce a finite differ of potential in crossing it, we should get formulæ for the intensi the reflected light and change in phase identical with those given.

Thus, assuming the electro-magnetic theory, the phenon of reflection and refraction are in accordance with the the that the electrostatic potential is propagated with a f velocity greater than the velocity of electro-dynamic fluence.

We shall now investigate th[illegible]f the energy in the conde tional wave produced by ref[illegible] the energy in the inci transverse wave producing it.

From equations (39) and (40), we have

$$A\,\frac{2\pi}{\lambda}\sin i\,(\gamma-\gamma')$$

$$= B\cos i\left\{1-\frac{\tan(i-r)}{\tan(i+r)}\,\frac{\left(1-\frac{\eta}{\xi}\tan(i-r)\right)}{1+\frac{\eta}{\xi}\tan(i+r)}\right.$$

$$\left.-\frac{2\cos r\sin r}{\sin(i+r)\cos(i-r)}\,\frac{1}{1+\frac{\eta}{\xi}\tan(i-r)}\right\}$$

$$= 2B\,\frac{\eta}{\xi}\,\frac{\cos i\tan(i-r)}{1+\frac{\eta}{\xi}\tan(i-r)},$$

or, since

$$\frac{\eta}{\xi}=\frac{\gamma-\gamma'}{\gamma\cot i'+\gamma'\cot r'},$$

$$A=\frac{\lambda}{2\pi}\,\frac{2B\cot i\tan(i-r)}{1+\frac{\eta}{\xi}\tan(i+r)}\,\frac{1}{\gamma\cot i'+\gamma'\cot r'} \quad \text{.........(46).}$$

The mean electrostatic energy, in unit volume of the incident transverse wave, is the mean value of

$$\frac{K}{8\pi}\left(\frac{dF}{dt}\right)^2 \quad\text{or}\quad \frac{K}{16\pi}B^2p^2,$$

where K is the specific inductive capacity of the medium, and

$$p=\frac{2\pi}{\lambda}v.$$

The mean electrostatic energy in unit volume of the condensational wave is

$$\frac{K}{10\pi^2}\,\frac{4\pi^2}{\lambda_1^2}A^2.$$

The ratio of this to the energy in the transverse wave is

$$\frac{A^2}{B^2}\,\frac{4\pi^2}{p^2\lambda^2},$$

or, substituting for A/B its value from equation (46),

$$\frac{4\cot^2 i\tan^2(i-r)}{\left(1+\frac{\eta}{\xi}\tan(i-r)\right)^2}\,\frac{v^2}{v_0^2p^2}\,\frac{1}{(\gamma\cot i'+\gamma'\cot r')^2}.$$

On the Development of certain Elliptic Functions as Cont Fractions. By L. J. ROGERS, M.A.

[*Read January 12th*, 1888.]

1. In Crelle's *Journal*, Vol. XXXII., Heine has established c theorems in connection with series of the form

$$1+\frac{\sin\lambda\alpha}{\sin\lambda}\cdot\frac{\sin\lambda\beta}{\sin\lambda\gamma}x+\frac{\sin\lambda\alpha\,\sin\lambda\,(\alpha+1)\,\sin\lambda\beta\,\sin\lambda\,(\beta+1)}{\sin\lambda\,\sin 2\lambda\,\sin\lambda\gamma\,\sin\lambda\,(\gamma+1)}x^2$$

analogous to Gauss's Hyper-geometric Series

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha\,(\alpha+1)\,.\,\beta\,(\beta+1)}{1\,.\,2\,.\,\gamma\,(\gamma+1)}x^2+\ldots,$$

and formed from the latter by substituting for every factor i coefficients the sine of some constant angle λ multiplied by factor.

Heine writes his series in the form

$$1+\frac{1-q^\alpha}{1-q}\cdot\frac{1-q^\beta}{1-q^\gamma}x+\frac{1-q^\alpha}{1-q}\cdot\frac{1-q^{\alpha+1}}{1-q^2}\cdot\frac{1-q^\beta}{1-q^\gamma}\cdot\frac{1-q^{\beta+1}}{1-q^{\gamma+1}}x^2+\ldots,$$

to which the above can be reduced by an easy substitution. I however, thought it advisable to treat of the series in the trig metrical form, in order to keep before the eye the analogy bet Heine's series and that of Gauss.

If we call the above trigonometrical series

$$F\sin\lambda\left\{\alpha,\beta,\gamma,x\right\},$$

Heine shows that the fraction

$$F\sin\lambda\left\{\alpha,\beta+1,\gamma+1,x\right\}\div F\sin\lambda\left\{\alpha,\beta,\gamma,x\right\}$$

can be reduced to a continued fraction of the form

$$\frac{e_0}{1-}\;\frac{e_1x}{1-}\;\frac{e_2x}{1-}\ldots,$$

by a method precisely similar to that used by Gauss in establis his analogous theorem for hyper-geometric series, explaine Todhunter's *Algebra*, § 801.

In fact we have

$$e_{2n-1} = \frac{\sin\lambda\,(\alpha+n-1)\sin\lambda\,(\gamma+n-1-\beta)}{\sin\lambda(\gamma+2n-2)\sin\lambda\,(\gamma+2n-1)},$$

$$e_{2n} = \frac{\sin\lambda\,(\beta+n)\sin\lambda\,(\gamma+n-\alpha)}{\sin\lambda\,(\gamma+2n-1)\sin\lambda\,(\gamma+2n)}.$$

It is my object in the following pages to show how certain elliptic functions can, by means of Heine's theorem, be brought into the form of a continued fraction.

2. It can be proved by induction or otherwise that the series

$$1+\frac{\sin\lambda n}{\sin\lambda}x+\frac{\sin\lambda n}{\sin\lambda}\cdot\frac{\sin\lambda\,(n-1)}{\sin 2\lambda}x^2+\dots \qquad \dots\dots\dots\dots (1),$$

formed by the process explained above from the binomial series, is decomposable into factors, provided n be a positive integer.

If $n = 2m$, it is equal to

$$(1+2x\cos\lambda+x^2)(1+2x\cos 3\lambda+x^2)(1+2x\cos 5\lambda+x^2)\dots \text{ to } m \text{ factors.}$$

If $n = 2m+1$, we get

$$(1+x)(1+2x\cos 2\lambda+x^2)(1+2x\cos 4\lambda+x^2)(1+2x\cos 6\lambda+x^2)\dots$$
$$\text{to } (m+1) \text{ factors.}$$

In either case its logarithm is

$$\frac{\sin n\lambda}{\sin\lambda}x-\tfrac{1}{2}\frac{\sin 2n\lambda}{\sin 2\lambda}x^2+\tfrac{1}{3}\frac{\sin 3n\lambda}{\sin 3\lambda}x^3-\dots.$$

If now, for brevity, we write the series (1)

$$1+C_1x+C_2x^2+\dots,$$

we can immediately deduce, by writing $x\sqrt{-1}$ for x, that

$$e^{(\sin n\lambda)\,xi/(\sin\lambda)+\frac{1}{2}(\sin 2n\lambda)\,x^2/(\sin 2\lambda)-\dots} = 1+C_1xi-C_2x^2+\dots = e^{Li+M}, \text{ say.}$$

Hence, equating real and imaginary parts,

$$\tan L = \frac{C_1x-C_3x^3+C_5x^5-\dots}{1-C_2x^2+C_4x^4-\dots},$$

which can easily be shown to be equal to

$$C_1 x \frac{F \sin 2\lambda \left\{-\frac{n-1}{2}, -\frac{n-2}{2}, \frac{3}{2}, -x^2\right\}}{F \sin 2\lambda \left\{-\frac{n-1}{2}, -\frac{n}{2}, \frac{1}{2}, -x^2\right\}},$$

and this, by § 1, can be reduced to continued fraction form. get after some reduction

$$\tan\left(\frac{\sin n\lambda}{\sin\lambda}x - \frac{\sin 3n\lambda}{3\sin 3\lambda}x^3 + \frac{\sin 5n\lambda}{5\sin 5\lambda}x^5 - \dots\right)$$

$$= \frac{x\sin n\lambda}{\sin\lambda -}\;\frac{x^2(\sin^2 n\lambda - \sin^2\lambda)}{\sin 3\lambda -}\;\frac{x^2(\sin^2 n\lambda - \sin^2 2\lambda)}{\sin 5\lambda -}.$$

We have obtained this result by supposing n to be a positiv teger; but, by the principle of permanence of algebraical form may suppose it to be true for any value of n, consistent with vergency in the series, and definiteness in the continued fractior

3. If $\lambda = 0$, we get the known form

$$\tan(n\tan^{-1}x) = \frac{nx}{1-}\;\frac{x^2(n^2-1^2)}{3-}\;\frac{x^2(n^2-2^2)}{5-}\;\frac{x^2(n^2-3^2)}{7-}\dots$$

If $n = 0$, we get forms already investigated by Heine.

Let $\frac{x}{2i}e^{n\lambda i} = y$, and let n increase and x decrease indefinitely,

y remains constant.

Then $$x\sin n\lambda = \frac{x}{2i}e^{n\lambda i}(1-e^{-2n\lambda i}) = y,$$

$$x^3\sin 3n\lambda = \frac{x^3}{2i}e^{3n\lambda i}(1-e^{-6n\lambda i}) = -2^2 . y^3, \text{ \&c.};$$

therefore

$$\tan\left(\frac{y}{\sin\lambda} + \frac{2^2 y^3}{3\sin 3\lambda} + \frac{2^4 y^5}{5\sin 5\lambda} + \dots\right) = \frac{y}{\sin\lambda -}\;\frac{y^2}{\sin 3\lambda -}\;\frac{y^2}{\sin 5\lambda}$$

....

Writing $\frac{x}{2i}$ for y, and $\sqrt{q}$ for $e^{\lambda i}$, we get

$$\tan\left(\frac{x\sqrt{q}}{1-q}-\frac{1}{3}\cdot\frac{x^3\sqrt{q^3}}{1-q^3}+\frac{1}{5}\frac{x^5\cdot\sqrt{q^5}}{1-q^5}-\dots\right)$$

$$=\frac{x\sqrt{q}}{1-q-}\;\frac{x^2q^2}{1-q^3-}\;\frac{x^2q^4}{1-q^5-}\;\frac{x^2q^6}{1-q^7-}\dots$$

Again in § 2, (2), put $n\lambda = \frac{\pi}{2}$,

then we have

$$\tan\left(\frac{x}{\sin\lambda}+\frac{x^3}{3\sin 3\lambda}+\frac{x^5}{5\sin 3\lambda}+\dots\right)=\frac{x}{\sin\lambda-}\;\frac{x^2\cos^2\lambda}{\sin 3\lambda-}\;\frac{x^2\cos^2 2\lambda}{\sin 5\lambda-}\dots$$

$$\dots\dots\dots\dots(2).$$

Or, putting $x = 2y$,

$$\tan\left(\frac{2y}{\sin\lambda}+\frac{2^3y^3}{3\sin 3\lambda}+\dots\right)=\frac{2y}{\sin\lambda-}\;\frac{4y^2\cos^2\lambda}{\sin 3\lambda-}\dots$$

Now, the bracketed series on the left side is just twice that in (1), so that, if we call either side of that equation y, each side of (2) must be equal to $\frac{2y}{1-y^2}$ or $\frac{2}{\frac{1}{y}-y}$.

Hence $$\frac{\sin\lambda}{2y}-\frac{2y\cos^2\lambda}{\sin 3\lambda-}\;\frac{4y^2\cos^2 2\lambda}{\sin 5\lambda-}\dots$$

$$=\frac{1}{2y}-\frac{y}{2}=\frac{\sin\lambda}{2y}-\frac{\frac{1}{2}y}{\sin 3\lambda-}\;\frac{y^2}{\sin 5\lambda-}\dots-\frac{\frac{1}{2}y}{\sin\lambda-}\;\frac{y^2}{\sin 3\lambda-}\dots$$

i.e., $$\frac{1}{\sin\lambda-}\;\frac{y^2}{\sin 3\lambda-}\;\frac{y^2}{\sin 5\lambda-}\dots+\frac{1}{\sin 3\lambda-}\;\frac{y^2}{\sin 5\lambda-}\dots$$

$$=\frac{4\cos^2\lambda}{\sin 3\lambda-}\;\frac{4y^2\cos^2 2\lambda}{\sin 5\lambda-}\;\frac{4y^2\cos^2 3\lambda}{\sin 7\lambda-}\dots$$

a curious result, as we have a linear relation connecting three infinite continued fractions. Transforming into q functions, we get, after some easy reductions,

$$\frac{q}{1-q+}\;\frac{zq^2}{1-q^3+}\;\frac{zq^4}{1-q^5+}\;\frac{zq^6}{1-q^7+}\dots+\frac{q^2}{1-q^3+}\;\frac{zq^4}{1-q^5+}\;\frac{zq^6}{1-q^7+}\dots$$

$$=\frac{q\,(q+1)^2}{1-q^3+}\;\frac{zq^2\,(1+q^2)^2}{1-q^5+}\;\frac{zq^3\,(1+q^3)^2}{1-q^7+}\dots\dots\dots(3),$$

an identity which holds good provided $q < 1$, $z < 1$.

Either side, when expanded as far as q^6, is found to be

$$q+2q^2-(z-1)\,q^3-(2z-1)\,q^4+(z^2-3z+2)\,q^5+(3z^2-6z+1)\,q^6+$$

4. We now proceed to the investigation of certain elli functions, when converted into continued fractions.

In § 2, (2), let

$$e^{\lambda i}=q^{-\frac{1}{2}},\quad x=1,\quad n\lambda=\theta,$$

then
$$\tan 2i\left(\frac{\sqrt{q}\sin\theta}{1-q}-\frac{1}{3}\,\frac{\sqrt{q^3}\sin 3\theta}{1-q^3}+\dots\right)$$

$$=\frac{2i\,\sqrt{q}\sin\theta}{1-q-}\;\frac{q^2\,\{4\sin^2\theta+q^{-1}(1-q)^2\}}{1-q^3-}\;\frac{q^4\,\{4\sin^2\theta+q^{-2}(1-q^2)^2}{1-q^5-}$$

Or, calling the bracketed series S,

$$i\,\frac{e^{4S}-1}{e^{4S}+1}=\text{the same fraction as before}$$

$$=\frac{2i\,\sqrt{q}\sin\theta}{1-q+}\;\frac{q\,(1-2q\cos 2\theta+q^2)}{1-q^3+}\;\frac{q^2\,(1-2q^2\cos 2\theta+q^4)}{1-q^5+\dots}$$

Now, $$4S=\log\sqrt{\frac{1+k\,\mathrm{sn}\,\frac{2K\theta}{\pi}}{1-k\,\mathrm{sn}\,\frac{2K\theta}{\pi}}}=\log\frac{1+k\,\mathrm{sn}\,\frac{2K\theta}{\pi}}{\mathrm{dn}\,\frac{2K\theta}{\pi}}$$ (Jacobi, *Fund. No* p. 156

therefore
$$\frac{e^{4s}-1}{e^{4s}+1}=\frac{1+k\,\mathrm{sn}\,\frac{2K\theta}{\pi}-\mathrm{dn}\,\frac{2K\theta}{\pi}}{1+k\,\mathrm{sn}\,\frac{2K\theta}{\pi}+\mathrm{dn}\,\frac{2K\theta}{\pi}},$$

which
$$=\frac{k\,\mathrm{sn}\,\frac{2K\theta}{\pi}}{1+\mathrm{dn}\,\frac{2K\theta}{\pi}}=\frac{k\,\mathrm{sn}\,\frac{K\theta}{\pi}\,\mathrm{cn}\,\frac{K\theta}{\pi}}{\mathrm{dn}\,\frac{K\theta}{\pi}}.$$

We shall always write $u=\frac{2K\theta}{\pi}$ for shortness.

Hence
$$\frac{k\,\mathrm{sn}\,u}{1+\mathrm{dn}\,u}=\frac{2\sqrt{q}\sin\theta}{1-q+}\;\frac{q\,(1-2q\cos 2\theta+q^2)}{1-q^3+\dots}$$

$$=\frac{2\sqrt{q}\sin\theta}{F},\ \text{say}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Now, $$\frac{k \operatorname{sn} u}{1+\operatorname{dn} u} = \frac{1-\operatorname{dn} u}{k \operatorname{sn} u} = \frac{2\sqrt{q} \sin \theta}{F};$$

and, since $$\frac{1+\operatorname{dn} u}{k \operatorname{sn} u} = \frac{F}{2\sqrt{q} \sin \theta},$$

we see that $\dfrac{2}{k \operatorname{sn} u}$ is equal to the sum of two continued fractions.

This reduces to the equation

$$\frac{4\sqrt{q} \sin \theta}{k \operatorname{sn} u} = 1-q+\frac{q(1-2q\cos 2\theta+q^2)}{1-q^3+}\,\frac{q^2(1-2q^2\cos 2\theta+q^4)}{1-q^5+\dots}$$

$$+\frac{4q\sin^2\theta}{1-q+}\,\frac{q(1-2q\cos 2\theta+q^2)}{1-q^3+\dots} \quad \dots\dots\dots\dots\dots\dots(2),$$

while $$\frac{4\sqrt{q}\sin\theta}{k}\cdot\frac{\operatorname{dn} u}{\operatorname{sn} u} = 1-q+\frac{q(1-2q\cos 2\theta+q^2)}{1-q^3+}\dots$$

$$-\frac{4q\sin^2\theta}{1-q+}\,\frac{q(1-2q\cos 2\theta+q^2)}{1-q^3+\dots} \quad \dots\dots(3).$$

Adding $\dfrac{\pi}{2}$ to θ, and K to u, we get

$$\frac{4k'\sqrt{q}\cos\theta}{k \operatorname{cn} u} = 1-q+\frac{q(1+2q\cos 2\theta+q^2)}{1-q^3+}\,\frac{q^2(1+2q^2\cos 2\theta+q^4)}{1-q^5+}$$

$$-\frac{4q\cos^2\theta}{1-q+}\,\frac{q(1+2q\cos 2\theta+q^2)}{1-q^3+\dots} \quad \dots\dots\dots\dots(4).$$

Changing k, k', q, $\operatorname{cn} u$ respectively into

$$\frac{ik}{k'},\quad \frac{1}{k'},\quad -q,\quad \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

we get

$$\frac{4\sqrt{q}\cos\theta}{k}\,\frac{\operatorname{dn} u}{\operatorname{cn} u} = 1+q-\frac{q(1-2q\cos 2\theta+q^2)}{1+q^3+}\,\frac{q^2(1+2q^2\cos 2\theta+q^4)}{1+q^5-\dots}$$

$$+\frac{4q\cos^2\theta}{1+q-}\,\frac{q(1+2q\cos 2\theta+q^2)}{1+q^3+\dots} \quad \dots\dots\dots\dots(5).$$

Adding $\dfrac{\pi}{2}$ to θ, and K to u, we get

$$\frac{4\sqrt{q}\sin\theta}{k \operatorname{sn} u} = 1+q-\frac{q(1+2q\cos 2\theta+q^2)}{1+q^3+}\,\frac{q^2(1-2q^2\cos 2\theta+q^4)}{1+q^5-}$$

$$+\frac{4q\sin^2\theta}{1+q-}\,\frac{q(1-2q\cos 2\theta+q^2)}{1+q^3+\dots} \quad \dots\dots\dots\dots\dots(6).$$

We have thus by a series of transformations brought the left-han side of the equation (2) into its original form, while the right-han side is different. This is due to certain identities which exi between certain continued fractions, and which will be established the following section.

5. The left-hand side of § 4, (1) is easily shown by elementar principles to be equal to

$$k \operatorname{sn} \frac{u}{2} \operatorname{cn} \frac{u}{2} \Big/ \operatorname{dn} \frac{u}{2},$$

so that, by writing $2u$ for u, 2θ for θ, we get

$$\frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{2\sqrt{q}\sin 2\theta}{1-q+} \; \frac{q\,(1-2q\cos 4\theta+q^2)}{1-q^3+} \; \frac{q^2\,(1-2q^2\cos 4\theta+q^4)}{1-q^5+\dots} \quad \dots\dots\dots\dots\dots\dots (1)$$

By changing q into $-q$, we change $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ respectively int

$$\frac{k' \operatorname{sn} u}{\operatorname{dn} u}, \quad \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \frac{1}{\operatorname{dn} u},$$

and (1) becomes

$$\frac{k \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{2\sqrt{q}\sin 2\theta}{1+q-} \; \frac{q\,(1+2q\cos 4\theta+q^2)}{1+q^3+} \; \frac{q^2\,(1-2q^2\cos 4\theta+q^4)}{1+q^5-\dots} \quad \dots\dots\dots\dots\dots\dots (2)$$

The right sides of (1) and (2) equated give an identity which ma be written

$$2q = \frac{q\,(1+2q\cos 2\theta+q^2)}{1+q^3+} \; \frac{q^2\,(1-2q^2\cos 2\theta+q^4)}{1+q^5-\dots} + \frac{q\,(1-2q\cos 2\theta+q^2)}{1-q^3+} \; \frac{q^2\,(1-2q^2\cos 2\theta+q^4)}{1-q^5+\dots} \quad \dots\dots\dots (3)$$

and by which we may establish the identity of equations (2) and (6 in the preceding section.

It is known, moreover, that by transforming k into $\dfrac{1-k}{1+k}$, w change u, K, K', q, $\operatorname{sn} u$ into

$$(1+k')\,u, \quad \tfrac{1}{2}(1+k')\,K, \quad (1+k')\,K', \quad q^2, \quad \frac{(1+k')\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

Conversely, (4) becomes

$$\sqrt{k}\operatorname{sn} u = \cfrac{2\,.\,\sqrt[4]{q}\sin\theta}{1-\sqrt{q}+\cfrac{\sqrt{q}\,(1-2\sqrt{q}\cos 2\theta+q)}{1-\sqrt{q^3}+\cfrac{q\,(1-2q\cos 2\theta+q^2)}{1-\sqrt{q^5}+\dots}}} \quad \dots\dots\dots (4)$$

Adding K to u, we get a form for $\frac{\operatorname{cn} u}{\operatorname{dn} u}$, from which, by retransformation, we get

$$\sqrt{\frac{1+k'}{1-k'}}\,\frac{\operatorname{dn}^2 u-k'}{\operatorname{dn}^2 u+k'}=\frac{2\sqrt{q}\cos 2\theta}{1-q+}\;\frac{q\,(1+2q\cos 4\theta+q^2)}{1-q^3+\dots}\;\dots\dots\dots(5).$$

6. Again, in § 2, (2), let $e^{\lambda i}=q^{-\frac{1}{2}}$, $x=q^{\frac{1}{2}}$, $n\lambda=\theta$, so that

$$\tan 2i\left(\frac{q\sin\theta}{1-q}-\frac{1}{3}\cdot\frac{q^3\sin 3\theta}{1-q^3}+\dots\right)$$

$$=\frac{2iq\sin\theta}{1-q+}\;\frac{q^2\,(1-2q\cos 2\theta+q^2)}{1-q^3+}\;\frac{q^3\,(1-2q^2\cos 2\theta+q^4)}{1-q^5+\dots}$$

$$=Hi,\text{ say}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1).$$

Then $$\tan 4i\left(\frac{q\sin\theta}{1-q}-\dots\right)=\frac{2Hi}{1+H^2},$$

which by Jacobi, *Fund. Nov.*, p. 156, becomes

$$\frac{\dfrac{1+\operatorname{sn} u}{1-\operatorname{sn} u}-\dfrac{1+\sin\theta}{1-\sin\theta}}{\dfrac{1+\operatorname{sn} u}{1-\operatorname{sn} u}+\dfrac{1+\sin\theta}{1-\sin\theta}}=\frac{2H}{1+H^2}$$

that is, $$\frac{\operatorname{sn} u-\sin\theta}{1-\operatorname{sn} u\sin\theta}=\frac{2H}{1+H^2}.$$

This ultimately reduces to

$$\frac{4q\cos^2\theta\sin\theta}{\operatorname{sn} u-\sin\theta}=1-q+4q\sin^2\theta+\frac{q^2\,(1-2q\cos 2\theta+q^2)}{1-q^3+\dots}$$

$$+\frac{4q^2\sin^2\theta}{1-q+}\;\frac{q^2\,(1-2q\cos 2\theta+q^2)}{1-q^3+\dots}\;\dots\dots\dots\dots(2).$$

From this formula may be deduced a series of like formulæ for

$$\frac{\operatorname{cn} u}{\operatorname{dn} u},\quad \operatorname{cn} u,\quad \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

by the same methods as were used in the last section.

For instance,

$$\frac{4q\sin^2\theta\cos\theta}{\cos\theta-\operatorname{cn} u}=1+q-4q\cos^2\theta+\frac{q^2\,(1-2q\cos 2\theta+q^2)}{1+q^3-\dots}$$

$$+\frac{4q^2\cos^2\theta}{1+q+}\;\frac{q^2\,(1-2q\cos 2\theta+q^2)}{1+q^3-\dots}\;\dots\dots\dots\dots(3).$$

Again, with the same notation, we have

$$\frac{\dfrac{1+\operatorname{sn} u}{\operatorname{cn} u}-\dfrac{1+\sin\theta}{\cos\theta}}{\dfrac{1+\operatorname{sn} u}{\operatorname{cn} u}+\dfrac{1+\sin\theta}{\cos\theta}}=H.$$

This reduces to

$$\frac{\dfrac{1+\operatorname{sn} u}{\operatorname{cn} u}-1}{\dfrac{1+\operatorname{sn} u}{\operatorname{cn} u}+1}=\cot\frac{\theta}{2}-\frac{\cot^2\dfrac{\theta}{2}-1}{\cot\dfrac{\theta}{2}+H},$$

or
$$\tan\frac{\theta}{2}\,\frac{\operatorname{sn}\dfrac{u}{2}\operatorname{dn}\dfrac{u}{2}}{\operatorname{cn}\dfrac{u}{2}}=1-\frac{1-\tan^2\dfrac{\theta}{2}}{1+H\tan\dfrac{\theta}{2}}\;\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence

$$\tan\theta\,\frac{\operatorname{sn} u\operatorname{dn} u}{\operatorname{cn} u}=1-\frac{1-\tan^2\theta}{1+}\,\frac{4q\sin^2\theta}{1-q+}\,\frac{q^2(1-2q\cos 4\theta+q^2)}{1-q^3+\ldots}\ldots($$

By adding K to u, we do not get any new identity; but, by chang q into $-q$, we have

$$\tan\theta\,\frac{k'\operatorname{sn} u}{\operatorname{cn} u\operatorname{dn} u}=1-\frac{1-\tan^2\theta}{1-}\,\frac{4q\sin^2\theta}{1+q+}\,\frac{q^2(1+2q\cos 4\theta+q^2)}{1+q^3+\ldots}$$

7. From equations § 5, (1), (5), we get

$$\frac{2kK}{\pi}=\frac{4\sqrt{q}}{1-q+}\,\frac{q(1-q)^2}{1-q^3+}\,\frac{q^3(1-q^2)^2}{1-q^5+\ldots},$$

$$\frac{k}{1+k'}=\frac{2\sqrt{q}}{1-q+}\,\frac{q(1+q)^2}{1-q^3+}\,\frac{q^2(1+q^2)^2}{1-q^5+\ldots},$$

$$\frac{2K}{\pi}+1=\frac{4q}{1-q+}\,\frac{q^2(1-q)^2}{1-q^3+}\,\frac{q^4(1-q^2)^2}{1-q^5+\ldots};$$

but they can be as well derived from Heine's results, after replac the left-hand sides by their known equivalents as q-series.

Moreover, we may derive many identities by comparing the differ fractional expression obtained for $\operatorname{sn} u$ in §§ 4 and 5, and ded values for K, &c. as continued fractions.

8. Again, since

$$\log\operatorname{tn} u=\log\tan\theta-\log\sqrt{k'}-\frac{4q^2}{1-q^2}\cos 2\theta-\frac{4q^6}{1-q^6}\,\frac{\cos 6\theta}{3}-\ldots,$$

we get, by § 2, (2),

$$\frac{\tan\theta-\surd k'\,\mathrm{tn}\,u}{\tan\theta+\surd k'\,\mathrm{tn}\,u}=\frac{2q^2\cos 2\theta}{1-q^2+}\;\frac{q^4(1-2q^2\cos 4\theta+q^4)}{1-q^6+}\;\frac{q^6(1-2q^4\cos 4\theta+q^8)}{1-q^8+\dots},$$

by the same method as before.

This may be written

$$\surd k'\,\mathrm{tn}\,u+\tan\theta=\frac{2\tan\theta}{1+}\;\frac{2q^2\cos 2\theta}{1-q^2+}\;\frac{q^4(1-2q^2\cos 4\theta+q^4)}{1-q^6+\dots}\quad\dots(1).$$

In the same way, we get from the equation

$$\log\mathrm{dn}\,u=\log\surd k'+\frac{4q\cos 2\theta}{1-q^2}+\frac{4q^3\cos 6\theta}{3(1-q^6)}+\dots,$$

that

$$\frac{\mathrm{dn}\,u-\surd k'}{\mathrm{dn}\,u+\surd k'}=\frac{2q\cos 2\theta}{1-q^2+}\;\frac{q^2(1+2q^2\cos 4\theta+q^4)}{1-q^6+}\;\frac{q^4(1+2q^4\cos 4\theta+q^8)}{1-q^{10}+\dots}$$

or $$\mathrm{dn}\,u+\surd k'=\frac{2\surd k'}{1-}\;\frac{2q\cos 2\theta}{1-q^2+}\;\frac{q^2(1+2q^2\cos 4\theta+q^4)}{1-q^6+\dots}\quad\dots\dots(2).$$

From the above, we get

$$\frac{2K}{\pi}\surd k'+1=\frac{2}{1+}\;\frac{2q^2}{1-q^2+}\;\frac{q^4(1-q^2)^2}{1-q^6+\dots},$$

$$\frac{1-\surd k'}{1+\surd k'}=\frac{2q}{1-q^2+}\;\frac{q^2(1+q^2)^2}{1-q^6+}\;\frac{q^4(1+q^4)^2}{1-q^{10}+\dots},$$

as may be proved by Heine's theorem.

9. The question of convergency, as regards the continued fractions above employed, is of easy solution.

It is known that a continued fraction

$$\frac{1}{1-}\;\frac{e_1}{1-}\;\frac{e_2}{1-}\;\frac{e_3}{1-\dots}$$

is convergent, provided the limit of e_n is numerically less than unity. (See Todhunter's *Algebra*, Art. 783.)

In all the fractions met with above in the later sections, the value of e_n is of the form

$$\frac{q^m(1-2q^n\cos 2\theta+q^{2n})}{(1-q^r)(1-q^s)},$$

where m, n, r, s all tend to infinite limit. Hence e_n tends to zero

limit, since $q<1$. Similarly, in § 3 (3), all the continued fracti
are convergent if $zq<1$. We may therefore suppose that $z=1$.

10. There are a very large number of algebraic identities wh
may be derived from the several equations established in the ab
section.

We have already obtained one in § 5.

If we make $2\theta=\pi$ in the identity § 5 (3), we get

$$2q=\frac{q(1+q^2)}{1+q^3+}\,\frac{q^2(1+q^4)}{1+q^5-}\cdots+\frac{q(1+q^2)}{1-q^3+}\,\frac{q^2(1+q^4)}{1-q^5+\ldots}.$$

But, by § 3 (3), we can split up each of these fractions into t
others, so that we have ultimately

$$2q=\frac{q}{1-q+}\,\frac{q^2}{1-q^3+}\cdots+\frac{q^2}{1-q^3+}\,\frac{q^4}{1-q^5+\ldots}$$

$$+\frac{q}{1+q+}\,\frac{q^2}{1+q^3+}\cdots-\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+\ldots}.$$

Again, from § 5, if we multiply the fraction in (4) wi
that obtained by adding K to u, we shall get a product equal
$\frac{k\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}$. Comparing this with § 5 (2), we get the followi
identity after some substitutions and reductions—

$$1-q^2+\frac{q^2(1-2q^2\cos 2\theta+q^4)}{1-q^6+}\,\frac{q^4(1-2q^4\cos 2\theta+q^8)}{1-q^{10}+\ldots}$$

$$=\left\{1-q+\frac{q(1-2q\cos\theta+q^2)}{1-q^3+}\,\frac{q^2(1-2q^2\cos\theta+q^4)}{1-q^5+\ldots}\right\}$$

$$\times\left\{1-q+\frac{q(1+2q\cos\theta+q^2)}{1-q^3+}\,\frac{q^2(1+2q^2\cos\theta+q^4)}{1-q^5+\ldots}\right\}.$$

If $\cos\theta=0$, we have

$$\left\{1-q+\frac{q(1+q^2)}{1-q^3+}\,\frac{q^2(1+q^4)}{1-q^5+\ldots}\right\}^2=1-q^2+\frac{q^2(1+q^2)^2}{1-q^6+}\,\frac{q^4(1+q^4)^2}{1-q^{10}+\ldots}.$$

We may get many other identities, *e.g.*, by comparing § 8 (2) wi
§ 5 (5). We do not arrive at very interesting results, as they are f
the most part very complicated and involve more than three infini
continued fractions.

On the c- and p-Discriminants of Ordinary Integrable Differential Equations of the First Order. By M. J. M. Hill, M.A., Professor of Mathematics at University College, London.

[*Read June* 14*th*, 1888.]

Introduction.

The theory of singular solutions of ordinary integrable differential equations of the first order, as at present accepted, was first given by Professor Cayley, in the *Messenger of Mathematics*, Vol. II., 1872, pp. 6—12.

It is there shown that, if $f(x, y, c_1, c_2 \dots c_m)$ be a rational integral indecomposable algebraical function of x, y, and the parameters $c_1, c_2 \dots c_m$, which parameters are connected by $(m-1)$ algebraical relations; then, treating x, y as Cartesian coordinates, the equation $f(x, y, c_1, c_2 \dots c_m) = 0$ will represent a system of plane curves, such that, if the discriminant of $f(x, y, c_1, c_2 \dots c_m)$ with regard to the c's be formed, it will be made up of the envelope-, node-, and cusp-loci of the system of curves.

If, further, the differential equation of the system of curves be

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0,$$

and if the discriminant of $\phi(x, y, p)$ be formed with regard to p, then the p-discriminant locus is made up of the envelope-, cusp-, and tac-loci.

In this paper the discussion will be limited to the case where there is a single arbitrary parameter c, *i.e.*, when m in the above is unity. The curves will then be what Prof. Henrici has called in his paper on the "Singularities of Curve Envelopes" in Vol. II. of the *Proceedings of the London Mathematical Society*, page 181, a Unicursal Series of Curves; and for such curves it will be shown that the c-discriminant locus contains the envelope-locus as a factor once, the node-locus twice, and the cusp-locus thrice (results which will be obtained by interpreting certain theorems given by Prof. Henrici in his paper "On certain Formulæ concerning the Theory of Discriminants, with applications to Discriminants of Discriminants, and to the Theory of Polar Curves," in Vol. II. of the *Proceedings of the London Mathematical Society*, page 108); and, further, that the p-discriminant locus con-

tains the envelope-locus as a factor once, the cusp-locus once, a tac-locus twice.

These theorems were stated by Professor Cayley without pr the *Messenger of Mathematics*, Vol. II., 1872, pp. 11, 12; and Vo 1882, p. 3.

The discussion in the paper is also limited to general Exceptional cases will be occasionally noticed, but an exhaustiv cussion of them will not be attempted.

[Throughout this paper d will denote total differentiation of y regard to x; δ partial differentiation when x, y are indepe variables; ∂, or a suffix, partial differentiation when x, y, c are pendent variables.]

1. *To investigate the condition that every curve of the s* $f(x, y, c) = 0$ *may have a double point, i.e., that there may be a locus; and to show how to determine the direction of the tangent t node-locus at any point on it.*

For every value of c the curve $f(x, y, c) = 0$ has by hypothe double point. When $c = \gamma$, let the coordinates of the double be ξ, η; then the following equations are simultaneously true,—

$$f(\xi, \eta, \gamma) = 0, \quad \frac{\partial f(\xi, \eta, \gamma)}{\partial \xi} = 0, \quad \frac{\partial f(\xi, \eta, \gamma)}{\partial \eta} = 0 \ldots\ldots\ldots$$

Next, when $c = \gamma + \partial\gamma$, let the coordinates of the double poin $\xi + \partial\xi$, $\eta + \partial\eta$. Hence

$$f(x, y, c) = 0, \quad \frac{\partial f(x, y, c)}{\partial x} = 0, \quad \frac{\partial f(x, y, c)}{\partial y} = 0,$$

where x, y, c are to be replaced respectively by

$$\xi + \partial\xi, \quad \eta + \partial\eta, \quad \gamma + \partial\gamma.$$

Hence, adopting the suffix notation,

$$f + f_\xi \partial\xi + f_\eta \partial\eta + f_\gamma \partial\gamma + \ldots = 0 \ldots\ldots\ldots\ldots\ldots (\text{I}$$

$$f_\xi + f_{\xi\xi} \partial\xi + f_{\xi\eta} \partial\eta + f_{\xi\gamma} \partial\gamma + \ldots = 0 \ldots\ldots\ldots\ldots\ldots (\text{I}$$

$$f_\eta + f_{\eta\xi} \partial\xi + f_{\eta\eta} \partial\eta + f_{\eta\gamma} \partial\gamma + \ldots = 0 \ldots\ldots\ldots\ldots\ldots (\text{I}$$

Hence (II.), (III.), (IV.) become, by means of (I.),

$$f_\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V}$$

$$f_{\xi\xi} \partial\xi + f_{\xi\eta} \partial\eta + f_{\xi\gamma} \partial\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V}$$

$$f_{\eta\xi} \partial\xi + f_{\eta\eta} \partial\eta + f_{\eta\gamma} \partial\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V}$$

The equation corresponding to (V.) must also be satisfied by the coordinates of the node at $\xi+\partial\xi$, $\eta+\partial\eta$ on the curve whose parameter is $\gamma+\partial\gamma$.

Hence $$f_\gamma+f_{\xi\gamma}\partial\xi+f_{\eta\gamma}\partial\eta+f_{\gamma\gamma}\partial\gamma = 0 \quad \text{..................(VIII.).}$$

Hence $$f_{\xi\gamma}\partial\xi+f_{\eta\gamma}\partial\eta+f_{\gamma\gamma}\partial\gamma = 0 \quad \text{..................(IX.).}$$

From any two of the equations (VI.), (VII.), (IX.) the direction of the tangent to the node-locus may be obtained by eliminating $\partial\gamma$.

In order that these equations may be consistent, the equation

$$\frac{\partial\,\{f_\xi, f_\eta, f_\gamma\}}{\partial\,\{\xi, \eta, \gamma\}} = 0 \quad \text{.........................(X.)}$$

must hold.

[In the above it is assumed that the coordinates of the node depend on the value of c; hence the case in which all the curves have a node at the same point is excluded from consideration. In this case the node-locus shrinks into a single point (or a definite number of points). I am indebted to Professor Henrici for the remark that the proper way to treat such singularities is to use line-coordinates, and then the number of times that the equation of the point or points into which the singular locus reduces occurs in the c-discriminant will be the question to be investigated.]

2. *To investigate the condition that every curve of the system should have a cusp, i.e., that there should be a cusp-locus; and to show how to determine the direction of the tangent to the cusp-locus at any point on it.*

Since the cusp is a double point, the equations (I.), (V.), (VI.), (VII.), (IX.) remain true as before. But, as the double point is a cusp, it follows that

$$f_{\xi\xi}f_{\eta\eta}-f_{\xi\eta}^2 = 0 \quad \text{............................(XI.).}$$

Now eliminate $\partial\eta$ from equations (VI.) and (VII.); therefore

$$(f_{\xi\xi}f_{\eta\eta}-f_{\xi\eta}^2)\,\partial\xi+(f_{\xi\gamma}f_{\eta\eta}-f_{\xi\eta}f_{\eta\gamma})\,\partial\gamma = 0.$$

Hence, by (XI.), it follows that

$$f_{\xi\eta}f_{\eta\gamma}-f_{\eta\eta}f_{\xi\gamma} = 0 \quad \text{.........................(XII.).}$$

Combining (XI.) and (XII.), it follows that

$$f_{\xi\xi} : f_{\xi\eta} : f_{\xi\gamma} = f_{\eta\xi} : f_{\eta\eta} : f_{\eta\gamma} \text{..................(XIII.).}$$

Now take equations (VII.) and (IX.), and eliminate $\partial\eta$; therefore

$$(f_{\xi\eta}f_{\eta\gamma}-f_{\xi\gamma}f_{\eta\eta})\,\partial\xi+(f_{\eta\gamma}^2-f_{\eta\eta}f_{\gamma\gamma})\,\partial\gamma=0.$$

Hence, by (XII.), it follows that

$$f_{\eta\gamma}^2-f_{\eta\eta}f_{\gamma\gamma}=0 \quad\text{.........................}(\text{XIV.}).$$

From (XII.) and (XIV.), it follows that

$$f_{\xi\eta}:f_{\eta\eta}:f_{\eta\gamma}=f_{\xi\gamma}:f_{\eta\gamma}:f_{\gamma\gamma}\text{.....................}(\text{XV.}).$$

Hence (XIII.) and (XV.) show that, in the case of the cusp-locus, the equations (VI.), (VII.) and (IX.) amount to but one equation, and therefore do not determine the direction of the tangent to the cusp-locus at any point on it.

Another equation can, however, be found to effect the required determination.

The relation $\qquad f_{\xi\xi}f_{\eta\eta}-f_{\xi\eta}^2=0$

is satisfied at the cusp ξ, η on the curve $c=\gamma$.

As there is a cusp at the point $\xi+\partial\xi$, $\eta+\partial\eta$ on the curve for which $c=\gamma+\partial\gamma$, this relation becomes for that cusp $\left(\text{writing } D \text{ as an abbreviation for } \partial\xi\frac{\partial}{\partial\xi}+\partial\eta\frac{\partial}{\partial\eta}+\partial\gamma\frac{\partial}{\partial\gamma}\right)$

$$D\,(f_{\xi\xi}f_{\eta\eta}-f_{\xi\eta}^2)=0.$$

Eliminating $\partial\gamma$ between this and one of the three equations (VI.), (VII.), (IX.), the direction of the tangent to the cusp-locus is determined.

Another form which will be used afterwards is as follows :—

Since there is a cusp at ξ, η on the curve for which $c=\gamma$, therefore

$$\frac{f_{\xi\xi}}{f_{\xi\eta}}=\frac{f_{\xi\eta}}{f_{\eta\eta}}=\frac{f_{\xi\gamma}}{f_{\eta\gamma}},$$

and the corresponding relations for the cusp at $\xi+\partial\xi$, $\eta+\partial\eta$ on the curve for which $c=\gamma+\partial\gamma$, will be

$$D\frac{f_{\xi\xi}}{f_{\xi\eta}}=D\,\frac{f_{\xi\eta}}{f_{\eta\eta}}=D\,\frac{f_{\xi\gamma}}{f_{\eta\gamma}}\quad\text{......................}(\text{XVI.}).$$

Eliminating $\partial\gamma$ from the two equations (XVI.), a single equation remains for the ratio $\frac{\partial\eta}{\partial\xi}$, which gives the direction of the cusp-locus. The most convenient form for the value of $\frac{\partial\eta}{\partial\xi}$, for use in this paper, will be given in Art. 11.

3. *Geometrical representation of the preceding results.*

The symmetry of the results, both in the case of the node-locus and cusp-locus with regard to ξ, η, γ, suggests that a geometric representation may be obtained by treating c as a third coordinate.

If this be done, $f(x, y, c) = 0$ is the equation of a surface in space. Its plane sections parallel to the plane of x, y represent the curves of the system.

The envelope of the curves is represented by the projection on the same plane of the enveloping cylinder, whose generating lines are parallel to the axis of c; the node-locus is represented by the projection on the same plane of a nodal line on the surface; the cusp-locus by the projection on the same plane of a line on the surface, which is such that the tangent cone at every point on it degenerates into two coincident planes.

4. *Formation of the c-discriminant of $f(x, y, c)$, and of its partial differential coefficients of the first and second orders with regard to x and y.*

Let $$f(x, y, c) \equiv a_m c^m + a_{m-1} c^{m-1} + \ldots + a_1 c + a_0,$$
where $a_0, a_1 \ldots a_m$ are functions of x, y, not c.

Let the roots of the equation
$$\frac{\partial f(x, y, c)}{\partial c} = 0,$$
regarded as an equation in c, be $c_1, c_2, \ldots c_{m-1}$; then these roots are functions of x, y; and the c-discriminant is
$$a_m^{m-1} f(x, y, c_1)\, f(x, y, c_2) \ldots f(x, y, c_{m-1}).$$
Call this Ξ; then Ξ is a rational integral function of x, y.

It should be noticed here that, if $m = 1$, then
$$\frac{\partial f(x, y, c)}{\partial c} = 0$$
is not an equation in c at all. In this case
$$f(x, y, c) = 0 \quad \text{is} \quad a_1 c + a_0 = 0,$$
and $$\frac{\partial f(x, y, c)}{\partial c} = 0 \quad \text{becomes} \quad a_1 = 0.$$
Hence the points of contact of the curves with their envelope, and the nodes and cusps of the curves must lie on $a_1 = 0$, and therefore also on $a_0 = 0$. Hence they are the intersections of the curves $a_0 = 0$ and $a_1 = 0$; and the singular loci therefore in this case degenerate into

one or more points, and are consequently excluded from consideration in this paper (see remark at end of Art. 1).

Put, for brevity,

$$Q = a_m^{m-1} f(x, y, c_2) \dots f(x, y, c_{m-1});$$

then

$$\Xi = Qf(x, y, c_1),$$

therefore

$$\frac{\delta \Xi}{\delta x} = \left(\frac{\partial f(x, y, c_1)}{\partial x} + \frac{\partial f(x, y, c_1)}{\partial c_1} \frac{\delta c_1}{\delta x}\right) Q + f(x, y, c_1) \frac{\delta Q}{\delta x}.$$

To calculate $\frac{\delta c_1}{\delta x}$, there is the equation

$$\frac{\partial f(x, y, c_1)}{\partial c_1} = 0,$$

giving

$$\frac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1} + \frac{\partial^2 f(x, y, c_1)}{\partial c_1^2} \frac{\delta c_1}{\delta x} = 0,$$

and

$$\frac{\partial^2 f(x, y, c_1)}{\partial y \partial c_1} + \frac{\partial^2 f(x, y, c_1)}{\partial c_1^2} \frac{\delta c_1}{\delta y} = 0;$$

therefore

$$\frac{\delta \Xi}{\delta x} = Q \frac{\partial f(x, y, c_1)}{\partial x} + f(x, y, c_1) \frac{\delta Q}{\delta x},$$

with a similar expression for $\frac{\delta \Xi}{\delta y}$, and

$$\begin{aligned} \frac{\delta^2 \Xi}{\delta x^2} &= Q \left\{ \frac{\partial^2 f(x, y, c_1)}{\partial x^2} + \frac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1} \frac{\delta c_1}{\delta x} \right\} \\ &\quad + 2 \frac{\partial f(x, y, c_1)}{\partial x} \frac{\delta Q}{\delta x} + \frac{\partial f(x, y, c_1)}{\partial c_1} \frac{\delta c_1}{\delta x} \frac{\delta Q}{\delta x} \\ &\quad + f(x, y, c_1) \frac{\delta^2 Q}{\delta x^2} \\ &= Q \left\{ \frac{\partial^2 f(x, y, c_1)}{\partial x^2} - \frac{\left(\dfrac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1}\right)^2}{\dfrac{\partial^2 f(x, y, c_1)}{\partial c_1^2}} \right\} \\ &\quad + 2 \frac{\partial f(x, y, c_1)}{\partial x} \frac{\delta Q}{\delta x} + f(x, y, c_1) \frac{\delta^2 Q}{\delta x^2}, \end{aligned}$$

with a similar expression for $\frac{\delta^2 \Xi}{\delta y^2}$.

$$\frac{\delta^2 \Xi}{\delta x\, \delta y} = Q \left\{ \frac{\partial^2 f(x, y, c_1)}{\partial x \partial y} + \frac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1} \frac{\delta c_1}{\delta y} \right\}$$

$$+ \frac{\partial f(x, y, c_1)}{\partial x} \frac{\delta Q}{\delta y} + \frac{\partial f(x, y, c_1)}{\partial y} \frac{\delta Q}{\delta x}$$

$$+ \frac{\partial f(x, y, c_1)}{\partial c_1} \frac{\delta c_1}{\delta y} \frac{\delta Q}{\delta x} + f(x, y, c_1) \frac{\delta^2 Q}{\delta x\, \delta y},$$

therefore

$$\frac{\delta^2 \Xi}{\delta x\, \delta y} = Q \left\{ \frac{\partial^2 f(x, y, c_1)}{\partial x \partial y} - \frac{\dfrac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1} \dfrac{\partial^2 f(x, y, c_1)}{\partial y \partial c_1}}{\dfrac{\partial^2 f(x, y, c_1)}{\partial c_1^2}} \right\}$$

$$+ \frac{\partial f(x, y, c_1)}{\partial x} \frac{\delta Q}{\delta y} + \frac{\partial f(x, y, c_1)}{\partial y} \frac{\delta Q}{\delta x} + f(x, y, c_1) \frac{\delta^2 Q}{\delta x\, \delta y}.$$

5. *To prove that, if ξ, η be the coordinates of any point on the envelopes of the curves $f(x, y, c) = 0$, and if x, y be put equal to ξ, η in the c-discriminant of $f(x, y, c)$, then this discriminant will vanish; and that consequently this discriminant will in general contain the envelope-locus once, and only once, as a factor.*

For, if $x = \xi$, $y = \eta$, then $c_1, c_2, \dots c_{m-1}$ become roots of the equation

$$\frac{\partial f(\xi, \eta, c)}{\partial c} = 0.$$

But ξ, η being on the envelope, the two equations

$$f(\xi, \eta, c) = 0 \quad \text{and} \quad \frac{\partial f(\xi, \eta, c)}{\partial c} = 0,$$

are satisfied by a common value of c, say $c = \gamma$; and suppose that it is c_1 which becomes equal to γ, when x, y are put equal to ξ, η. (Then γ is the parameter of the curve which touches the envelope at ξ, η.)

Hence Ξ will contain the factor $f(\xi, \eta, \gamma)$ when x, y are put equal to ξ, η; and consequently it will vanish.

Hence Ξ, which is a rational integral function of x, y will vanish, whenever x, y are put equal to ξ, η, where ξ, η are the coordinates of any point whatever on the envelope of the curves $f(x, y, c) = 0$. This can only happen when Ξ contains the envelope-locus as a factor once at least. It does not contain it more than once in general; for, if x, y be put equal to ξ, η, and consequently $c_1 = \gamma$, in $\frac{\delta \Xi}{\delta x}$, $\frac{\delta \Xi}{\delta y}$; then

$\frac{\partial \Xi}{\partial x}$ becomes equal to the value of $Q\frac{\partial f(x, y, c_1)}{\partial x}$, and $\frac{\partial \Xi}{\partial y}$ to the v

of $Q\frac{\partial f(x, y, c_1)}{\partial y}$, when $x = \xi$, $y = \eta$, $c_1 = \gamma$.

Hence $\frac{\partial \Xi}{\partial x}$, $\frac{\partial \Xi}{\partial y}$ do not vanish in general when $x = \xi$, $y = \eta$.

But $\frac{\partial \Xi}{\partial x}$, $\frac{\partial \Xi}{\partial y}$ would both vanish when $x = \xi$, $y = \eta$, if Ξ conta the envelope-locus more than once as a factor. For, let $\phi = 0$ be envelope-locus, and let $\Xi = \phi^m \psi$, where m is a positive integer gre than unity. Therefore

$$\frac{\partial \Xi}{\partial x} = m\phi^{m-1}\frac{\partial \phi}{\partial x}\psi + \phi^m \frac{\partial \psi}{\partial x},$$

which vanishes when $x = \xi$, $y = \eta$, because $m > 1$.

Hence Ξ contains the envelope-locus once, and only once, in gen as a factor.

Some cases of exception will be noticed later on.

6. *To prove that, if* ξ, η *be any point on the node-locus of the cu* $f(x, y, c) = 0$; *then* Ξ $\frac{\partial \Xi}{\partial x}$, $\frac{\partial \Xi}{\partial y}$ *will all vanish when* x, y *are equal to* ξ, η; *and that consequently* Ξ *will in general contain the n locus twice, and twice only, as a factor.*

Let c_1 become γ (the parameter of the curve having its nod ξ, η), when x, y are put equal to ξ, η.

The values $x = \xi$, $y = \eta$, $c = \gamma$ satisfy all the equations

$$f = 0, \quad \frac{\partial f}{\partial c} = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

The same proof as in the case of the envelope-locus shows t these values make Ξ vanish. Substituting $x = \xi$, $y = \eta$, $c_1 = \gamma$ the values of $\frac{\partial \Xi}{\partial x}$, $\frac{\partial \Xi}{\partial y}$ given in Art. 4, it follows that they both va when $x = \xi$, $y = \eta$.

It will next be shown that, if the coordinates of every point on node-locus make

$$\Xi = 0 \quad \text{and} \quad \frac{\delta \Xi}{\delta x} = 0;$$

then they must also make $\quad \frac{\delta \Xi}{\delta y} = 0.$

For, let ξ, η be a point on the node-locus, and $\xi+\delta\xi$, $\eta+\delta\eta$ a neighbouring point also on the node-locus. Then

$$x = \xi+\delta\xi, \quad y = \eta+\delta\eta$$

must satisfy
$$\Xi = 0, \quad \frac{\delta\Xi}{\delta x} = 0;$$

therefore
$$\Xi + \frac{\delta\Xi}{\delta\xi}\,\delta\xi + \frac{\delta\Xi}{\delta\eta}\,\delta\eta + \ldots = 0,$$

$$\frac{\delta\Xi}{\delta\xi} + \frac{\delta^2\Xi}{\delta\xi^2}\,\delta\xi + \frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\,\delta\eta + \ldots = 0.$$

Retaining only the most important terms in the first of these two equations, it follows that

$$\frac{\delta\Xi}{\delta\eta} = 0.$$

Hence it is sufficient to use only the conditions that $x=\xi$, $y=\eta$ satisfy

$$\Xi = 0 \quad \text{and} \quad \frac{\delta\Xi}{\delta x} = 0.$$

It will now follow that Ξ contains the node-locus twice as a factor.

Because $\Xi = 0$, whenever x, y are put equal to ξ, η, the coordinates of any point on the node-locus, it follows (as in the case of the envelope-locus) that Ξ contains the node-locus once at least as a factor. Let $\phi = 0$ be the node-locus, and $\Xi = \phi R$; therefore

$$\frac{\delta\Xi}{\delta x} = \frac{\delta\phi}{\delta x}\,R + \phi\,\frac{\delta R}{\delta x}.$$

Put $x = \xi$, $y = \eta$; therefore

$$\frac{\delta\Xi}{\delta\xi} = \frac{\delta\phi}{\delta\xi}\,(R)_{\substack{x=\xi \\ y=\eta}}.$$

Now, $\dfrac{\delta\phi}{\delta\xi}$ will not, in general, vanish whenever x, y are put equal to ξ, η, the coordinates of any point on the node-locus.

Hence R must vanish whenever $x = \xi$, $y = \eta$.

Therefore R contains ϕ once as a factor at least.

Therefore Ξ contains ϕ twice as a factor at least.

Further, Ξ will not, in general, contain ϕ more than twice as a factor; for, if it did, then would $\dfrac{\delta^2\Xi}{\delta x^2}$, $\dfrac{\delta^2\Xi}{\delta x\,\delta y}$, $\dfrac{\delta^2\Xi}{\delta y^2}$ all vanish, whenever $x = \xi$, $y = \eta$; which is not the case, as it appears on substituting these values in the expressions given for the second differential co-

efficients in Art. 4 that $\frac{\delta^2 \Xi}{\delta x^2}$ would become equal to the value of

$$Q\left\{\frac{\partial^2 f(x, y, c_1)}{\partial x^2} - \frac{\left(\frac{\partial^2 f(x, y, c_1)}{\partial x \partial c_1}\right)^2}{\frac{\partial^2 f(x, y, c_1)}{\partial c_1^2}}\right\},$$

when $x = \xi$, $y = \eta$, $c_1 = \gamma$; and would not therefore vanish.

But $\frac{\delta^2 \Xi}{\delta x^2}$ would vanish when $x = \xi$, $y = \eta$, if Ξ contained the node-locus more than twice as a factor. For, let $\phi = 0$ be the node-locus and let $\Xi = \phi^m \psi$, where m is a positive integer greater than 2.

$$\text{Therefore}\quad \frac{\delta^2 \Xi}{\delta x^2} = m(m-1)\,\phi^{m-2}\left(\frac{\delta\phi}{\delta x}\right)^2 \psi + m\phi^{m-1}\frac{\delta^2 \phi}{\delta x^2}\psi$$

$$+2m\phi^{m-1}\frac{\delta\phi}{\delta x}\frac{\delta\psi}{\delta x} + \phi^m \frac{\delta^2\psi}{\delta x^2},$$

which vanishes when $x = \xi$, $y = \eta$, because $m > 2$.

Hence Ξ contains the node-locus twice, and twice only, as a factor, in general.

Some cases of exception will be noticed later on.

7. *To prove that, if* ξ, η *be any point on the cusp-locus of the curves* $f(x, y, c) = 0$; *then* Ξ, $\frac{\delta \Xi}{\delta x}$, $\frac{\delta \Xi}{\delta y}$, $\frac{\delta^2 \Xi}{\delta x^2}$, $\frac{\delta^2 \Xi}{\delta x \delta y}$, $\frac{\delta^2 \Xi}{\delta y^2}$ *will all vanish when* x, y *are put equal to* ξ, η; *and that consequently* Ξ *will, in general, contain the cusp-locus thrice, and thrice only, as a factor.*

As before, let, when $x = \xi$, $y = \eta$, c_1 become γ; then $x = \xi$, $y = \eta$, $c = \gamma$ satisfy all the equations

$$\left.\begin{array}{c} f = 0,\ \frac{\partial f}{\partial c} = 0,\ \frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial^2 f}{\partial x^2} : \frac{\partial^2 f}{\partial x \partial y} : \frac{\partial^2 f}{\partial x \partial c} = \frac{\partial^2 f}{\partial y \partial x} : \frac{\partial^2 f}{\partial y^2} : \frac{\partial^2 f}{\partial y \partial c} = \frac{\partial^2 f}{\partial c \partial x} : \frac{\partial f}{\partial c \partial y} : \frac{\partial^2 f}{\partial c^2} \end{array}\right\} \ldots (\text{A}).$$

In this case Ξ, $\frac{\delta \Xi}{\delta x}$, $\frac{\delta \Xi}{\delta y}$ vanish (as in the previous case), when $x = \xi$, $y = \eta$.

Further, when $x = \xi$, $y = \eta$, $c_1 = \gamma$, it follows that

$$\frac{\delta^2 \Xi}{\delta \xi^2} = \left[\left\{\frac{\partial^2 f(x, y, c_1)}{\partial x^2} - \frac{\left(\dfrac{\partial^2 f(x, y, c_1)}{\partial x\, \partial c_1}\right)^2}{\dfrac{\partial^2 f(x, y, c_1)}{\partial c_1^2}}\right\} Q\right]_{\substack{x = \xi \\ y = \eta \\ c_1 = \gamma}}.$$

[This result is equivalent to that given in Prof. Henrici's paper in Vol. II., page 108, of the *Proceedings of the London Mathematical Society*. Its interpretation leads to the theorem that the c-discriminant contains the cusp-locus thrice.]

Hence, by equations (A), it follows that

$$\frac{\delta^2 \Xi}{\delta \xi^2} = 0.$$

Similarly,

$$\frac{\delta^2 \Xi}{\delta \eta^2} = \left[\left\{\frac{\partial^2 f(x, y, c_1)}{\partial y^2} - \frac{\left(\dfrac{\partial^2 f(x, y, c_1)}{\partial y\, \partial c_1}\right)^2}{\dfrac{\partial^2 f(x, y, c_1)}{\partial c_1^2}}\right\} Q\right]_{\substack{x = \xi \\ y = \eta \\ c_1 = \gamma}}.$$

Hence, by equations (A), $\dfrac{\delta^2 \Xi}{\delta \eta^2} = 0.$

Again,

$$\frac{\delta^2 \Xi}{\delta \xi\, \delta \eta} = \left[\left\{\frac{\partial^2 f(x, y, c_1)}{\partial x \partial y} - \frac{\partial^2 f(x, y, c_1)}{\partial x\, \partial c_1}\, \frac{\dfrac{\partial^2 f(x, y, c_1)}{\partial y\, \partial c_1}}{\dfrac{\partial^2 f(x, y, c_1)}{\partial c_1^2}}\right\} Q\right]_{\substack{x = \xi \\ y = \eta \\ c_1 = \gamma}}.$$

Hence, by equations (A), $\dfrac{\delta^2 \Xi}{\delta \xi\, \delta \eta} = 0.$

It will now be shown that, of the three conditions

$$\frac{\delta^2 \Xi}{\delta x^2} = 0, \quad \frac{\delta^2 \Xi}{\delta x\, \delta y} = 0, \quad \frac{\delta^2 \Xi}{\delta y^2} = 0,$$

when x, y are put equal to ξ, η, it is sufficient to take the first only.

For $$\Xi = 0, \quad \frac{\delta \Xi}{\delta x} = 0, \quad \frac{\delta \Xi}{\delta y} = 0,$$

when x, y are put equal to ξ, η; and also when x, y are] put equal to $\xi + \delta\xi$, $\eta + \delta\eta$, the coordinates of a neighbouring point on the cusp-locus.

Therefore

$$\Xi+\frac{\delta\Xi}{\delta\xi}\delta\xi+\frac{\delta\Xi}{\delta\eta}\delta\eta+\tfrac{1}{2}\left(\frac{\delta^2\Xi}{\delta\xi^2}\delta\xi^2+2\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\xi\delta\eta+\frac{\delta^2\Xi}{\delta\eta^2}\delta\eta^2\right)+\ldots=0,$$

$$\frac{\delta\Xi}{\delta\xi}+\frac{\delta^2\Xi}{\delta\xi^2}\delta\xi+\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\eta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

$$\frac{\delta\Xi}{\delta\eta}+\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\xi+\frac{\delta^2\Xi}{\delta\eta^2}\delta\eta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0.$$

Hence, using $\Xi=0,\quad \dfrac{\delta\Xi}{\delta\xi}=0,\quad \dfrac{\delta\Xi}{\delta\eta}=0,\quad \dfrac{\delta^2\Xi}{\delta\xi^2}=0,$

these become
$$\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\xi\delta\eta+\tfrac{1}{2}\frac{\delta^2\Xi}{\delta\eta^2}\delta\eta^2+\ldots=0,$$

$$\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\eta+\ldots\ldots\ldots\ldots=0,$$

$$\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}\delta\xi+\frac{\delta^2\Xi}{\delta\eta^2}\delta\eta+\ldots\ldots=0,$$

or, retaining only the most important terms in each equation,

$$\frac{\delta^2\Xi}{\delta\xi\,\delta\eta}=0,\quad \frac{\delta^2\Xi}{\delta\eta^2}=0.$$

Now, it follows (as in the case of the node-locus) that Ξ contain the cusp-locus twice at least as a factor.

Let $\phi=0$ be the cusp-locus, and let $\Xi=\phi^2 S$.

Therefore $\dfrac{\delta^2\Xi}{\delta x^2}=2S\left(\dfrac{\delta\phi}{\delta x}\right)^2+$terms which vanish when $\phi=0$.

But
$$\frac{\delta^2\Xi}{\delta x^2}=2S\left(\frac{\delta\phi}{\delta x}\right)^2=0,$$

when x, y are put equal to ξ, η; which values, therefore, make $\phi=($

Hence $S\left(\dfrac{\delta\phi}{\delta x}\right)^2=0$ when x, y are put equal to ξ, η.

But $\left(\dfrac{\delta\phi}{\delta x}\right)^2$ does not in general vanish when x, y are put equal to ξ, η

Hence $S=0$ when x, y are put equal to ξ, η, the coordinates of an point on the cusp-locus $\phi=0$.

Hence S contains ϕ once at least as a factor.

Therefore Ξ contains ϕ thrice at least as a factor; it does not con tain it more often in general; for, if i[illegible]en its differentia

coefficients of the third order, at least, would vanish when x, y are put equal to ξ, η; which is not the case.

Some cases of exception will be noticed later on.

8. *On the Intersections of Consecutive Curves.*

It has now been shown that the c-discriminant of $f(x, y, c)$ in general contains the envelope-locus of the curves $f(x, y, c) = 0$ once as a factor, the node-locus twice, and the cusp-locus thrice.

There is an article,* No. 89a, in the Third Edition of Salmon's *Higher Plane Curves*, which shows how these results might be expected. What follows is the analytical equivalent of this article.

If c be eliminated between the equations

$$f(x, y, c) = 0, \quad \frac{\partial f(x, y, c)}{\partial c} = 0,$$

it is implied that the values of x, y are the same in each. In other words, x, y are points of intersection of the two curves

$$f(x, y, c) = 0 \text{ and } f(x, y, c+\partial c) = 0,$$

where ∂c is infinitely small. Two such curves will be called consecutive curves of the system, and their intersections will now be examined.

Let ξ, η be a point on the curve

$$f(x, y, \gamma) = 0,$$

and $\xi + X$, $\eta + Y$ a point in which this curve intersects the consecutive curve

$$f(x, y, \gamma+\partial\gamma) = 0,$$

where $\partial\gamma$ is infinitely small. Then

$$f(\xi, \eta, \gamma) = 0,$$

$$f(\xi+X, \eta+Y, \gamma) = 0,$$

$$f(\xi+X, \eta+Y, \gamma+\partial\gamma) = 0.$$

Therefore

$$0 = f(\xi, \eta, \gamma),$$

$$0 = f(\xi, \eta, \gamma) + Xf_\xi + Yf_\eta + \tfrac{1}{2}(X^2 f_{\xi\xi} + 2XYf_{\xi\eta} + Y^2 f_{\eta\eta}) + \dots,$$

$$\begin{aligned} 0 = f(\xi, \eta, \gamma) &+ Xf_\xi + Yf_\eta + \partial\gamma f_\gamma \\ &+ \tfrac{1}{2}\{X^2 f_{\xi\xi} + Y^2 f_{\eta\eta} + (\partial\gamma)^2 f_{\gamma\gamma} + 2XYf_{\xi\eta} + 2X\partial\gamma f_{\xi\gamma} + 2Y\partial\gamma f_{\eta\gamma}\} \\ &+ \dots . \end{aligned}$$

* For this reference I am indebted to Professor Cayley.

Therefore $0 = Xf_\xi + Yf_\eta + \frac{1}{2}(X^2 f_{\xi\xi} + 2XYf_{\xi\eta} + Y^2 f_{\eta\eta}) + \dots,$

$$0 = f_\gamma + \tfrac{1}{2}(\partial\gamma f_{\gamma\gamma} + 2Xf_{\xi\gamma} + 2Yf_{\eta\gamma}) + \dots.$$

The second equation shows that X, Y are in general finite. They will, however, be infinitely small of the order of $\partial\gamma$, if $f_\gamma = 0$. Hence a curve intersects its consecutive curve in points whose coordinates satisfy both the equations

$$f(\xi, \eta, \gamma) = 0, \quad \frac{\partial f(\xi, \eta, \gamma)}{\partial\gamma} = 0.$$

Hence these points of intersection lie—

(1) On the envelope, by what is proved in treatises on the Differential Calculus.

(2) On the node- and cusp-loci, by what is proved analytically in Art. 1 [see equation (V.)] of this paper.

It is, therefore, necessary to find out in how many points a curve intersects its consecutive curve when the point of intersection lies— (1) on the envelope-locus, (2) on the node-locus, (3) on the cusp-locus.

(1) Let the point of intersection be on the envelope; then, by the equations given above, it follows that, when X, Y are indefinitely small of the order $\partial\gamma$, in the limit

$$Xf_\xi + Yf_\eta = 0,$$

$$\partial\gamma f_{\gamma\gamma} + 2Xf_{\xi\gamma} + 2Yf_{\eta\gamma} = 0.$$

These give only one value for X, and one for Y; hence there is only one point of intersection here.

(2) Let the point of intersection be on the node-locus; then in the limit the equations are

$$X^2 f_{\xi\xi} + 2XYf_{\xi\eta} + Y^2 f_{\eta\eta} = 0,$$

$$\partial\gamma f_{\gamma\gamma} + 2Xf_{\xi\gamma} + 2Yf_{\eta\gamma} = 0.$$

These give two values for X, and two corresponding values for Y. Hence there are two points of intersection here.

(3) Let the point of intersection be on the cusp-locus; then in the limit the equations are

$$X^2 f_{\xi\xi} + 2XYf_{\xi\eta} + Y^2 f_{\eta\eta} + \text{terms of the third degree in } X, Y = 0,$$

$$\partial\gamma f_{\gamma\gamma} + 2Xf_{\xi\gamma} + 2Yf_{\eta\gamma} + \text{terms of the second degree in } \partial\gamma, X, Y = 0.$$

Now, by means of the equations (A) of Art. 7,

$$X^2 f_{\xi\xi} + 2XYf_{\xi\eta} + Y^2 f_{\eta\eta} = \frac{f_{\xi\xi}}{f_{\xi\gamma}^2}(Xf_{\xi\gamma} + Yf_{\eta\gamma})^2.$$

Hence the equations are

$$\frac{f_{\xi\xi}}{f_{\xi\gamma}^2}(Xf_{\xi\gamma}+Yf_{\eta\gamma})^2+\text{terms of the third degree in } X,\ Y=0,$$

$$\partial\gamma f_{\gamma\gamma}+2Xf_{\xi\gamma}+2Yf_{\eta\gamma}+\text{terms of the second degree in } \partial\gamma,\ X,\ Y=0.$$

As a first approximation the last equation gives

$$Y=-\frac{1}{2f_{\eta\gamma}}(2Xf_{\xi\gamma}+\partial\gamma f_{\gamma\gamma}).$$

Hence, substituting in the first equation, it follows that

$$\tfrac{1}{4}\frac{f_{\xi\xi}f_{\gamma\gamma}^2}{f_{\xi\gamma}^2}(\partial\gamma)^2+\text{terms of the third degree in } X,\ \partial\gamma=0\ ;$$

but

$$f_{\xi\xi}=\frac{f_{\xi\gamma}f_{\xi\gamma}}{f_{\eta\gamma}}=\frac{f_{\xi\gamma}^2}{f_{\gamma\gamma}}.$$

Hence this equation becomes

$$\tfrac{1}{4}f_{\gamma\gamma}(\partial\gamma)^2+\text{terms of the third degree in } X,\ \partial\gamma=0.$$

This is a cubic for X, and there are three corresponding values of Y. Hence there are three intersections here.

Hence the number of intersections of consecutive curves at any point of the envelope-, node-, or cusp-locus, is the same as the number of occurrences of the corresponding locus as a factor in the c-discriminant.

These results will now be applied to explain some cases in which the number of times that a singular locus appears in the c-discriminant exceeds the number of times given above, viz., once for an envelope-locus, twice for a node-locus, and thrice for a cusp-locus—cases, therefore, for which the preceding theory would appear to be incomplete. It was pointed out at the end of Articles 5, 6, 7 that, though the theorems there proved were true in general, yet there were cases of apparent exception, and it is desirable to show that some at least of these can be shown to be included in the general theory. (It is beyond the scope of this paper to examine all possible cases of exception.)

Example I.—Consider the curves

$$[\chi(x,y,c)]^2-[\phi(x,y)]^2\,\psi(x,y)=0,$$

where ϕ, ψ are rational integral functions of x, y; and χ is a rational integral function of x, y, c.

Let ϕ, ψ, χ be of degrees m, n, p, respectively, in x, y; and let χ be of degree q in c.

Hence, to find the node-locus, it is necessary to satisfy at the time the equations

$$\chi^2-\phi^2\psi = 0,$$

$$2\chi\chi_x-2\phi\phi_x\psi-\phi^2\psi_x = 0,$$

$$2\chi\chi_y-2\phi\phi_y\psi-\phi^2\psi_y = 0.$$

These can all be satisfied by taking $\chi = 0$, $\phi = 0$; and, in ge these will give all the common solutions. Hence, as ϕ does not tain c, $\phi = 0$ is the node-locus.

Each curve has mp nodes, viz., the points where $\phi = 0$, $\chi =$ tersect. At each point of the node-locus there are q nodes of consecutive curves; for, if ξ, η be a point on $\phi = 0$, it is a node curves of the system corresponding to the q values of c, whic roots of the equation $\qquad \chi(\xi, \eta, c) = 0.$

Each of the q non-consecutive curves intersects its consecutive in two points, thus making $2q$ intersections at ξ, η. Hence the locus $\phi = 0$ may be expected to occur $2q$ times in the c-discrimi

To verify this, calculate the c-discriminant, *i.e.*, eliminate c fr

$$\chi^2-\phi^2\psi = 0, \quad 2\chi\chi_c = 0.$$

The second equation in c has q roots which make $\chi = 0$, and $(q$ roots which make $\chi_c = 0$. Hence the c-discriminant is

$$(-\phi^2\psi)^q \times \big[\text{the factor corresponding to the } (q-1) \text{ roots of } \chi_c =$$

Hence the node-locus ϕ occurs $2q$ times in the c-discriminant.

Again, $\psi = 0$ is a part of the envelope, and it appears q time cause, at each point ξ, η of the locus $\psi = 0$, it is touched by q c of the system, viz., those corresponding to the q values of c d mined by the equation $\qquad \chi(\xi, \eta, c) = 0.$

EXAMPLE II.—Consider the curves

$$(c^2-cy+x)^2-(x+y)^3 = 0.$$

To find the c-discriminant, eliminate c from this equation, and

$$2(c^2-cy+x)(2c-y) = 0.$$

Hence the c-discriminant is

$$(x+y)^6\left[(x-\tfrac{1}{4}y^2)^2-(x+y)^3\right] = 0.$$

Here the cusp-locus appears six times, because at each point ξ, η there are two cusps belonging to non-consecutive curves of the sy

viz., those whose parameters satisfy the equation $c^2 - c\eta + \xi = 0$, where $\xi + \eta = 0$. Each of these curves intersects its consecutive in three points at ξ, η. Hence there are altogether six intersections at ξ, η. Hence the cusp-locus appears six times.

EXAMPLE III.—The following example is worthy of notice, as it shows that the cusps are distributed over the cusp-locus in such a manner that, though each curve has two cusps, yet there is never more than one cusp at a single point of the cusp-locus.

Consider the curves $(x^2 + y^2 - c)^2 - (y-1)^3 = 0.$

The cusp-locus is $y = 1.$

Each curve of the system has two cusps, which are situated at the intersections of $y - 1 = 0$ and $x^2 + y^2 - c = 0$; so that the cusps are always on opposite sides of the axis of y, and therefore at each point of the cusp-locus there is but one cusp of the system of curves. The curve to which this cusp belongs intersects its consecutive in three points: hence the cusp-locus may be expected to appear three times in the c-discriminant, as it does; for the c-discriminant is $(y-1)^3$.

On the Loci of Contacts of Parallel Tangents.

9. Let the differential equation of the curves $f(x, y, c) = 0$, be

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0.$$

Let
$$p = \frac{dy}{dx}.$$

Then the process frequently adopted for finding the singular solution is to find the p-discriminant of $\phi(x, y, p)$. Comparing this with the other method, viz., the finding of the c-discriminant of $f(x, y, c)$, it appears that p here takes the place of c in the previous case. But

$$p = \frac{dy}{dx}.$$

Hence the function $\frac{dy}{dx}$ is used here in the same way as the constant was used before.

This suggests the importance of discussing the loci of points on the curves $f(x, y, c) = 0$, at which $\frac{dy}{dx}$ is constant, *i.e.*, at which the tangent lines all make the same angle with the axis of x.

If the differential equation be $\phi(x, y, p) = 0$, then the lo given by $\phi(x, y, a) = 0$, where a is an arbitrary parameter.

It is useful, however, to examine these loci without reference differential equation at all. They may be called the loci of co of parallel tangents, and the following theorems will be prov regard to them—

(1) These loci all touch the envelope of the curves $f(x, y, c$
(2) These loci all touch the cusp-locus of the curves $f(x, y, c$
(3) The tac-locus of the curves $f(x, y, c) = 0$ is, in genera node-locus, or a part of the node-locus, of the loci of tacts of parallel tangents.

10. *To find the direction of the locus of contacts of parallel tange any point, which is not a singular point of the curves* $f(x, y, c) = ($ *to prove that the loci of contacts of parallel tangents touch the envel the curves* $f(x, y, c) = 0$.

Let α be the inclination to the axis of x of the tangent to the $f(x, y, \gamma) = 0$ at the point ξ, η on it, ξ, η being supposed to be singular point of this curve.

Therefore $$\tan\alpha = -\left(\frac{f_x}{f_y}\right)_{\substack{x=\xi\\ y=\eta\\ c=\gamma}}.$$

Let $\xi+\partial\xi$, $\eta+\partial\eta$ be the coordinates of a point on the curve
$$f(x, y, \gamma+\partial\gamma) = 0,$$
at which the tangent also makes an angle α with the axis

Therefore $$\tan\alpha = -\left(\frac{f_x}{f_y}\right)_{\substack{x=\xi+\partial\xi\\ y=\eta+\partial\eta\\ c=\gamma+\partial\gamma}}.$$

Hence $$-\left(\frac{f_x}{f_y}\right)_{\substack{x=\xi\\ y=\eta\\ c=\gamma}} = -\left(\frac{f_x}{f_y}\right)_{\substack{x=\xi+\partial\xi\\ y=\eta+\partial\eta\\ c=\gamma+\partial\gamma}}.$$

Hence, using D with the same meaning as in Art. 2,
$$D\left(\frac{f_x}{f_y}\right) = 0.$$

Again, $\xi+\partial\xi$, $\eta+\partial\eta$ is a point on the curve
$$f(x, y, [illegible]) = 0;$$
therefore $$f(\xi+\partial\xi, [illegible]\gamma) = 0.$$
But also f [illegible]

Hence $$Df = 0.$$

From the two equations

$$Df = 0, \quad D\left(\frac{f_\xi}{f_\eta}\right) = 0,$$

it follows that

$$\frac{\partial\xi}{f_\gamma \frac{\partial}{\partial\eta}\left(\frac{f_\xi}{f_\eta}\right) - f_\eta \frac{\partial}{\partial\gamma}\left(\frac{f_\xi}{f_\eta}\right)} = \frac{\partial\eta}{f_\xi \frac{\partial}{\partial\gamma}\left(\frac{f_\xi}{f_\eta}\right) - f_\gamma \frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right)}$$

$$= \frac{\partial\gamma}{f_\eta \frac{\partial}{\partial\xi}\left(\frac{f_\xi}{f_\eta}\right) - f_\xi \frac{\partial}{\partial\eta}\left(\frac{f_\xi}{f_\eta}\right)} \quad \text{......(XVII.)}.$$

The ratio $\partial\eta : \partial\xi$ gives the direction of the locus of contacts of parallel tangents at ξ, η.

If now ξ, η be a point on the envelope,

$$f_\gamma = 0;$$

therefore $$\frac{\partial\xi}{-f_\eta \frac{\partial}{\partial\gamma}\left(\frac{f_\xi}{f_\eta}\right)} = \frac{\partial\eta}{f_\xi \frac{\partial}{\partial\gamma}\left(\frac{f_\xi}{f_\eta}\right)};$$

therefore $$f_\xi \partial\xi + f_\eta \partial\eta = 0.$$

But this same equation determines the direction of the envelope. Hence the loci of contacts of parallel tangents touch the envelope.*

11. *To find the direction of the locus of contacts of parallel tangents at a point on the cusp-locus of the curves $f(x, y, c) = 0$, and to show that the loci of contacts of parallel tangents touch the cusp-locus.*

In this article it is useful to adopt the following notation. Denote $\frac{\partial f(\xi, \eta, \gamma)}{\partial\xi}$ by (ξ), $\frac{\partial^2 f(\xi, \eta, \gamma)}{\partial\xi\, \partial\eta}$ by (ξ, η), and so on.

The direction of the tangent at the cusp situated at the point ξ, η is given by

$$(\xi, \xi)(\partial x)^2 + 2(\xi, \eta)\, \partial x\, \partial y + (\eta, \eta)(\partial y)^2 = 0,$$

which, being a perfect square, reduces to

$$(\xi, \xi)\, \partial x + (\xi, \eta)\, \partial y = 0.$$

Therefore, if a be the inclination of this tangent to the axis of x,

$$\tan a = -\frac{(\xi, \xi)}{(\xi, \eta)}.$$

* For a geometrical proof of this theorem see Note at end.

Now, let $\xi+\partial\xi$, $\eta+\partial\eta$ be that point on the curve

$$f(x, y, \gamma+\partial\gamma) = 0$$

at which the tangent is parallel to the tangent at the cusp at ξ, η

Then
$$-\frac{(\xi, \xi)}{(\xi, \eta)} = -\left(\frac{f_x}{f_y}\right)_{\substack{x=\xi+\partial\xi \\ y=\eta+\partial\eta \\ c=\gamma+\partial\gamma}}$$

$$= -\frac{(\xi)+(\xi,\xi)\,\partial\xi+(\xi,\eta)\,\partial\eta+(\xi,\gamma)\,\partial\gamma+\frac{1}{2}R_2}{(\eta)+(\eta,\xi)\,\partial\xi+(\eta,\eta)\,\partial\eta+(\eta,\gamma)\,\partial\gamma+\frac{1}{2}S_2}$$

................(XVI]

where $R_2 = (\xi,\xi,\xi)\,\partial\xi^2+(\xi,\eta,\eta)\,\partial\eta^2+(\xi,\gamma,\gamma)\,\partial\gamma^2$
$+2\,(\xi,\xi,\eta)\,\partial\xi\,\partial\eta+2\,(\xi,\xi,\gamma)\,\partial\xi\,\partial\gamma+2\,(\xi,\eta,\gamma)\,\partial\eta$

and $S_2 = (\eta,\xi,\xi)\,\partial\xi^2+(\eta,\eta,\eta)\,\partial\eta^2+(\eta,\gamma,\gamma)\,\partial\gamma^2$
$+2\,(\eta,\xi,\eta)\,\partial\xi\,\partial\eta+2\,(\eta,\xi,\gamma)\,\partial\xi\,\partial\gamma+2\,(\eta,\eta,\gamma)\,\partial\eta$

Now
$$(\xi) = 0, \quad (\eta) = 0.$$

Also
$$\frac{(\xi,\xi)}{(\xi,\eta)} = \frac{(\xi,\eta)}{(\eta,\eta)} = \frac{(\xi,\gamma)}{(\eta,\gamma)}.$$

Hence (XVIII.) becomes $R_2-\dfrac{(\xi,\xi)}{(\xi,\eta)}S_2 = 0;$

or, putting
$$A = (\xi,\xi,\xi)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\xi,\xi),$$

$$B = (\xi,\eta,\eta)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\eta,\eta),$$

$$C = (\xi,\gamma,\gamma)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\gamma,\gamma),$$

$$F = (\xi,\eta,\gamma)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\eta,\gamma),$$

$$G = (\xi,\xi,\gamma)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\xi,\gamma),$$

$$H = (\xi,\xi,\eta)-\frac{(\xi,\xi)}{(\xi,\eta)}(\eta,\xi,\eta),$$

(XVIII.) becomes

$$A\,\partial\xi^2+B\,\partial\eta^2+C\,\partial\gamma^2+2F\,\partial\eta\,\partial\gamma+2G\,\partial\gamma\,\partial\xi+2H\,\partial\xi\,\partial\eta = 0\ldots(\mathrm{XI}$$

Further, since ξ, η is on the curve $f(x, y, \gamma)=0$, and $\xi+\partial\xi$, $\eta+\partial$

the curve $f(x, y, \gamma+\partial\gamma) = 0$,

$$f(\xi, \eta, \gamma) = 0, \quad f(\xi+\partial\xi, \eta+\partial\eta, \gamma+\partial\gamma) = 0\,;$$

whence it follows that

$$\begin{aligned}(\xi)\,\partial\xi+(\eta)\,\partial\eta+(\gamma)\,\partial\gamma \\ +\tfrac{1}{2}\big[(\xi,\xi)\,\partial\xi^2+(\eta,\eta)\,\partial\eta^2+(\gamma,\gamma)\,\partial\gamma^2 \\ +2\,(\eta,\gamma)\,\partial\eta\,\partial\gamma+2\,(\gamma,\xi)\,\partial\gamma\,\partial\xi+2\,(\xi,\eta)\,\partial\xi\,\partial\eta\big] = 0.\end{aligned}$$

But at a point on the cusp-locus

$$(\xi) = 0, \quad (\eta) = 0, \quad (\gamma) = 0,$$

$$(\xi,\xi)(\eta,\eta) = (\xi,\eta)^2, \quad (\xi,\xi)(\gamma,\gamma) = (\xi,\gamma)^2, \quad (\xi,\xi)(\eta,\gamma) = (\xi,\eta)(\xi,\gamma).$$

Hence the last equation is equivalent to

$$\begin{aligned}\frac{1}{(\xi,\xi)}\big[(\xi,\xi)^2\,\partial\xi^2+(\xi,\eta)^2\,\partial\eta^2+(\xi,\gamma)^2\,\partial\gamma^2 \\ +2\,(\xi,\eta)(\xi,\gamma)\,\partial\eta\,\partial\gamma+2\,(\xi,\xi)(\xi,\gamma)\,\partial\xi\,\partial\gamma+2\,(\xi,\xi)(\xi,\eta)\,\partial\xi\,\partial\eta\big] = 0.\end{aligned}$$

Hence $\qquad (\xi,\xi)\,\partial\xi+(\xi,\eta)\,\partial\eta+(\xi,\gamma)\,\partial\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots(\text{XX.}).$

If $\partial\gamma$ be eliminated between (XIX.) and (XX.), an equation of the second degree in $\partial\eta : \partial\xi$ will be found. It will be shown first that the roots are equal, and afterwards the value of the equal roots will be calculated.

First, to prove that the roots are equal, let $\partial\xi$, $\partial\eta$, $\partial\gamma$ be treated as trilinear coordinates, and all the other quantities in (XIX.) and (XX.) as constants; then (XIX.) is the equation of a conic, and (XX.) of a straight line; and it is necessary to prove that the straight line touches the conic, *i.e.*, to show that

$$\begin{vmatrix} A & H & G & (\xi,\xi) \\ H & B & F & (\xi,\eta) \\ G & F & C & (\xi,\gamma) \\ (\xi,\xi) & (\xi,\eta) & (\xi,\gamma) & 0 \end{vmatrix} = 0,$$

i.e.,

$$\begin{vmatrix} A-H\dfrac{(\xi,\xi)}{(\xi,\eta)} & H-B\dfrac{(\xi,\xi)}{(\xi,\eta)} & G-F\dfrac{(\xi,\xi)}{(\xi,\eta)} & 0 \\ H-G\dfrac{(\xi,\eta)}{(\xi,\gamma)} & B-F\dfrac{(\xi,\eta)}{(\xi,\gamma)} & F-C\dfrac{(\xi,\eta)}{(\xi,\gamma)} & 0 \\ G & F & C & (\xi,\gamma) \\ (\xi,\xi) & (\xi,\eta) & (\xi,\gamma) & 0 \end{vmatrix} = 0.$$

To prove this relation. The equations (XVI.) of Art. 2 ma written

$$\frac{D(\xi,\xi)}{(\xi,\eta)} - \frac{(\xi,\xi)}{(\xi,\eta)^2} D(\xi,\eta)$$

$$= \frac{D(\xi,\eta)}{(\eta,\eta)} - \frac{(\xi,\eta)}{(\eta,\eta)^2} D(\eta,\eta)$$

$$= \frac{D(\xi,\gamma)}{(\eta,\gamma)} - \frac{(\xi,\gamma)}{(\eta,\gamma)^2} D(\eta,\gamma);$$

or, if $\partial\xi$, $\partial\eta$ be elements of the cusp-locus corresponding to an alt tion $\partial\gamma$ in γ, then the above become

$$\frac{(\xi,\xi,\xi)\,\partial\xi + (\xi,\xi,\eta)\,\partial\eta + (\xi,\xi,\gamma)\,\partial\gamma}{(\xi,\eta)}$$

$$- \frac{(\xi,\xi)}{(\xi,\eta)^2} \{(\xi,\eta,\xi)\,\partial\xi + (\eta,\eta,\xi)\,\partial\eta + (\gamma,\eta,\xi)\,\partial\gamma\}$$

$$= \frac{(\xi,\xi,\eta)\,\partial\xi + (\eta,\xi,\eta)\,\partial\eta + (\gamma,\xi,\eta)\,\partial\gamma}{(\eta,\eta)}$$

$$- \frac{(\xi,\eta)}{(\eta,\eta)^2} \{(\xi,\eta,\eta)\,\partial\xi + (\eta,\eta,\eta)\,\partial\eta + (\gamma,\eta,\eta)\,\partial\gamma\}$$

$$= \frac{(\xi,\xi,\gamma)\,\partial\xi + (\eta,\xi,\gamma)\,\partial\eta + (\gamma,\xi,\gamma)\,\partial\gamma}{(\eta,\gamma)}$$

$$- \frac{(\xi,\gamma)}{(\eta,\gamma)^2} \{(\xi,\eta,\gamma)\,\partial\xi + (\eta,\eta,\gamma)\,\partial\eta + (\gamma,\eta,\gamma)\,\partial\gamma\}$$

But this with the symbols A, B, C, F, G, H may be written

$$\frac{A\partial\xi + H\partial\eta + G\partial\gamma}{(\xi,\eta)} = \frac{H\partial\xi + B\partial\eta + F\partial\gamma}{(\eta,\eta)} = \frac{G\partial\xi + F\partial\eta + C\partial\gamma}{(\gamma,\eta)},$$

which, by means of equations (A), can be re-written

$$\frac{A\partial\xi + H\partial\eta + G\partial\gamma}{(\xi,\xi)} = \frac{H\partial\xi + B\partial\eta + F\partial\gamma}{(\xi,\eta)} = \frac{G\partial\xi + F\partial\eta + C\partial\gamma}{(\xi,\gamma)}$$

...............(XX

therefore

$$\left[A - \frac{(\xi,\xi)}{(\xi,\eta)} H\right]\partial\xi + \left[H - \frac{(\xi,\xi)}{(\xi,\eta)} B\right]\partial\eta + \left[G - \frac{(\xi,\xi)}{(\xi,\eta)} F\right]\partial\gamma = 0,$$

$$\left[H - \frac{(\xi,\eta)}{(\xi,\gamma)} G\right]\partial\xi + \left[B - \frac{(\xi,\eta)}{(\xi,\gamma)} F\right]\partial\eta + \left[F - \frac{(\xi,\eta)}{(\xi,\gamma)} C\right]\partial\gamma = 0,$$

and equation (VI.) gives

$$(\xi,\xi)\,\partial\xi \quad + \quad (\xi,\eta)\,\partial\eta \quad + \quad (\xi,\gamma)\,\partial\gamma \quad = 0.$$

Hence, eliminating $\partial\xi$, $\partial\eta$, $\partial\gamma$, it follows that

$$\begin{vmatrix} A-\dfrac{(\xi,\xi)}{(\xi,\eta)}H & H-\dfrac{(\xi,\xi)}{(\xi,\eta)}B & G-\dfrac{(\xi,\xi)}{(\xi,\eta)}F \\ H-\dfrac{(\xi,\eta)}{(\xi,\gamma)}G & B-\dfrac{(\xi,\eta)}{(\xi,\gamma)}F & F-\dfrac{(\xi,\eta)}{(\xi,\gamma)}C \\ (\xi,\xi) & (\xi,\eta) & (\xi,\gamma) \end{vmatrix} = 0.$$

Hence the required relation is satisfied.

This being so, the direction of the locus of contacts of parallel tangents is to be determined by eliminating $\partial\gamma$ from (XIX.) and (XX.).

Multiply (XIX.) by $(\xi,\gamma)^2$; therefore

$$(\xi,\gamma)^2\left[A\partial\xi^2+B\partial\eta^2+2H\partial\xi\partial\eta\right]$$

$$+(\xi,\gamma)(\xi,\gamma)\partial\gamma\left[2G\partial\xi+2F\partial\eta\right]+C\left[(\xi,\gamma)\partial\gamma\right]^2=0;$$

therefore

$$(\xi,\gamma)^2\left[A\partial\xi^2+B\partial\eta^2+2H\partial\xi\partial\eta\right]$$

$$-2(\xi,\gamma)\left[(\xi,\xi)\partial\xi+(\xi,\eta)\partial\eta\right]\left[G\partial\xi+F\partial\eta\right]$$

$$+C\left[(\xi,\xi)\partial\xi+(\xi,\eta)\partial\eta\right]^2=0;$$

therefore

$$(\partial\xi)^2\left[A(\xi,\gamma)^2-2G(\xi,\xi)(\xi,\gamma)+C(\xi,\xi)^2\right]$$

$$+2\partial\xi\partial\eta\left[H(\xi,\gamma)^2-F(\xi,\gamma)(\xi,\xi)-G(\xi,\gamma)(\xi,\eta)+C(\xi,\xi)(\xi,\eta)\right]$$

$$+(\partial\eta)^2\left[B(\xi,\gamma)^2-2F(\xi,\gamma)(\xi,\eta)+C(\xi,\eta)^2\right]=0.$$

This has been shown to be a perfect square. Hence, multiplying by the coefficient of $(\partial\xi)^2$, and extracting the square root, it follows that the direction of the locus of contacts of parallel tangents at a cusp of the curves $f(x, y, c) = 0$ is given by the equation

$$\partial\xi\left[A(\xi,\gamma)^2-2G(\xi,\xi)(\xi,\gamma)+C(\xi,\xi)^2\right]$$

$$+\partial\eta\left[H(\xi,\gamma)^2-F(\xi,\gamma)(\xi,\xi)-G(\xi,\gamma)(\xi,\eta)+C(\xi,\xi)(\xi,\eta)\right]=0.$$

Again, the direction of the cusp-locus may be determined from (VI.) and the first and third ratios of equations (XXI.), *i.e.*,

$$(\xi,\xi)\partial\xi+(\xi,\eta)\partial\eta+(\xi,\gamma)\partial\gamma=0,$$

and

$$\partial\xi\left\{A(\xi,\gamma)-G(\xi,\xi)\right\}+\partial\eta\left\{H(\xi,\gamma)-F(\xi,\xi)\right\}$$

$$+\partial\gamma\left\{G(\xi,\gamma)-C(\xi,\xi)\right\}=0.$$

Eliminating $\partial\gamma$, it follows that

$$\partial\xi\left[A(\xi,\gamma)^2-2G(\xi,\xi)(\xi,\gamma)+C(\xi,\xi)^2\right]$$
$$+\partial\eta\left[H(\xi,\gamma)^2-F(\xi,\xi)(\xi,\gamma)-G(\xi,\eta)(\xi,\gamma)+C(\xi,\eta)(\xi,\xi)\right]=($$

Hence the loci of contacts of parallel tangents touch the cusp-loci

12. To *prove that the tac-locus of the curves $f(x, y, c)=0$ is, in gene the node-locus, or a part of the node-locus, of the loci of contacts of para tangents to the curves $f(x, y, c) = 0$.*

At a point ξ, η on the tac-locus where the curve $f(x, y, \gamma')$ = touches $f(x, y, \gamma'') = 0$, the following relations hold:—

$$f(\xi,\eta,\gamma')=0,\quad f(\xi,\eta,\gamma'')=0,$$

$$-\frac{\dfrac{\partial f(\xi,\eta,\gamma')}{\partial\xi}}{\dfrac{\partial f(\xi,\eta,\gamma')}{\partial\eta}}=-\frac{\dfrac{\partial f(\xi,\eta,\gamma'')}{\partial\xi}}{\dfrac{\partial f(\xi,\eta,\gamma'')}{\partial\eta}}.$$

Now, the direction of the locus of contacts of parallel tangents the point ξ, η on the curve $f(x, y, c) = 0$, is given by

$$\frac{\partial x}{f_c\dfrac{\partial}{\partial\eta}\left(\dfrac{f_\xi}{f_\eta}\right)-f_\eta\dfrac{\partial}{\partial c}\left(\dfrac{f_\xi}{f_\eta}\right)}=\frac{\partial y}{f_\eta\dfrac{\partial}{\partial c}\left(\dfrac{f_\xi}{f_\eta}\right)-f_c\dfrac{\partial}{\partial\xi}\left(\dfrac{f_\xi}{f_\eta}\right)}.$$

The values of $\dfrac{\partial y}{\partial x}$, obtained by putting $c=\gamma', \gamma''$ respectively, will in general, different. Hence the locus of contacts of parallel tange passes, in general, in two non-coincident directions from a point the tac-locus. Hence the tac-locus of the curves $f(x, y, c) = 0$ is, general, at least a part of the node-locus of the loci of contacts parallel tangents. It will be shown in the next article it is, in gene the whole of that node-locus.

[If the directions given for the tangents to the loci of contacts parallel tangents at a point on the tac-locus should coincide, then t above theorem will require modification; but this is an exceptio case, and is accordingly excluded from consideration in this paper

13. *To show that the p-discriminant of the differential equation of curves $f(x, y, c) = 0$ contains, in general, the envelope-locus once as factor, the cusp-locus once, and the tac-locus twice.*

Prof. Cayley has shown that the p-discriminant contains the envelope-, cusp-, and tac-loci as factors.

The theorems given in Arts. 10, 11, 12 will determine the number of times that each factor is repeated in general.

If the differential equation of the curves be $\phi(x, y, p) = 0$, then the loci of contacts of parallel tangents are given by $\phi(x, y, a) = 0$, where a is an arbitrary parameter. It has been shown (Art. 10) that these curves all touch the envelope-locus of the curves $f(x, y, c) = 0$. Hence that envelope is at least a part of the envelope of the curves $\phi(x, y, a) = 0$. Hence, by Art. 5, the a-discriminant of the curves $\phi(x, y, a) = 0$, which is the same as the p-discriminant of $\phi(x, y, p) = 0$, will contain the envelope-locus of the curves $f(x, y, c) = 0$ once as a factor, and only once in general.

Again, it has been shown (Art. 11) that the curves $\phi(x, y, a) = 0$ all touch the cusp-locus of the curves $f(x, y, c) = 0$. Hence that cusp-locus is at least a part of the envelope of the curves $\phi(x, y, a) = 0$. Hence, by Art. 5, the a-discriminant of the curves $\phi(x, y, a) = 0$, which is the same as the p-discriminant of $\phi(x, y, p) = 0$, will contain the cusp-locus of the curves $f(x, y, c) = 0$ once as a factor, and only once in general.

Further, it has been shown (Art. 12) that the tac-locus of the curves $f(x, y, c) = 0$ is, in general, at least a part of the node-locus of the curves $\phi(x, y, a) = 0$. Hence, by Art. 6, the a-discriminant of the curves $\phi(x, y, a) = 0$, which is the same as the p-discriminant of $\phi(x, y, p) = 0$, will contain the tac-locus of the curves $f(x, y, c) = 0$ twice as a factor, and twice only in general.

Hence, in general, the p-discriminant of $\phi(x, y, p) = 0$ contains the envelope-locus once as a factor, the cusp-locus once, and the tac-locus twice; and, in general, there are no other factors.

14. It is shown in Prof. Cayley's paper that, in general, the p-discriminant is made up of the factors corresponding to the envelope-, cusp-, and tac-loci; *i.e.*, that these factors are the only ones which will arise when the curves considered have the ordinary singularities, viz., nodes and cusps. It is, therefore, conceivable that singularities of a higher order may give rise to other factors in the p-discriminant.

Further, $\phi(x, y, p)$ is rational and integral in p. Hence it is possible at once to write down a form of $\phi(x, y, p)$, such that its p-discriminant may contain a cubic factor; and that such a differential equation may be derivable from an algebraic primitive of the limited form considered in the present paper, will be evident from an example, which will be examined in detail, and the factors of its c- and p-discriminants explained upon the principles employed in the general case.

Consider the equation $$y^3p^2 = x^3,$$
for which the p-discriminant is x^3y^3.

In this case $$y^{\frac{3}{2}}p = \pm x^{\frac{3}{2}},$$

therefore $$y^{\frac{5}{2}} = \pm x^{\frac{5}{2}} \pm c^{\frac{1}{2}},$$

therefore $$y^5 = x^5 \pm 2c^{\frac{1}{2}}x^{\frac{5}{2}} + c,$$

therefore $$(y^5-x^5-c)^2 = 4cx^5,$$

therefore $$(x^5-y^5)^2-2c(x^5+y^5)+c^2 = 0.$$

The c-discriminant is x^5y^5.

It is necessary to explain the meaning of the factors in the c p-discriminants.

To do this, it must be observed that the curves considered each, at the points $x = 0$, $y = c^{\frac{1}{5}}$, and $y = 0$, $x = c^{\frac{1}{5}}$, a singu which is equivalent to a cusp and a double point.

A penultimate form of each curve of the family may be obt by replacing x^5 by $x^3(x-a)^2$, and y^5 by $y^3(y-b)^2$, where a, b are s The equation will then be

$$(X-Y)^2-2c(X+Y)+c^2 = 0,$$

where $$X = x^3(x-a)^2, \quad Y = y^3(y-b)^2.$$

The c-discriminant is $XY = 0$, *i.e.*,

$$x^3(x-a)^2y^3(y-b)^2 = 0.$$

To get the p-discriminant, form first the differential equation, tre Y as dependent, and X as independent variable. This reduces, rejecting factors which do not contain any differential coefficient

$$-Y+\left(\frac{dY}{dX}\right)^2 X = 0;$$

and this, in like manner, reduces to

$$y(5y-3b)^2\left(\frac{dy}{dx}\right)^2 - x(5x-3a)^2 = 0;$$

therefore the p-discriminant is

$$xy(5x-3a)^2(5y-3b)^2.$$

And the c-discriminant was shown to be

$$x^3(x-a)^2y^3(y-b)^2.$$

Hence $$x = a,\ y = b \text{ are node-loci,}$$
$$x = 0,\ y = 0 \text{ are cusp-loci,}$$
$$x = \frac{3a}{5},\ y = \frac{3b}{5} \text{ are tac-loci.}$$

Now, diminish a, b each indefinitely; then the c-discriminant contains x as a factor five times—twice as node-locus, thrice as cusp-locus; and y in like manner.

Also the p-discriminant contains x as a factor three times—once as cusp-locus, and twice as tac-locus, and y in like manner.

From this example it is clear that curves of the limited form considered in this paper may give rise to differential equations whose p-discriminants contain factors of degree higher than the second. To investigate the converse question,—viz., given that the p-discriminant has a factor of a given degree, to find out how it arises,—is beyond the scope of the present paper. It will only be remarked that it may arise in different ways; for example, a factor of the first degree has been shown to be due either to an envelope-locus or a cusp-locus. A factor of the second degree may be due to a tac-locus directly, as in the general case which has been explained, or indirectly, as in the case of a higher singularity which is equivalent to two nodes, as in the following example (or in some other way).

EXAMPLE.—Consider the curves

$$a^3 (y-c)^2 = (x+b)\, x^4,$$

where a, b are fixed constants, and c is the arbitrary parameter.

The c-discriminant is $\qquad (x+b)\, x^4.$

The differential equation of the curves is

$$4a^3 (x+b)\, p^2 - x^2 (5x+4b)^2 = 0.$$

Therefore the p-discriminant is

$$(x+b)\, x^2\, (5x+4b)^2.$$

To investigate the meaning of the factors, take as the penultimate curves $$a^3 (y-c)^2 = (x+b)(x^2-e^2)^2,$$

where e is small. The c-discriminant is

$$(x+b)(x-e)^2\, (x+e)^2.$$

The differential equation of the curves is

$$4a^3 (x+b)\, p^2 = (5x^2+4bx-e^2)^2.$$

Therefore the p-discriminant is

$$(x+b)(5x^2+4bx-e^2)^2.$$

Hence the envelope-locus is $x+b=0$,

the node-loci are $x+e=0,\ \ x-e=0$,

the tac-loci are $5x^2+4bx-e^2=0$,

or $$x=-\frac{2b}{5}\pm\frac{\sqrt{2b^2+5e^2}}{5}.$$

Hence, when e vanishes,

the envelope-locus is $x+b=0$,

the node-loci are $x=0,\ \ x=0$,

the tac-loci are $x=-\frac{4b}{5},\ \ x=0$;

therefore the c-discriminant should be

$$(x+b)\,x^2\,.\,x^2,$$

the p-discriminant should be

$$(x+b)\left(x+\frac{4b}{5}\right)^2 x^2,$$

which is the case (neglecting numerical factors). So that, in t discriminant, the factor x^2 is due to a tac-locus indirectly involv the higher singularity formed by two nodes.

Note added February, 1889.

[The theorem in Art 10, viz., that the loci of contacts of pa tangents touch the envelope of the curves $f(x, y, c)=0$, admit simple geometrical proof, as follows :—

Adopting the geometrical representation of Art. 3, let a cyl whose generating lines are parallel to the straight line $y=x$ $z=0$, be drawn enveloping the surface $f(x, y, z)=0$. The c in which this cylinder touches the surface, projects into the c $\phi(x, y, \tan\alpha)=0$ on the plane of x, y. Consider a point on th velope of the curves $f(x, y, c)=0$, at which the tangent mak angle α with the axis of x. This point is the projection of a po on the surface $f(x, y, z)=0$, which lies on the enveloping cyl which projects into the envelope of the curves $f(x, y, c)=0$. the tangent plane to the surface $f(x, y, z)=0$, the tangent pla

* $\phi(x, y, p)=0$ denoting the differential equation of the curves $f(x, y, c$

the enveloping cylinder which projects into the envelope of the curves $f(x, y, c) = 0$, and the tangent plane to the enveloping cylinder whose curve of contact projects into $\phi(x, y, \tan\alpha) = 0$, are all coincident. Hence the envelope of the curves $f(x, y, c) = 0$, and the curve $\phi(x, y, \tan\alpha) = 0$, touch at the point which is the projection of P.]

Presents received during the Recess (August, 1888) :—

"Royal Society, Proceedings of the," Nos. 268 and 269.

"Physical Society of London, Proceedings," Vol. IX., Part 3.

"Educational Times" for July and August.

"Royal Society of Edinburgh, Proceedings of the," Nos. 115 to 125.

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"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 6 and 7.

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Presents received during the Recess (October, 1888) :—

"Royal Society—Proceedings," No. 270, Vol. XLIV.

"Educational Times" for September.

"Royal Dublin Society—Scientific Transactions," Vol. III., No. 14; Vol. IV., No. 1; 4to; Dublin, 1887 and 1888.

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"A New Theory of Parallels," Part i., by C. L. Dodgson (Two copies), 8vo; London, 1888.

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"Annales de l'École Polytechnique de Delft," Tome iv., 1 and 2 Livr.; Leiden, 1888.

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"The Journal of the College of Science, Imperial University, Japan," Vol. ii., Part 1.

"Memorias de la Sociedad Cientifica - Antonio Alzate," Tomo i., Núm. 12; Tomo ii., Núm. 1.

"Über Lamésche Funktionen mit Komplexen Parametern," Inaugural-Dissertation der Albertus-Universität zu Königsberg i. Pr., von Fritz Cohn, 8vo pamphlet; Königsberg, 1888.

A number of pamphlets (12) by Maurice d'Ocagne, M. Émile Lemoine, Dr. Voss (Zur Erinnerung an Axel Harnack), and G. Heinricius and H. Kronecker.

"An Elementary Treatise on Algebra," Part i., by S. C. Basu, B.A.; Calcutta, 1888. From the Author.

(i.) "On Poisson's Integral." (ii.) "A General Theorem of the Differential Equations of Trajectories." (iii.) "Remarks on Monge's Differential Equation to all Conics." By Prof. Mukhopâdhyây, M.A. From the Author.

APPENDIX.

Mr. Basset's communication on "The Motion of a Sphere in a Viscous Liquid" (*cf.* Vol. XVIII., pp. 388, 392) is published in the *Proceedings of the Royal Society*, Vol. XLIII., No. 260, pp. 174, 175; and Mr. Walker's memoir "On the Diameters of a Plane Cubic" (*cf.* Vol. XVIII., l.c.) has appeared in Vol. CLXXIX. A. (pp. 151—203) of the same Society's *Philosophical Transactions.*

In connection with Prof. J. J. Thomson's paper "On Electrical Oscillations in Cylindrical Conductors" (Vol. XVII., Nos. 272, 273), we refer readers to a criticism by Mr. Oliver Heaviside in a "Note on a paper on Electro-magnetic Waves" (*Phil. Mag.*, March 1888, pp. 202—210), wherein the writer impugns the accuracy of some of Prof. Thomson's results.*

Mr. Forsyth draws attention to his footnote on p. 28, and points out that a history of the theorem there quoted is given by Dr. T. Muir, in the *Phil. Mag.*, for Nov. 1884, "An overlooked discoverer in the Theory of Determinants." The relation of the form given by Sylvester to the general form is given in § 11 of Dr. Muir's paper.

Mr. John Brooksmith was born at Huddersfield, on the 17th of July, 1824. He received his early education at Huddersfield College, whence he proceeded to Edinburgh University. Here he obtained the Gold Medal for Mathematics, and then passed to St. John's College, Cambridge. At the University he displayed mathematical ability of a high order, but ill-health prevented him from entering for the Mathematical Tripos. In due course he took his M.A. degree. Having selected the scholastic profession for his life's work, he obtained, in 1850, the appointment at Cheltenham College, which he held till within a month of his death (on the 5th May, 1888), viz., the Second Mastership on the Modern Side. How utterly unexpected was his death may be inferred from the fact that he vacated the above post to undertake that of Headmaster of the Modern Department, and that, until the last month of his life, he had not for twenty

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* *Cf.* p. 534 of this Volume for Prof. Thomson's correction.

years been apart from his class a single day through indisposit As a boarding-master his popularity was most marked, so that f June 1853, when he opened Boyne House, until the date of his de that establishment has been constantly full, and indeed for m years quite inadequate to meet the applications of parents ' desired that their sons should be under Mr. Brooksmith's care. Charles Warren, Sir Charles Wilson, the late Col. Barrow, of Sou reputation, and many others of all professions, but more especiall the army, were his pupils. In 1872, he published his *Arithmeti Theory and Practice*, which has passed into a seventh edition. Bes this well-known work, he was the author of several pamphlets mathematics. The actual cause of death is somewhat obscure, bu was on the Saturday morning that a rupture, doubtless of the nat of aneurism, occurred, causing death with appalling suddenness.*

Mr. Arthur Buchheim, the eldest son of Professor Buchheim King's College, was born in the year 1859, and was educated at City of London School, whence, in 1877, he obtained an open Ma matical Scholarship at New College, Oxford. He obtained a F Class in Mathematical Moderations in December, 1878, and a F Class in the Mathematical Final Schools, in December, 1880. 1881, he was elected to the Senior University Mathematical Scho ship. From Easter to Christmas, 1881, he attended Prof. Kle lectures, in the University of Leipzig, and was a member of seminary. From September, 1882, to Christmas, 1887, he wa Mathematical Master at the Manchester Grammar School. Leipzig episode in his life " no doubt contributed to widening his tellectual horizon, but at the same time had the unfortunate effec getting him out of the style of ordinary English university exam tions, in consequence of which he abstained, although strongly pre by the authorities to do so, from offering himself as a candidate f vacant fellowship at the College of which he was a scholar."† " was a man of singular modesty and goodness of heart, which m him beloved by all who were brought into connection with hi Mr. Buchheim was also an Oriental student of some promise. I feared his life may have been shortened by his intense applicatio study, as, we have been informed, after the arduous labours of day, he would sit long at night to study Sanskrit, Persian, Chin and Russian. His father writes, "I think one of the favourite pl of my beloved son was to write a history of mathematics."

* Our authority for the above account is the *Cheltenham College Magazine*, w furnishes other details of interest, and also gives an account of the funeral.

† Prof. Sylvester, in *Nature*, Sept. 27th, 1888.

He was elected a member of the Society on the 9th of March, 1882, and a member of Council on the 10th of November, 1887; and died, during his tenure of office, on September 9th, 1888. Those members who saw him at the June meeting of the Society, plainly felt that death had already marked him as a victim, yet he contributed several apposite remarks to the discussion, with his usual fulness of knowledge on the subjects which he had made peculiarly his own. This was more particularly the case at the Council meeting, with reference to a paper for which he had acted as referee. Dr. Buchheim, in his letter to the Council, remarks that, "he highly valued from the beginning his connection with your Society, which served him as a stimulus in his mathematical studies, and the fact that he was honoured last year with a seat on the Council was very encouraging to him in his laborious pursuits." There is some likelihood of his papers being collected and published, to serve as a literary remembrance of him.

Mr. Buchheim contributed the following papers to our *Proceedings*:—

On the Extension of certain Theories relating to Plane Cubics to Curves of any Deficiency (Vol. xiii.).

On the Theory of Screws in Elliptic Space (four papers in Vols. xv., xvi., xvii., xviii.).

On the Theory of Matrices (Vol. xvi.).

On Clifford's Theory of Graphs (Vol. xvii.).

To the *Messenger of Mathematics* he contributed several Notes which are printed in Vols. xi. (p. 143), xii. (p. 129), xiii. (pp. 62, 120), xiv. (pp. 74, 127, 143, 167).

In the *Philosophical Magazine* (Nov., 1884), he gave a "Proof of Prof. Sylvester's Third Law of Motion." Of this Prof. Sylvester writes "'The three laws of motion,' of which it forms one, were formulated by me in one of the Johns Hopkins Circulars, and it is a proof of the keenness of his research, that the subject of this notice (probably the only mathematician in Europe) should have made himself so well acquainted with them as to be able to write an independent paper on the subject. They have no direct connection (except in a Hegelian sense) with mechanical principles, but are three cardinal principles in my Theory of Universal Algebra, between which and Newton's Three Laws of Motion I considered that I had succeeded in establishing a one-to-one correspondence." (*Nature*, l.c.)

One other paper he contributed to the *American Journal of Mathematics* (Vol. vii., No. 4), entitled "A Memoir on Biquaternions."

It is very grateful to us to know that Arthur Buchheim "always

spoke in the friendliest terms" of us, and our somewhat f
correspondence with him quite bore out the testimony we hav
above as to his modesty and goodness of heart.

23rd January, 1889. R.

The following corrections should be made in the present vol

p. 42, lines 3—8, *for* the + sign within the brackets, *read* − (thus cha
sign of the second term in the coefficients).

p. 261, line 3 from foot, on sinister, *for* $-c'$ *read* c'.

p. 266, Art. 119, line 1, *for* 96, 97 *read* 116, 117.

p. 270, line 3, in last denominator, *for* $D-a$ *read* $D-a-2$.

p. 270, line 4, *for* a *read* a or b.

Additional errata will be found on pp. 55, 56, 67.

The following additional errata occur in Vol. XVIII. :—

p. 183, line 5, accent the the negative or middle term of the sinister.

p. 191, line 4, *for* $\dot{p}+\frac{\dot{r}}{r}$ *read* $\dot{p}+\frac{\dot{r}'}{\dot{r}}$.

p. 193, line 4, *for* second *read* third.

p. 200, Art. 77, line 1, *for* This *read* The sinister of this.

p. 200, Art. 77, line 8, *for* c *read* $\dot{c}$, and *for* $\dot{c}$ *read* c.

By an oversight, Nos. 301—303 of Vol. XVIII. were not indexed.

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INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed* in extenso, *or an abstract of it given in the "Proceedings," or in the Appendix.*]

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